



# PRIMORDIAL COSMOLOGY

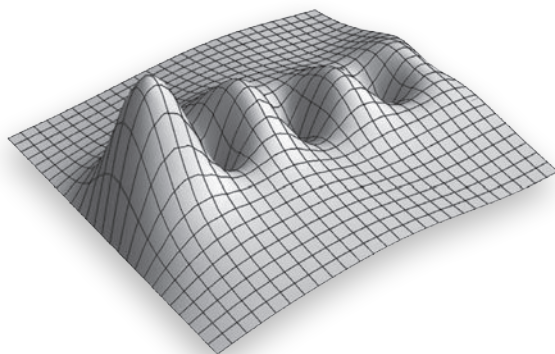
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The authors dedicate this Book to all the people who, like them, regarded and will regard the understanding of the Early Universe as the greatest challenge of their lives.

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# Preface

Understanding the Origin and Evolution of the Universe is certainly one of the most ambitious and fascinating attempts of the human intelligence. This intellectual and scientific adventure is not only the major approach to give answers to fundamental questions concerning our existence but also this activity provides our intelligence with the special role to be the only (known) tool the Universe has to *investigate* itself, acquiring awareness on the Reality essence.

Despite this ambitious path could discourage from addressing any perspective, the scientific investigation of the Universe has reached surprising and impressive achievements and offers many reliable answers about our origin.

This volume summarizes the most important results of the scientific cosmology, describing the observational knowledge about the Universe evolution and how it allowed the derivation of a theoretical paradigm, able of predictions beyond detected phenomena.

Our analysis is based on a rigorous mathematical characterization of the cosmological topics, finding in the Einstein theory of General Relativity the privileged descriptive physical tool. For the wide and coherent cosmological scenario, this Book is built up as a reference for both students interested to an introductory path, and also for specialists who desire to deepen selected topics.

This Book faces the analysis of many aspects of Modern Cosmology, starting from the presentation of well-grounded assessments on the observed Universe and their theoretical interpretation, up to the discussion of very speculative topics concerning the nature of the *Cosmological singularity*, which are timely for the scientific debate.

The content can be successfully approached by any reader having a



certain familiarity with the concepts and the formalisms at the ground of General Relativity. The presentation is purposely self-consistent when characterizing some canonical topics with a pedagogical perspective, as well as when serving an advanced profile for the subjects requiring a wider background knowledge.

Some peculiar approaches to modern cosmology are treated in their general aspects and included only to provide the reader with a broad vision of the contemporary lines of research, although averting from the core perspective of the Book, and referring to the specific literature for details.

We widely illustrate a series of modern cosmological issues, re-enforcing the idea that the highly symmetric nature of our Universe, as observed at very large scales, is not a primordial notion but results from an evolutionary process of very general initial conditions.

The first part of the Book (Chapters 1 and 2) is devoted to a historical picture of the notion of Universe across the centuries and then addresses the fundamental formalism of General Relativity and differential geometry for the modern approaches to the Early Universe providing the basic notions for the non-specialist reader. The so-called *Physical Cosmology* (Chapters 3-6) is faced in details, introducing the structure and the evolution of the isotropic Universe, according to the *Standard Cosmological Model*. Particular attention is dedicated to the phenomenology of the Universe with respect to the implications for the interpretation of the features implied by the theoretical prescriptions of different models. This aim is also pursued by analyzing the inflationary paradigm as well as a detailed treatment of the density inhomogeneities faced by the perturbation theory to the isotropic Universe and by the paradigm of the *quasi-isotropic solution*.

The part of the Book entitled *Mathematical Cosmology* (Chapters 7-9) gives a wide discussion of the general features of the Universe near the singularity, when the isotropy and homogeneity assumptions are removed. We start with a geometrical characterization of the homogeneous three-dimensional spaces, as arranged in the Bianchi classification, whose dynamics is treated in the framework of General Relativity, also by means of a Hamiltonian formulation. In this respect, our presentation is focused on the study of the chaotic dynamics of the Bianchi type VIII and IX models near the singularity. Here, the discussion clearly separates well-established results from timely reformulations of the problem or aspects yet open to scientific investigation. Finally, we endow this part by a deep analysis of the generic inhomogeneous solution near the cosmological singularity,

derived in analogy to the chaotic homogeneous model and implementing the parametric role played by the space coordinates. The nature of the space-time foam, characterizing the generic Universe asymptotically to the singular point, is outlined toward a proper statistical picture. This study of cosmologies more general than the isotropic Universe is also extended to multidimensional issues, in view of the interest raised through the recent literature.

The last part of the Book is focused on *Quantum Cosmology* (Chapters 10-12) and it touches very timely questions in applying Quantum Gravity proposals to the Universe origin and its primordial evolution. We discuss different approaches to the quantization of the gravitational field, concentrating the investigation on the canonical method, both in the original Wheeler-DeWitt and in the more promising Loop Quantum Gravity approaches. However, different points of view, like the path integral quantization and the generalized Heisenberg non-commutative algebras procedures are also taken into account and implemented on specific models. Our quantum description of the Universe is very general and addresses all the most relevant features, especially in view of the interpretative shortcomings of the different approaches. The recent and outstanding success of Loop Quantum Cosmology in determining the existence of a Big Bounce at the Universe birth is eventually traced with care. Indeed, the quantum nature of the cosmological singularity (its removal or survival) is a central theme and many different issues are contrasted in view of their motivating hypotheses. A part of such Section treats the interpretation of a semiclassical Universe and its tendency to isotropization.

The material presented in the Chapters on quantum cosmology clarifies the motivation for an intense study of the mathematical cosmology, because the general character of the cosmological models in the regime asymptotic to the initial singularity makes them as the most appropriate for the implementation of a quantum theory to the Universe birth. The idea that the classical Universe comes out of the Planck epoch via a semiclassical limit, say when its volume expectation value is large enough, implies that we must have a clear understanding of such general dynamics already at a classical level. The possibility to link the anisotropic and inhomogeneous cosmologies to the isotropic Universe, underlying the Standard Cosmological model, can be recognized in the inflation scenario. In fact, the vacuum energy responsible for the de Sitter phase of the inflating Universe is a strong isotropic term, able to stretch the inhomogeneities at scales much

larger than the physical horizon and to suppress anisotropic features.

This scenario prescribes that the Universe is born from a singularity-free generic inhomogeneous model; when its volume is probabilistically large enough, it transits to a quasi-classical dynamics, and then it is reconciled to the Standard Cosmological Model by the inflationary process. Such a point of view constitutes the leading perspective suggested and partially demonstrated by this Book.

We would like to express our gratitude to Dr Massimiliano Lattanzi for his precious contribution to the part of this Book devoted to *Physical Cosmology*, especially in view of the effort made to link the theoretical framework to the present knowledge of the observed Universe.

Dr Nakia Carlevaro and Dr Francesco Cianfrani are thanked for their help in writing Sec. 3.6 on the Lemaître-Tolmann-Bondi model. F.C. is also thanked for his comments and suggestions on the part of the Book concerning *Quantum Cosmology*.

We would like to acknowledge Dr Simone Speziale for his valuable suggestions on that part in which we review the main features of the Loop Quantum Gravity theory.

Finally, a special thought of G.M. is devoted to the memory of Prof. Kensuke Yoshida, who appreciably encouraged him to write the review article<sup>1</sup>, from which the project of this Book arose.

*Giovanni Montani*  
*Marco Valerio Battisti*  
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*Giovanni Imponente*

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<sup>1</sup>Montani, G., Battisti, M.V., Benini, R. and Imponente, G.(2008). Classical and Quantum Features of the Mixmaster Singularity, *Int. J. Mod. Phys. A* **23**, pp. 2353 - 2503, doi:10.1142/S0217751X08040275.

$\kappa = 8\pi G$	Einstein constant
$\hbar = c = 1$	natural units
$(+, -, -, -)$	metric signature
$g_{ij}$	metric tensor
$h_{\alpha\beta}$	spatial metric tensor
$ds^2$	space-time line element
$dl^2$	spatial line element
$\mathcal{M}$	space-time manifold
$\Sigma$	space-like surface
$S$	action
$\mathcal{L}$	Lagrangian density
$\mathcal{H}$	Hamiltonian density
$H$	Hubble parameter
$\rho$	energy density
$P$	pressure
$p$	particle momentum
$i, j, k, \dots$	space-time indices
$\alpha, \beta, \gamma, \dots$	spatial indices
$I, J, K, \dots$	internal 4-d indices
$a, b, c, \dots$	internal 3-d indices

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## **PART 1**

# **Historical and Basic Notions**

In these Chapters, we present the fundamental concepts relevant for the further developments of the topics.

Chapter 1 is devoted to provide a historical picture concerning the notion of Universe through the centuries.

Chapter 2 gives a pedagogical review of the fundamental formalisms required for the self-consistency of the presentation.

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## Chapter 1

# Historical Picture

This Chapter is devoted to draw the historical path of Cosmology from the first written evidences in the ancient cultures up to contemporary science and serves as an introduction to the topics of modern Cosmology which constitute the main core of the Book.

Although synthesized, we will provide a vision of how the concept of Cosmology as a realm outside the daily experience evolved from a religious or philosophical task toward a well-grounded scientific subject of investigation and discussion. The history of this evolution has been slow since the experimental hints pushing to find explanation to natural phenomena have been very limited up to the last century, leaving Cosmology to the area of Astronomy at most, and only in regions of the world where the social environment supported such kind of science. Finally, since the object of investigation cannot be reproduced in a laboratory or allows any “second try” for any test, the theoretical approach has often been influenced by personal beliefs, traditions and not-scientific related issues.

To give a peculiar introduction to Theoretical Cosmology, this Chapter stresses how the relationship of the human beings evolved with respect to the celestial phenomena, enlarging their (and our) view of the space surrounding the planet where we live and following how they pushed further and further the borders of the Universe around them.

### 1.1 The Concept of Universe Through the Centuries

#### 1.1.1 *The ancient cultures*

The perception of the nature as vital environment had a variegated evolution and aim, either with the passing of the centuries or in the different geo-

graphical areas. As a first instance, the modern science finds its two main roots in the Greek-Roman basin and in the Oriental world. Nevertheless, when the latter flourished, showing a written tradition which dated back to around 1300–1200 B.C. with the religious and philosophical setting of Hinduism with the first doctrines in the *Rig Veda* sacred books and thereafter in the *Upanishad* around the IX–VIII century B.C., the former was still in an embryonic stage. The principal effort of the understanding pursued by the philosophers in the Eastern area was devoted to deal with existential problems, considering the sense of knowledge towards ones' salvation and freedom. The role of the Universe was a paradigm, a reference for the human behavior. On the other side, around the Mediterranean sea, other cultures showed attention to the natural phenomena starting from the Chaldee who, even before 2000 B.C., addressed much efforts towards the comprehension of celestial phenomena, taking records of eclipses, constellations shapes, astral conjunctions, and considering a primordial Zodiac. In a nearby area, the Egyptians, strong of the emerging mathematical techniques, developed an accurate record of stars and constellations arising in particular from the precise orientation evident in the pyramid foundations, as well as also in Mesopotamia, under the kingdom of Hammurabi of Babylon. Such cultures received a strong influence from the Oriental connections, were the privilege of a sacerdotal caste and relied on myths regarding the origin of the Universe, plenty of fantastic representations with a strong religious and traditionalistic role. Much differently, on the Greek side, grown in the Hellenistic age, the basic demand was toward a break with the preceding beliefs, in a rational investigation of the natural phenomena. Although it is natural that the Greek science was aware of other traditions, like for example a catalogue of the possible astronomical phenomena, Egyptian and Mesopotamians used them to study the cosmic order and its influence on the daily life of their kingdoms (essentially as horoscopes, *observation of time*). The Greeks tended to a more theoretical approach, with the purpose of understanding the causes of the astral movements, the origin of the eclipses, passing from a description to a tentative rational explanation, getting toward the fundamental abstraction provided by the concept of a *formula* and the *law*, i.e. the inclusion of the infinite possible cases of realizing a certain situation. The socio-economical environment of the Greek cities, where the republican society is much more dynamic than the absolute power kingdoms in other areas, will reflect in the vision of the Universe as an evolutionary process, rationally questionable, whose basis is not under the control of a religious group but can be debated by the free thought of

the philosophers.

The shape and the borders of the physical world changed with the size of the observable Universe, thus limited to the astronomical observable phenomena and leaving all the remaining to the metaphysical one where gods and myths had a crucial role, often communicating with the sensible experience. The concept of Universe was intended to include both of them and the principal effort was to point out its *origin*.

A common aspect of all cultures when trying to explain or describe the whole Universe is that it must be emerging *from* a limited and compact description, which can be a basic element (water, earth, fire) or made (from a breathe, or by a god) by a clearly defined subject.

The analysis of the human activities and the comprehension of the existing world from the natural events in relation to the social and economic strategies is pursued by a unique figure provided by the philosopher. Physics will be considered as a branch of philosophy up to the end of the XVII century and on many topics both of them remain deeply related to Religion.

### 1.1.2 *Ancient Greek and the Mediterranean*

Aristotle mentioned that the poet Hesiod (around 700 B.C.) was the first to search for a unique principle for all the things, «First existed chaos, then the earth . . . and finally the love among the gods»: although the origin of the Universe finds an answer from the myth, it opened the way to several explanations devoted to find a fundamental one, unrelated to the activities of the common life but essentially metaphysical, *beyond the nature*. The actual Universe is regarded as an ordered state and the Cosmology is devoted to find the unity which guarantees its origin and its equilibrium.

During the VI century B.C. in the Ionian region, flourished a strong society based in several important centers such as Miletus, Ephesus, Colophon, Samos, with a class of merchants willing to expand in the Mediterranean area, going from Black Sea to Egypt, to the Caucasus, Sicily, Spain and France with a mentality open to overcome the limits of the magic beliefs and devoted to a more accurate care to the rational observations of the natural phenomena. In this environment a group of thinkers called Pre-Socratic (also known as Pre-Sophists) is formed, whose main topic of investigation is the *cosmological problem*: the Ionians pose at the basis of everything a *unique* and *eternal* reality, provided by the *arché* (i.e. a principle) for the matter from which everything comes out. The primordial force is provided

of an intrinsic force allowing all movements.

This cultural current has been initiated by Thales (around 585 B.C.) who seeks in the water the basic element from which everything comes out. In the same years Anaximander didn't look at a specific material element but to the *infinity*, the infinite amount of matter, the origin of all what is observed. This principle is never ending and undying, it is outside the world and comprises: he developed an idea for the process of the matter generation through successive separations, breaking the uniqueness of the primordial infinity and providing a law governing the nature. His idea of the Earth is in the form of a cylinder and the human beings come from inside fishes where some of them developed and finally they were thrown outside to live by themselves.

In the years 530–490 B.C. Pythagoras and his School were active in Magna Grecia (across southern Italy) and funded the modern mathematics, introducing basic concepts, the first rigorous demonstrations providing abstractions from many empirical situations. Every geometrical figure is a deployment of points and the numbers measure the order of everything. The true nature of the world comes as a measurable order of basic elements and the opposition which is manifest in the real world can be comprehended in a natural cosmic harmony from which everything proceeds.

A deep crisis of the Pythagoreans philosophers sprung with the discovery of the infinity in mathematics and in numerics. Their astronomical knowledge was rather evolved, as based on the sphericity of the Earth and of the celestial bodies. Such shape, as maximum expression of harmony, was rephrased also in the model for the sky: the Earth moving around a central Fire, together with all other celestial bodies. All celestial bodies were classified as the sky of the fixed stars, the five planets (Saturn, Jupiter, Mars, Mercury, Venus), the Sun, Moon, Earth and anti-Earth (hypothetical planet in order to reach the sacred number of 10). Only few decades later Aristarchus from Samos (III century B.C.) proposed the Sun as the center of the celestial spheres, thus clearly anticipating the Copernican heliocentrism.

As we see from what described, the concept itself of Universe was limited to what was the knowledge of existence beyond the Earth: in this view we have the two extremes of the local astronomy on one side and on the other a more effective ontological concept about the origin of everything, with the concept of far (fixed) stars in between. Although in the subsequent evolution the astronomical knowledge deviated toward an Earth-centric system following a peculiar line of debate, this frame remained almost unchanged

up to the first astronomical evidences of different orbiting systems and celestial bodies beyond our galaxy.

The concept of eternal evolution, within the harmony of the opposites, dominates the thought of Heraclitus from Ephesus (VI–V cent. B.C.), whose idea of *becoming* identifies alternating eras of destruction-production, assessing a strong difference with the philosophies from the East.

At this stage, the concept of Universe and that of Cosmos are still deeply different from the idea in the readers' mind: Cosmos, in ancient Greek "*ornament*" is considered as a decoration of the celestial sphere and its physical reality is deeply mixed and indistinct from its ontological counterpart.

One of the first proposals regarding the origin of matter derived from the physical experience lies in the idea of Anaxagoras (mid V century B.C.). He argued about the absence of either a minimal or a maximal size, from the tiniest pieces of matter to the whole existing Universe, where the divine *Nous*, intelligence, orders the original seeds properly mixing them as they appear in the world. From the original chaos of such seeds a swirling movement produced the Earth and the stars result from the lightening of the particles coming from it and even the Sun.

The concept of a minimum dimension for the particles constituting the matter has been proposed by Democritus (ca. 460 B.C. ca. 370 B.C.), with the atoms (i.e. non divisible particles) which are unchangeable, eternal and indestructible. They chaotically move in every direction, whirling in infinite ways and assembling in an infinite number of ways which are born and die perpetually. The impulse to a mechanistic attitude in the investigation of the nature lies in the search for the causes of the events, looking for quantitative properties and definition of objective properties.

The influence on the evolution of thought of Socrates first and then of his disciple Plato in the environment of the Academia in Athens relied mainly on the investigation of ethics and other problems focused on man, his interior and relations, recovering in the cosmological picture the primordial link between celestial bodies and the divinities.

A new effort toward the investigation about the system of the nature and science was by the Plato disciple Aristotle (about 384–322 B.C.). A systematic classification of all branches of the knowledge provided a clear statement of the role of Physics, well separated from Metaphysics, Theology and other branches of philosophy. The fundamental topic of his investigation is the *being* in movement with all its qualities and properties: he classified three species of motion, from the center of the world upward, from



the high downward, referred to generation and corruption of compounded substances, perishable and mutating. The third species is the circular one, which has no contraries, thus the substances moving with this peculiar motion are necessarily unchangeable, un-generable, incorruptible. Aristotle regards the *ether* as compounding celestial bodies and being the only element in circular movement, different from all other elements. This opinion regarding a material for the celestial objects distinct from the remainder of the Universe and therefore not subject to birth, death and alteration, as indeed for the matter encountered by experience, will last for a long time in the western culture, finally revised and abandoned in the XV century by Nicolaus Cusanus (Nicholas of Kues). For Aristotle, the physical Universe, comprising the sky made of ether and the sub-lunar world made by the four elements (water, air, earth, fire), is perfect, unique, finite and eternal. The basic elements are displaced in a *natural* order: at the center of the world the earth is the heaviest element; around it there are the spheres referring to the decreasing order of weight, water, air and fire. The fire constitutes the outer zone of the sub-lunar sphere and around it we have the first ethereal (or celestial) sphere, the one of the Moon. In his view, the Universe is *a priori* perfect and thus *finite*. In fact, infinity would be related to an uncompleted and unfinished property, lacking a part and which could eventually be added of something. Moreover, for him a real thing cannot be infinite as anything in the real life has a direction and a well defined position in the space: since no physical reality can be infinite, the sphere of the fixed stars marks the *limits* of the Universe beyond which there is no space. This is the maximum volume and no line can cross its diameter: other worlds cannot exist besides our and since the space cannot be empty, even vacuum cannot exist, either infra-cosmic (between common objects) either extra-cosmic, as one allocating the Universe itself. In this scheme, if it is meaningful to ask where is an object, this is not true for the Universe: it is indeed the container of all what exists. This is a revolutionary view that strongly adheres with the modern speculations. The concept of time, strongly related to that of space, is analyzed as the property of the becoming and the changing of the common things. From this point of view, the world as a whole is perfect and finiteness is eternal, without an origin and without an end. Eternity is seen as different from the infinite duration of time, it is the atemporal existence of the immutable, thus the world has never been generated and will never be destroyed, comprising all its alterations. Aristotle does not formulate a cosmogony since time is eternal and comprises all single local events: the world is eternal as well and has not

an origin.

The ideas of Aristotle had an enormous impact on the following thinkers on several topics related to science, such as biology and logics, for several centuries: although in the Middle Age the Arab philosophers were more influential, they considered him as the principal expression of the human reason, he will shape and feed the philosophy up to the XVI century all over Europe in the university studies. A deep criticism to his ideas started with the birth of the modern science, when his astronomical and physical theories were found unable to describe the world under the evidence of the discoveries performed in such years.

### 1.1.3 *The Hellenistic era*

The IV century B.C., in particular after the death of Alexander the Great in 323 B.C., saw his immense empire divided in three big reigns as Macedonia, Egypt and Asia characterized by a universalistic cultural progression, with the birth of several new places for the social and cultural life. In this fertile environment of social changes, there was a great attention to the particular sciences, separated from the speculative philosophy of the past. Alexandria of Egypt became the paradigmatic city giving an enormous impulse to increase and feed the cultural activities, with the 700,000 volumes in its Bibliotheca, its Museum with a center of studies and research, an astronomical observatory, a zoo, a botanical garden and the first anatomy tables. It hosted scientists and teachers paid by the government which could devote their time to free research. Several other cities followed the example of Alexandria, whose magnificence will continue to be renowned even after its destruction in 641 A.C.

The peculiar environment of a society split between a diffuse wealthy governance and intellectual class and, on the other side, the abundance of slavery lead to a clear separation between science and technique, thus polarizing the investigations of philosophers toward very speculative attitudes. For example, Zeno of Citium (336-35 B.C.–264-63 B.C.), initiator of the Stoic school was very keen for the role of science, whose basic concept relies on an immutable, rational, perfect and necessary order, coinciding also with a religious point of view. The whole world life performs a cycle, even the stars turn around up to the same initial position: everything started with a conflagration and the destruction of all existing beings. At that point, another identical cycle restarts.

An important step ahead, with the purpose of containing the religious influence on the cosmic vision, was performed by Epicurus (341 B.C.–270 B.C.). He aimed at leaving out any role of the gods in the nature design, envisaged an infinite number of worlds, each one with a birth and a death and each of them constituted by a finite number of atoms moving in an infinite vacuum.

These cosmological speculations introduced a great bloom for all sciences in the Hellenistic era, lasting approximately from 300 B.C. to 145 B.C.: in this year the Museum was destroyed during the civil war, the intellectual elite had to abandon Alexandria starting a period of decadence, marked by a first fire in the Bibliotheca in 48 B.C. The only exceptions were offered during the II century A.D. by Claudius Ptolemaeus for astronomy and Galen for medicine.

The Alexandrine era (after III century B.C.), had three great mathematicians, Euclid, Archimedes and Apollonius, whose efforts in organizing the knowledge and rigorous approach to calculus and geometry provided the basis also of the important astronomic speculations which remained valid and undiscussed up to the XX century.

Although from Plato and Aristotle the geocentric prevailed on the heliocentric system, Heraclides Ponticus (387 B.C.–312 B.C.), disciple of Plato and friend of Aristotle, strongly supported a hybrid system: it was geocentric for the Sun and all planets except Venus and Mercury which, in order to explain their anomalies, turned around the Sun in uniform circular motion.

A few years later, Aristarchus of Samos (310 B.C.–230 B.C.) extended such system on three fundamental hypotheses: the absolute motionless of the fixed stars sphere, the perfect stillness of the Sun at its center and the annual movement of the Earth on a circle centered on the Sun. He admitted that the fixed stars sphere had a radius enormously larger than that of the terrestrial orbit. Such theory, based on different sphere dimensions for the motion of inner and outer planets, was shortly thereafter opposed as it was contrary to the initial appearance of observations. The efforts of the following scientists were devoted to support the religious tradition.

Hipparchus (190 B.C.–120 B.C.) made a first stellar catalogue, counting approximately 800-900 stars. Although he introduced new hypotheses to support the geocentrism, he admitted that the Earth was not exactly at the center of the celestial sphere and that the motion of planets was on epicycles, i.e. the combination of two circumferences moving one inside the other. This theory was eventually fit by Ptolemaeus (II century B.C.), on the basis of the summary of all existing observations of planetary motions.

In this period, for the first time, the notion of the number of celestial bodies, their nature and the radius of the spheres where they are supposed to move are the point of interest of the scientists and are critically considered. We will see how similar progressions in the astronomical observations will bring, several centuries after, to modern cosmology.

#### 1.1.4 *The philosophical point of view in the Old Age*

The first centuries (I–VII A.C.) are marked by the growth and consolidation of the new religion which diffuses around the Mediterranean with a strong impact on the evolution of the philosophy, with the efforts of Christianity and its evolution in the contemporary society: the concepts of Universe, space and time are rescaled to the human dimensions, the purpose of the research is to give a foundation to the new ideas and to defend, from the apologetics and through the Patristic, from the persecutions to the achievements got in the common knowledge. The purpose of the research is based on the dualism God/man, and thus reason, faith and soul. The speculations of the philosophers regarding the nature and its origin are formulated starting from the Biblical story: as in the work of Augustine of Hippo, saint, (354–430 A.C.), God, being the basis of all existing things, created the Universe and everything else. The speculations are led by ethical and religious reasoning, without pursuing a specific attention to the physical aspects of the narration, giving rise to an eternal and perfect Universe with all characteristics set according to some preliminary fundamental statements, simply regarding the role and attributes of God.

In the same framework, although the physical aspects of the Universe *creation* are not the focus of the research, a central point of investigation is the role of *time*: even admitting the action of God for creating the whole world, before this event time did not exist and thus the concept of *eternity* takes form, opposed to the fugacity of the temporal evolution over the human time-scales.

#### 1.1.5 *The Middle Age dogmas*

A change of attitude shapes the research of the philosophers in the Middle Age: the problem of the Scholasticism in the Christian Europe is to bring the man to comprehend the revealed truth: the religious tradition is the rule of the research and since everything has been revealed through the sacred Biblical books, philosophy has to support the common work towards the

divine revelation, excluding the need of formulating a new system or new explanations to explain some phenomena. This attitude perfectly reflects in the rigid hierarchies of the Medieval society. Starting from the VIII century, many economical and cultural exchanges fade up to starting again in the XI century, opening the way to a criticism of a rigid cosmic order. The idea of a supernatural relation to the human power and to the initiative of the single person takes place in the following years, up to the end of the XIII century.

In this environment, it is worth mentioning the role of Johannes Scotus Eriugena (c. 810–877) who for the first time denied that the sky was made of incorruptible and not-generable ether (according to Aristotle) and proposed an astronomical system with the Earth still in the center, but all planets orbiting around the sun.

For decades, most of the efforts remain devoted to address theological questions, especially regarding the role of the man in the created world, whose origin is granted as created by God at the beginning of time.

In the same period and across XI and XII centuries the Greek philosophy and science had been inherited also by the Arab world, where many thinkers gave a great impulse in astronomy and mathematics earlier than the western Schools. In particular, they had a rational approach to several basic problems which were considered as reasoning paradigms by the European philosophers, conscious of the limits and drawbacks provided by the heavy influence of the tradition. The translation and diffusion of the Greek works, served as basis for the development of two clear philosophical approaches represented by two prominent thinkers, Avicenna and Averroes, who were the most prominent exponents of the so-called Neoplatonic and Aristotelian evolutions.

Avicenna (Abu Ali Sina Balkhi or Ibn Sina), (around 980–1037), medical doctor and philosopher, exposed the principle of the necessity of the being: everything happens necessarily and could not happen in a different way. The role of God is to shape the natural events and is the first origin of all physical processes. The astrological predictions are thus fully justified since the action of God is directly on the asters and from them it expands to the other levels of the nature: if the human beings could have a perfect knowledge of the stellar (and planetary) evolution, they could know the events on the Earth without mistake. The predictions fail due to this imperfect knowledge of the details of the movements of the celestial bodies.

Although at this stage there is no clear distinction between stars and planets, the philosophers point to the sky as the intermediary place with

the divinity, recognizing its role either for the religious speculations, or for the human events.

The arab Spanish-born philosopher Averroes (Ibn-Rashid), (1126–1198), devoted much of his efforts to discuss the contemporary philosophers, particularly hostile to Avicenna, starting from the same principle of the necessity of all that exists. He considered the Aristotelian doctrine as the scientific and demonstrative counterpart of the Muslim religion, which on the other hand is seen as the simplified version suitable for uneducated people. Nevertheless, since the world itself is necessary because it is created by God, it is eternal as well and cannot have originated *within* the time. The order of the world is also necessary, and the human being has not capability or freedom of action. Although such last concept could seem incompatible with the free research or aim for new explanations of the natural events, it remains at the basis of Renaissance confidence to discover a necessary order in all manifestations of the nature.

The Arab and Judaic thinkers, mainly active in Spain and Egypt, strongly influenced the inheritance over the thinkers in central Europe, since the doctrines of Plato and Aristotle were transmitted through their studies. This had the effect of splitting the philosophers between those who opposed Aristotelianism in favor of Platonism and those who desired to merge several aspects of them.

In particular, we note the role of Robert Grosseteste (1175–1253) active in Oxford and bishop, for his speculations on natural philosophy. He stated that the study of the nature must be based on mathematics, and reduced the whole Physics to a theory of the light, which is the primary form of the bodies: since the light diffuses in all directions, it is equivalent to the corporeity itself, similar to the extension of the matter in the three space dimensions.

Thomas Aquinas (1225–1274) inserts himself in the Scholastic debate regarding theology and philosophy, essentially for the integration of rational thinking and faith, finding a solution in the subordination of the reason to faith, proving itself as an instrument for the theological truth. His approach to the demonstration of the existence of God finds the first proof as a *cosmological* one, relating the existence of any movement in the Universe to the existence of God: given that every movement of any object has been activated by that of another one, this provides a potentially infinite chain, i.e. the *first principles of action*, commonly attributed to God. Similarly, the movement of the celestial spheres also comes from Him. For what regards the creation, the only conclusion Thomas offered was the impossibility of

demonstrating either the beginning of time or the world eternity.

Towards the end of the XIII century a restored interest in the philosophy of nature arose across Europe, dividing theological questions from an autonomous effort of the reason for the problems of the physical world. In every cultural field, following the new Aristotelian spirit, the experimental research proposes new methods and new questions.

The most important representative of this experimental approach of the XIII century is Roger Bacon (c.1210–1292). Although his results in physics, in particular optics, astronomy and mathematics did not find outstanding originality, his great contribution relies on declaring the sources of any knowledge: reason and experience. Nevertheless considering natural (from external experience) and supernatural (from interiority experiences) truths and acting much as an alchemist looking for marvelous discoveries, he can be considered as the precursor of the modern science. In fact, he gave maximum value to the experimental approach to research, giving to mathematics the fundamental guiding role for it, in order to give certainty to the finding of other sciences.

The role of philosophy is to clarify the limits of the human science domain for John Duns Scotus (c. 1266–1308), who assessed how science and faith refer to different levels of truth. The human mind aims at comprehending the rational aspects of nature adopting a theoretical approach based on the freedom of reasoning, while the faith is based on a more practical level, regarding the human behavior and possibility of actions, needless of doubt. Following Aristotle and the Arab philosophers, he declared the ideal of a necessary science, fully relying on principles based on evidence and on rational demonstrations. Stating that all attributes regarding God are matter of faith and cannot be demonstrated, Duns opened the way to the rise of the Renaissance, proposing an approach of division between religion and science which was, on the contrary, the main issue of the early Scholasticism.

The rise of a new class of merchants and bankers, opposed to a static theological society infrastructure, manifested in a new interest toward nature and science. In a changing society divided among the secular role of the Catholic church fighting against the Emperor, William of Ockham (c. 1290– c. 1348) lived contributing to establishing a strong basis to a radical empiricism as the foundation for the philosophical investigation. For the first time, he exposed the limits of the theological proofs regarding the existence of God, on the basis of a total heterogeneity of science and faith, essentially bringing light on the lack of necessity in every step of the

proofs. In his criticism of the concept of cause and effects, which can be related among themselves only on the basis of the experience, the natural events evolve according to necessary laws, independently of the divine action. Metaphysics loses its power to explain everything, giving the researcher the role to describe *how* phenomena happen, avoiding to enquire about their *essence* or *purpose*. In his anti-metaphysics theology, the Universe has been created by God without any pre-existing logical rule, and therefore God could have given the world also a different set of rules, at his freedom: philosophers can only accept the world as it is, without trying to find a metaphysics explanation.

Since the nature is the domain of the human knowledge, the experience loses its magical character, as in the past, to be accessible to all human beings: Ockham, for the first time, rules out the belief of the different nature attributed to celestial bodies and sub-lunar ones. On the basis of a principle of economy (the so-called *razor*, i.e. to avoid all unnecessary concepts to describe some phenomena), he asserted how all bodies are made of the same kind of matter. He also admitted the possibility of a multi-world Universe, where all local characteristics (up, down, center, etc.) would appropriately have their meaning, in that local Universe. The whole Universe can be infinite and eternal, as any celestial revolution can indefinitely repeat and the creation itself can be excluded as being highly improbable, opening new ways to the philosophical investigation.

### 1.1.6 *The Renaissance revolution*

The XV and XVI centuries are characterized by a radical change toward the modern age, in an enlargement of the view of the world, coherently with new geographical discoveries, new inventions and new efforts by a urban society, powered by banks and social transformations.

The Humanism, preparing the way to the Renaissance, explicits the new culture which breaks the previous perspective of the human being with respect to the life and the world, eventually opening the way to different interpretations of the nature apart from the religion dogmas, though reconsidering the efforts made by the Greek philosophers. The culture is now organized by the local communities as *Academiae*, where the use of the latin language is considered as a means of intellectual and international way to circulate the ideas, also with the help of print. The *naturalism* at the basis of the Renaissance marks the active role of the humans as a full part of the nature and their attitude and interest in its study, though frequently the



nature will be approached by a magic perspective.

Nicolaus Chryppfs (Cusanus, 1401–1464), was the first prominent philosopher stating the role of mathematics to evaluate the proportion of knowledge with respect to ignorance. He was the first to refuse the idea that the celestial part of the world possesses an absolute perfection and thus be un-generable and incorruptible. The world (considered as the whole Universe) has no center nor circumference but comprises all the space. Since there is no center, the center is everywhere, the Earth is not at the center of the world and moves of a rather perfect circularity. The Sun is another star, made of more pure elements and all movements of the nature are arranged so as to guarantee the highest possible order, maximally approaching the circular shape.

Giovanni Pico della Mirandola (1463–1494) states how the Cabbala can help to penetrate the divine mysteries and the astrology can be used to understand the mathematical rules of the Universe, though the human actions cannot be influenced by the celestial bodies since they are free and full of dignity.

The main idea of Renaissance is deeply characterized by a re-birth of the human beings within the nature, fully introduced in the world and thus giving large space to investigation of the natural world, in an early stage through magic and finally with the philosophy of the nature.

### 1.1.7 *The Scientific Revolution*

The Scientific revolution marking the following centuries can be chronologically set between the publication of Copernicus *De revolutionibus orbium coelestium* (On the Revolutions of the Celestial Spheres, 1543) and Newton's *Principia* (1687). In this lapse of time some paradigms express the new approach to nature, in particular:

- the nature has an objective order, i.e. its character has nothing to do with a spiritual dimension of investigation, and thus with the human purposes and needs. The Universe has no human attributes nor qualities.
- The nature has a causal order, in the sense that there is a constant relation between one (or more) facts and the only type of cause is the efficient one. The Science does not investigate the purposes for which events happen, but is devoted to study the causes that produce them.

- The Nature is a set of causal relations and, finally,
- the facts are governed by laws, and the Nature is the compound of the laws which govern all phenomena and make them predictable.

Correspondingly, the Science is an experimental knowledge based on the experience, it is mathematical, as it can quantify and express itself by formulas, it is accessible to everyone and its only purpose is the objective knowledge of the world and its rules. Such knowledge permits to eventually modify the world according to the human purposes, thus providing strong links between scientists and technicians.

The scientific revolution marked a deep philosophical change in the approach to the vision of the Universe. In fact, the Greek-medieval cosmology, arising from Aristotle and Ptolemaeus, conceived the world as unique, closed, finite, made as concentric spheres, geocentric and divided into two qualitatively distinct parts. Such vision was based on a unique Universe, as the only existing one, with all possible matter aggregated in a single place. It was closed, as a limited sphere from the level of the fixed stars in the sky, and outside it there was nothing, except the realm of God. It was finite, since the concept of infinity could not be considered as reality. Finally, the Universe was represented as two different cosmic zones, one perfect and one imperfect. The former was the super-lunar world of the skies made of ether, a divine element incorruptible and perennial, characterized only by a circular movement. The sub-lunar world, on the other hand, was made by the four elements (air, water, earth, fire) having rectilinear motion and marked by generation and corruption. Such framework was coherent with the metaphysics and religious justification, linked to the creationist doctrine and the human role stated by the sacred books.

Nicolaus Copernicus (1473–1543) provided the first strong criticism to the geocentric Ptolemaic system in his *De Revolutionibus orbium coelestium*. As a theorist of celestial mechanics, he considered such system as too complex and reconsidered alternative theories in the books of the Greek philosophers, thus recovering the heliocentric idea which, in his view, offered a deep simplification in the mathematical evaluation of the celestial bodies movements. Although in his scheme the Solar system recovered what it is known today, with the Sun at the center and the planets orbiting around it, his vision of the Universe was limited to the sphere of the fixed stars, leaving a unique and closed spherical Universe with a perfect circular uniform motion.

Although several thinkers opposed the ideas of Copernicus, mainly on the religious and philosophical point of view, his ideas were extended by Tycho Brahe (1546–1601) who proposed a Solar system with the planets orbiting around the Sun and all together orbiting around the Earth in the center, overcoming any conflict with the Bible.

Johannes Kepler (1571–1630) started his investigations on astronomy exalting the harmony of a Solar system centered on the Sun as the image of God with the planets at the edges of a regular polyhedron. The mathematical difficulties in relating such assumptions with the astronomical observations led him to dismiss such Pythagorean approach to the symmetry of cosmos. In the following, he considered only physical forces instead of intelligent motions, and attributed to the world and to the matter necessarily a geometrical order. His major achievement springs from Tycho's observations stating the laws for the planets' motion over ellipses, thus confirming the role of mathematical proportions for the description of the natural objectivity. So far, the Universe is still limited to the solar system and to the fixed stars with their immensity and incommensurability. Although Copernicus left the discussion about the possible infinity of the Universe to philosophers, Kepler's Universe is still finite.

Although not an astronomer nor a mathematician, a new vision of the Universe was proposed by Giordano Bruno (1548–1600) who forced the Lucretius thought about ancient atomism toward the intuition of the infinity of the Universe based on the Copernican discoveries. His hypothesis was based on the similarity to the solar system for all stars in the sky, thus with a Universe composed of a multiplicity of systems similar to ours, with an unlimited number of Suns in different places of the space, though stating that they have never been observed. The revolution was initiated and ready to influence the following philosophers and astronomers.

The Universe has no borders and is open in all directions, while the fixed stars are dispersed in the unlimited space. Coherently, other parts of the sky are made of the same kind of matter, with no hierarchy about the quality of the component substances, in a unique and homogeneous space.

Such view deeply influenced the perception of the thinkers, either philosophers or poets, about the role of the human beings relative to the position of the Earth in the Universe, inducing a debate regarding the difficult coexistence of the new cosmology with the Bible sentences about the movement of the planets or the position of the Sun. Thus, a great impulse on the view of the world for the contemporary thinkers was given by a visionary proposal, opening the way to the following evolution (in particular

by Galilei) which would have shaped the following years, although its affirmation was based more on the fascination evoked by the new ideas than on the physical observations which were still far to come.

Although the Brunian proposal was included in the arguments for the magnification of the religious orthodoxy, with an infinite Universe testifying the infinity of the Being who created it, the Church continued to be reluctant to accept the Copernican view of the world, canceling the condemnation for the Copernican writers only in 1757, allowing the printing of books teaching the Earth motion in 1822 and taking off from the list of forbidden books the *De Revolutionibus* only in 1835.

We see how the new theories prepared the way to the modern view of the Universe, although yet without experimental ground, and changed the way the celestial space was perceived, leaving to the Earth (and the human beings on it) a marginal role in the whole context of infinite planets, stars and worlds.

Starting from mathematical studies, Galileo Galilei (1564–1642) was the first to use the *spyglass* (1609) for his discoveries in Astronomy enthusiastically announced in his *Sidereus Nuncius* (1610), promptly supported by Kepler. In contrast with the hierarchies of the Church, he received an ammunition (1616) which did not stop him from publishing *Il Saggiatore* about the problems of cometary motions. He also continued his work on the *Dialog over the two maximum systems of the world* (discussing the comparison between the Ptolemaic and the Copernican cosmological systems) which led him under trial (1632) and forced him to *abjure*, publicly dismissing his results in 1633 and to retire in Arcetri (near Florence) to write his last book about dynamics, finally published in Holland. His point of view about the relation between the religious truth and the scientific truth was explained in the *Copernican letters* where he detailed how both truths derive from God and any contradiction among them must be apparent, since the Bible and the science approach different aspects of the knowledge. Thus, for what regards the investigation of the Nature, the Bible interpretation must adapt to the scientific evidence.

His studies about Mechanics and the motion of bodies were the scientific counterpart to the Brunian intuitions about the cosmos. The principle of inertia was valid for the terrestrial as well as for the astronomical dynamics, though leaving the idea of a finite Cosmos, explaining the indefinite motion of all planets. The idea of circular orbits for the planetary motions was retained in his picture outlining a reminiscence of a theological influence.

The formulation of a unique science of motion, with the negation of a different nature between circular and rectilinear motions (so far considered typical of the lunar and sublunar worlds, respectively), brought to refuse the existence of a different structure between the sky and the Earth. So far, a possible difference was based on the possible motions on them.

Galilei was the first to explore, thanks to his spyglass, the Lunar surface, finding it similar to the terrestrial one, with valleys, craters, mountains and shadows. The discovery of the four Jupiter satellites, orbiting together with the parent planet around the Sun, suggested that also the Earth, together with the Moon, was orbiting around the Sun and could be *experimentally* verified by the observations done with the new telescope. The discovery of the Venus phases induced the idea that all the planets receive their light from the Sun orbiting around it. Finally, and most relevantly for our discussion, Galilei used the telescope to discover that after the fixed stars, visible by naked eye, a huge number of other stars existed, never seen before but observable with the new instrument. The galaxy itself was only an aggregation of uncountable stars scattered in groups, similarly to the nebulae which were composed of other stars.

The analysis of the structure of the Universe, at least the part surrounding the Earth, was ready to be boosted on an observational basis, drastically changing the view of cosmology and the evolution of science.

In the *Dialogue*, in 1632, Galilei confuted all the current theses against the Earth motion, introduced the principle of inertia, explained the Earth rotation and finally exposed his theory for the sea tides.

The convergence of the technical expertise in assembling the telescope with its scientific usage provided the formal scheme which imprinted the modern science, from phenomena observation to mathematical measurements of data, hypotheses, verification and finally formulation of a physical law. Galilei was persuaded about the mathematical structure of the Universe, with the necessity of a geometrical description for deciphering its real structure.

The impulse to the physical investigations were now related to a new crucial possibility to apply the observations also to Astronomy which, on a mathematical basis, was open to explore a natural world which was no longer split between the terrestrial and the one of the celestial spheres, but characterized by general laws where all parts were correlated by a causal relation.

René Descartes (1596–1650), mathematician, physicist and philosopher, formalized the scientific method founding the way to proceed in the research about the concept of *doubt*, an approach that would have marked the following centuries. His view of the Universe was based on a Euclidean space, arising from the identification of the concept of matter with its own extension. Thus, the infinity of the space implied the infinity of the matter, its infinite divisibility and continuity, without holes nor vacuum. He refused the concept of forces acting at distance among different bodies, leaving the Universe to the principle of inertia and of conservation of momentum, in a fully deterministic framework scaling up to include the first impulse from God in the evolution of the Universe. This idea led him to describe the motion of the celestial bodies as immersed in the ether filling the entire Universe, with all movements due to a series of vortices including all planets, in order to satisfy a mechanistic description for an infinite continuum. The unique engine of the big machine constituting the world is the original momentum, distributed in different ways among the bodies through their collisions.

A new vision of the cosmic order was proposed by Baruch de Spinoza (1632–77), who considered an identification of the Nature as the order governing all substances and their movements. Thus, it assumes a role as God-Nature, geometrically ordering the whole Universe with its laws. His criticism of the former philosophers, including Galilei, addressed the finalism, viewing it as a prejudice preventing a correct interpretation of the world scheme.

With Newton, the scientific Revolution initiated by Copernicus and Galilei gets its final form, either on the methodological approach, either for its contents, thus outlining the Universe picture which is familiar to the modern view and which, after Einstein, is called “classical Physics”.

Isaac Newton (1642–1727), mathematician and physicist, worked in optics, inventing the reflection telescope, and starting from the results of Huygens on pendulum oscillations and on the experimental techniques developed during the second half of XVII century, investigated several aspects of physics and gravitation, summarized in his *Philosophiae Naturalis Principia Mathematica*, published in 1687 with the support of the astronomer Halley.

In this work, he recognized the identity of the motion of the planets with the fall of heavy bodies on Earth, in the formulation of the *universal gravitation law*, finally explaining the planets’ motion around the Sun and of the satellites around their corresponding planets.

The infinitesimal calculus was the missing concept to unify Physics and Mathematics, and its use allowed Newton to correct the Kepler laws of motion, taking into consideration the attraction exerted by the Sun together with the attraction among the planets. This way he could explain a perturbative mean for the planets motion, so that, for example, the Earth is not moving along a perfect ellipsis but along one perturbed by the action of other planets around. On a broader point of view, he stated the similarity of the motion of bodies on Earth with that of planets in the sky, a picture in which still missed the initial principle of motion. Newton thus admitted as the first cause the creational action of the divinity, providing the initial momentum.

The dynamics received a definite form, once introduced the concept of *mass* for the generalization of the concept of *force*, consequently extending the laws of mechanics to the entire Universe.

A focal point for mechanics is the idea of an absolute motion, with reference to empty space, funded on absolute space and time, mathematically fluent uniformly, without relation to anything external, related to an absolute space, always similar to itself and stationary.

His formulation of mechanics and dynamics led to exclude other forces, apart from gravity, acting on the movement of celestial bodies. Moreover, the formulation of *scientific induction*, prescribing the extension of a law verified for a limited number of cases to all possible cases, opened to the description of all parts of the Universe with the same law. His method stated also that the propositions got by induction from phenomena must be considered exactly or approximately as true until other phenomena eventually confirm or show any exception.

The scientific method with the support of calculus provided a new basis to the description of the celestial phenomena. The Universe resulting from the Newton's investigations appears as a real physical environment whose phenomena are governed everywhere by the same laws, which can be formulated in a precise mathematical language. More than modifying the notion of Cosmology, Newton imprinted with his new method the possibility for the scientists to address the scientific observations in a self-consistent theoretical framework. However, the understanding of the laws governing the gravitational interactions at a local level allowed to extrapolate their validity everywhere in the Universe and therefore to approach the analysis of its structure on a new perspective.

The strong scientific imprint to the interpretation of natural phenomena did not prevent Newton from mixing some Hermeticism ideas (from the Hellenistic Egyptian tradition) about the relations of attraction and repulsion between particles, gained from a strong interest for alchemy. Thus, his view of an invisible force acting on large distances was seen as the attempt to introduce some occult component in the natural picture.

His relation with religion was twofold: his interest in the Bible was exerted trying to extract any information regarding nature or referable to some scientific measurement, though he viewed God as the clockmaker of the Universe. The complexity of the planetary motions could not be simply ascribed to natural phenomena but should have been designed by an intelligent being.

Nevertheless, this line of thinking would have found several crucial difficulties which prevented the birth of a modern notion of Cosmology before the formulation of General Relativity by Albert Einstein.

A new vision of the cosmos was under way: Galilei and Newton had opened the broadest possibility of development to astronomy.

The needs of navigation lead to address the new notions more accurately, thus inducing the foundation of new observatories. The first was established in Paris by Louis XIV and granted to the Italian astronomer Giandomenico Cassini. A few years later, in 1675 the Greenwich observatory was built by Charles II, directed by Edmond Halley (1656–1742). His studies were devoted to the motion of the comets, demonstrating how they belong to the solar system and move on eccentric orbits. In 1675 the Danish Olaf Römer, working at the Paris observatory, noted that the eclipses of the Jupiter satellites on certain periods of the year happened earlier and on other periods later than those predicted through the computed tables. Thus he ascribed such phenomenon to the different distance from the Earth of such satellites, finally providing a measurement of the speed of light as 308,000 km/s, indeed very accurate, a result which was vainly searched by Galilei.

### 1.1.8 *The Enlightenment Era*

During the XVIII century, a new and specific attitude to relate to the rationality sprung out as a philosophical counterpart of the Scientific Revolution. The Enlightenment was characterized by a new role of the reason in the society with an emerging role of the bourgeoisie, together with a novel



perception of the science and its influence on the real life of the people (including trade, economics and technical evolution), rising in the hierarchy of the activities related to knowledge.

The role of the experience in the philosophical investigation mitigates the rationalistic and idealistic impulse to the theoretical exploitation of the physical laws underlying the observable phenomena.

In the 1700, during the philosophical celebration of science and of its methods, an important development involved mathematics and astronomy.

Leonhard Euler (1707–1783) born in Swiss and active in Russia at the court of Catherine I the Great in St Petersburg, showed his trust in the mathematics devoting all his efforts in developing the infinitesimal calculus towards the application to several phenomena.

In the same years, Joseph Louis Lagrange (1736–1816) (born Giuseppe Lodovico Lagrangia, in Turin, Italy), first in Berlin and finally in Paris, was appreciated by Napoleon who named him to the Legion of Honour for his results in applied calculus. His work *Mécanique Analytique* (Berlin, 1788) summarized all topics of classical mechanics treated so far, since the epoch of Newton.

On the basis of the great impulse to mathematics, also the celestial mechanics established by Newton developed and flourished. The most prominent astronomer of this period was Friedrich Wilhelm Herschel (1738–1822), who discovered Uranus, enlarging the borders of the Solar system, limited so far to Saturn since the ancient times. After this, he discovered the Sun's motion and the dragging of all planets with it and proved the rotation of the Saturn ring, measured also its period. His work comprised the catalogue of a large number of nebulae and finally his studies about the Milky Way lead to view it as a quantity of stars disc-shaped with a diameter equal to five times its width.

In the same years, Giuseppe Piazzi (1746–1826) discovered Ceres, the first planetesimal between Jupiter and Mars, whose orbit would have been calculated several months later by the German mathematician C.F. Gauss.

The perception of the Cosmos was surpassing the limit related to the observational capacities which were increasing year after year, providing a novel consciousness about the perspectives on new ways to cover.

The experiments carried out by Henry Cavendish (1731–1810), opened the path to measuring the weight of the Earth and of the celestial bodies. Through the use of a torsion pendulum he computed the value of the gravitational constant  $g$  characterizing the Newton's law.

Finally, the role of the mathematician Gaspard Monge (1746–1818) pro-

vided a way to the description of phenomena thanks to the invention of the descriptive geometry, allowing to treat on a bidimensional surface (like a sheet of paper) tridimensional displacements of bodies.

Other scientific fields were improved under the impulse of the adoption of calculus and its extensions outside mathematics, as for thermology, electrology, chemistry and biological sciences. The effort to explain and organize the current knowledge in several topics lead to classify and organize the observed phenomena in catalogues and systematic classifications of the natural world, with applications to animals, plants, basic constituents of matter, celestial events.

A huge work of collection in this line was pursued by Georges-Louis Leclerc, Comte de Buffon (1707–1788) with his *Histoire naturelle*, in 44 volumes, including several volumes devoted to quadrupeds, birds and minerals and finally to the theory of the Earth and the general characters of the plants, of the animals and of humans. He explained the origin of the Earth from the impact of a comet with the Sun and this idea inspired several thinkers thereafter.

The most important thinker influenced by such ideas was Immanuel Kant (1724–1804) in his work *General History of Nature and Theory of the Heavens* (*Allgemeine Naturgeschichte und Theorie des Himmels*) (1755) describing the formation of the whole cosmic system from a primordial nebula, in accordance with the Newtonian physics. Moreover, he investigated the role of mathematics for the description of physical phenomena, overcoming the explanation provided by Galilei, who based his epistemology on the existence of God, but attributing to the nature of space and time an intrinsic geometrical and arithmetical configuration: if the concept of space itself is Euclidean, the theorems of Euclid's geometry apply to the whole phenomenological world.

Analogous theories about the formation of the solar system were also proposed, in the same years, by Johann Heinrich Lambert (1728–1777) and Pierre-Simon Laplace (1749–1827).

## 1.2 The XIX Century Knowledge

### 1.2.1 Geometrical formalisms

The new science arising from the scientific revolution expressed its potential during the 1800s. Despite the strong link between science and philosophy in the previous era, in the XIX century the latter tends to reduce the strong

link with the experimental foundation leaving the science fragmenting in several specific topics, often without communication between them. Around the mid of the century, thanks to the progress in the mathematical abstraction and modeling, the mechanistic ideas permeate again the different fields of Physics, posing the basis for the following unification approaches.

In particular, Laplace expressed the view of the current state of the Universe as the effect of the previous state and the cause of the following one (1812, *Théorie analytique des probabilités*). In his *Mécanique céleste*, published in five volumes from 1799 up to 1825, he addressed the stability of the Solar system, with the purpose to match the astronomical data under the point of view of the laws of motion. He also pursued the idea introduced by Kant about the origin of the Solar system based on the nebular hypothesis giving rise to a planetary system and discussing the possibility that gravitational forces could not act instantaneously, although without success.

The mechanism thus formulated gave rise to a vision of the Universe where reversibility would always be possible, thus preventing the idea of an evolution or degradation of the nature at large scales.

The independent path taken by the evolution of mathematics revealed the basis of a new formal approach to scientific problems, leading to a level of abstraction which would bring to a process of unification for the phenomena based on more generic formal structures.

For the first time, the mathematicians discussed non-Euclidean geometries, without the necessity of a strong link with the real world. In particular, the analysis of the fifth postulate of Euclidean geometry was reconsidered, attributed to Proclus Lycaeus (412–485): given a straight line, it is possible to draw one and only one line parallel to it and passing on a point external to the first line. Independently, Karl Friederich Gauss (1777–1855), Nicolaï Ivanovič Lobačevskij (1793–1856) and Janos Bolyai (1802–1860), founded the new hyperbolic geometry. Analogously, by the end of the XIX century, Bernard Riemann formulated the elliptic geometry which would have been fundamental for the modern gravitational physics introducing the General Relativity Theory and a new Cosmology.

From an evolutionary point of view, the history of the Universe was considered from a new perspective once the analysis of thermodynamics and entropy were applied to the Universe as a whole: time was seen as asymmetric and the new ideas lead to consider that if the Universe can be regarded as an isolated system, it must evolve toward a progressive thermal death. Scientists started to apply the consequences of mathematical abstraction

to the physical world thus posing new challenges to metaphysics.

The abstraction pursued by the research in mathematics brought to investigate the geometry as a new field, assuming the character of an *a priori* synthetic science. Bernhard Riemann (1826–1866), student of Gauss, lectured about a new multi-dimensional geometry *Über die Hypothesen welche der Geometrie zu Grunde liegen* (*On the hypotheses which lie at the foundation of geometry*), later published in 1868, describing manifolds with any dimension and any type of curvature, constant or variable.

A different approach to geometry, strongly based on an axiomatic perspective, independently on any hypothesis about the physical space, was investigated by David Hilbert (1862–1943) in his work *Grundlagen der Geometrie* (1899, *Foundations of Geometry*), developing non-Euclidean geometries by using purely formal methods.

### 1.2.2 *Difficulties for the birth of a real cosmology: Olbers' paradox*

In 1826, the German astronomer Heinrich Wilhelm Olbers (1758–1840) assessed the paradox regarding the consequences of an infinite Universe over the night sky: if the Universe has an infinite extension (as proposed by Newton to prevent a collapse), contains an infinite number of stars and exists from an infinite time, the sky should not be dark at night, since every point of the sky should be covered by the emission of the light by some star, though far and distant. Such paradox had already been stated by Kepler in 1610 but all possible explanations were destined to fail or to induce other paradoxes.

The solutions proposed over the years spanned different points of view: Otto von Guericke (1602–1686) had previously proposed that the darkness is caused by an endless void between the stars. Olbers thought that light would be absorbed by clouds of dust in the interstellar medium while William Thomson (1824–1907) (known as Lord Kelvin) a few years later introduced the idea that stars would have started their life a finite amount of time ago, thus introducing also a limit on the size of the observable Universe.

Notwithstanding such proposals, a definitive answer was awaiting from the observations made by Hubble in the Twenties of the XX century.

### 1.2.3 *Luigi Bianchi and the developments of differential geometry*

Under the influence of Riemann and Sophus Lie (1842–1899), across the change of century the Italian School of mathematics provided several efforts in the field of differential geometry, algebra and topology. In 1898, Luigi Bianchi (1865–1928) derived the classification which brings his name of the isometries classifying the Riemannian tridimensional spaces into nine non-equivalent (Lie) groups which is at the basis of the extension to Cosmology performed six decades later by the Landau School. Friend and colleague of Bianchi, Gregorio Ricci-Curbastro (1823–1925) promoted a group focusing on tensor calculus, involving also Tullio Levi-Civita (1873–1941), thus opening the way to the formalism of differential calculus with coordinates, later becoming the language for General Relativity.

### 1.2.4 *Einstein vision of space-time*

Addressing the new discoveries about the electromagnetism phenomena and about the nature of the light speed, Albert Einstein (1879–1955) published in 1905 a short memory *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt* (On the electrodynamics of bodies in motion) which, formalized in 1908 by the mathematician Hermann Minkowski (1864–1909), provided a geometrical interpretation of the basic postulates of Special Relativity. Einstein worked on extending the relativity principle to accelerated systems, exposing in 1916 the new theory of gravitation, known as General Relativity.

The ideas of Einstein constituted a conceptual revolution since the notions of space and time acquired an intrinsic relative character, being influenced by the matter field living in the background environment. This striking contrast with the Newton picture of spacetime (however contained in General Relativity in the proper non-relativistic limit) opened new perspectives to reconsider the mechanisms governing the Universe genesis and evolution.

The great impact of General Relativity on the concepts of matter and spacetime was mainly due to the synthesis of the geometrical description of the spacetime manifolds by the tensorial formulation of the laws of nature. On a physical ground, this correspondence is able to conjugate the General Relativity Principle (i.e. all the physical laws stand in the same form in all the reference systems) with the geometrodynamics (i.e. the gravitational in-

teraction implies the spacetime evolution coupled to the energy-momentum source describing the physical entities).

Einstein started with the idea of generalizing the Special Relativity theory to non-inertial systems, but his deep understanding of the Equivalence principle allowed the full development of the gravitational theory in a space-time geometrodynamics.

The formulation of Special Relativity started by the experimental evidence of a constant, frame independent, value of the speed of light. Hence, Einstein recognized that a limiting velocity of the signals implied the relative nature of two events simultaneity.

Despite what is often believed, General Relativity also arose from an experimental evidence, i.e. the proportionality of the inertial and the gravitational masses through a universal factor (conventionally set to unity).

Einstein deeply considered the unnatural physical scenario coming out from the equivalence of these very different concepts. Indeed the gravitational mass is nothing more than the charge of the gravitational field, in principle fully uncorrelated from the dynamical concept of inertial mass. Thus Einstein formulated the Equivalence Principle as the local physical correspondence between a non-inertial system and an inertial one, endowed with a suitable gravitational field: both the interactions (non-inertial forces and gravity) have the same property to exert a force independent of the inertial (i.e. gravitational) mass of the test body. The Equivalence Principle not only links the non-inertial to the gravitational force in a common physical scenario, but suggests the idea that these force fields have an *environment* character, well dressed by their geometrical origin.

Despite the ideas of the new theory were well defined in the Einstein mind, the walk to the proper mathematical formulation of General Relativity was somewhat long and also required the important contribution of Marcel Grossmann (1878–1936), who supported Einstein with his rigorous mathematical hints on the formal description of the physical phenomena. At the end of this conceptual and formal path, Einstein summarized the non-inertial and the gravitational forces into the metric tensor describing the spacetime manifold, with the fundamental distinction that the inertial forces arise from changes of reference system, while the gravitational field is associated to a spacetime curvature.

We emphasize how Einstein derived his revolutionary theories from very well-known facts (constant speed of light and mass equivalence), but he was able to cast these notions toward a new physics providing the “correct answers to the good questions”. However, we cannot forget how Einstein was

influenced by the spirit of his time, especially in the concept of a relativistic world. Indeed two important facts must support the clever intuition of a physicist: the existence of a philosophical background favorable to the development of his ideas and the possibility to adopt well-grounded mathematical formalisms. Einstein could make account on both factors.

### 1.3 Birth of Scientific Cosmology

Although Cosmology, in a strict sense, is one of the most ancient disciplines of speculation, only in the XX century it has acquired a proper scientific character, on the basis of the new conceptual instruments provided by the Theory of Relativity and from Particle Physics, together with the recent observational means allowed by new telescopes and from the introduction of radioastronomy.

The Newtonian notion of absolute space, together with the law of the static gravitational field were not able to give rise to a modern notion of Universe. In particular, Olbers' paradox constituted a very serious no-go argument to the construction of a coherent picture for the observed sky in the framework of a stable gravitational configuration.

General Relativity, in agreement to the idea of a dynamical space, deformed by the matter and energy distributions contained in it, offered a deeply new scenario to describe the origin of the Universe, born from a primordial phenomenon and emerging as an expanding space. In this context, Olbers' paradox is easily solved because an observer can receive light from the distance traveled by a photon from the Universe birth, which in this framework is finite. Furthermore, these photons, coming from far galaxies, are redshifted by the Universe expansion and they are observed with lower energy than they were emitted. In this respect, first the discovery of the galaxies (i.e. understanding that the observed nebulae were *outside* the Milky Way), and later the Hubble demonstration of their recession, were milestones in the definition of a modern view of the cosmological paradigm. Einstein himself, without the notion of expanding galaxies, had difficulties in accepting that his theory privileged non-stationary Universes with respect to the static one initially proposed.

Thus, we can say that the geometrical framework of the gravitational field in the Einstein picture was naturally able to read the *Book of the Origin*, in view of the link between the gravitational interaction and anything else present in the Universe; for this reason in the Einstein formulation

General Relativity is often called *environment interaction*.

### 1.3.1 *Einstein proposal of a static Universe*

From a theoretical point of view, the birth of modern Cosmology can be traced back to 1917 (on the wave of GR, started in 1914), when Einstein proposed a mathematical Cosmology based on his theory of General Relativity. As a first attempt to build a description of the Universe based on the equations of General Relativity, his model was based on three assumptions.

The first was that, on the largest scales, the Universe is spatially homogeneous and isotropic, i.e. that no preferred observers exist. One reason for this assumption was certainly philosophical in nature, since in some sense it embodies the Copernican Principle, i.e. the Earth does not occupy a special place in the Universe. This assumption is called the *cosmological principle*, a term coined by Edward Milne (1896–1950). Another reason for introducing the cosmological principle, and for its success before proving as a good approximation, was that it simplified the mathematical treatment of Einstein field equations. In fact, at the time of its introduction, the cosmological principle did not properly describe the observed Universe, until then limited to our galaxy. It was already well known that stars in the Milky Way are not homogeneously distributed. The cosmological principle was more than a theoretical prejudice or a simplifying working hypothesis, until the observations showed that there were other galaxies beyond the Milky way and that they were indeed homogeneously distributed.

The second assumption was that the Universe has a closed spatial geometry and thus a constant positive curvature, ensuring a finite volume, although without boundaries, like the surface of a sphere.

Finally, the third assumption was that the Universe is static, i.e. it does not change with time. This can also be considered either a theoretical/philosophical prejudice or a simplifying assumption. In particular, it avoided the embarrassment of dealing with a “creation” event. When taken together, the cosmological principle, expressing the space invariance of the Universe, and the static Universe assumption, expressing time invariance, are sometimes called the “perfect cosmological principle”.

As Einstein himself realized, a shortcoming of his static model is that the equations of General Relativity do not admit any solution compatible with these three assumptions. In order to obtain the static scheme, Einstein modified the field equations introducing a cosmological constant term, which can be interpreted as a gravitationally repulsive term acting at large



distances. Later, when the redshift of the galaxies was observed, thus proving that the Universe is not static but is indeed expanding, he regretted the introduction of the cosmological constant as the “greatest blunder” of his life<sup>1</sup>.

In 1917, the astronomer Willem de Sitter (1872–1934) found another solution to Einstein field equations (with a cosmological constant) that satisfies the cosmological principle and describes an expanding empty Universe. A notable feature of the de Sitter solution is that, even if expanding, it is however stationary since it admits a time-independent representation.

In the following years, other relativistic cosmological models were developed. In particular, both the Russian mathematician Aleksandr A. Friedmann (1888–1925) and the Belgian astronomer Georges Lemaître (1894–1966) independently discovered, in 1922 and 1927 respectively, the solutions to the Einstein equations that describe a Universe filled with matter. They assumed the validity of the cosmological principle but dropped the assumption of time-independence, considering both positively and negatively curved spaces. The Friedmann-Lemaître models predict that a pair of objects move with a relative velocity proportional to their distance, thus anticipating the discovery of the Hubble law. Friedmann and Lemaître also determined the physical conditions for an open Universe, indefinitely expanding, and on the other hand for a closed Universe, destined to a contraction, depending on the amount of matter contained in it.

### 1.3.2 *Galaxies and their expansion: The Hubble’s discovery*

A turning point along the path that led to the birth of modern Cosmology was the realization that the *spiral nebulae* (from the latin word for “cloud”, at that time used to denote any astronomical object with a diffuse appearance, as opposed to a star) visible in the sky are indeed galaxies, similar to the Milky Way, and discovering a whole new level in the hierarchical structure of the Universe.

The controversy about the galactic or extragalactic nature of the spiral nebulae was settled down in the 1920s. In 1922, the Estonian astronomer Ernest Öpik (1893–1985) estimated the distance of the Andromeda Nebula M31 placing it well outside the Milky Way<sup>2</sup>. The American astronomer

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<sup>1</sup>Ironically, recent observations suggest that the Universe is currently undergoing a phase of accelerated expansion that could be driven by a cosmological constant-like component.

<sup>2</sup>The obtained value,  $D \sim 450$  kpc is actually quite close to the modern determination of 770 kpc.

Vesto Slipher (1875–1969) showed that the recession velocities of the spiral nebulae, as estimated by the Doppler shift of their spectral lines, were far higher than those of the other known astronomical objects. The definitive evidence for the extragalactic nature of the nebulae was found in 1923–24 by Edwin Hubble (1889–1953), who resolved the stars inside the Andromeda and the Triangulum (M33) nebulae. In particular, since some of these stars were variable Cepheids, he could measure the distance of the two nebulae (the luminosity of Cepheid stars is correlated to the period of variation of the luminosity itself, making them standard candles for the determination of distances). Hubble found that both M31 and M33 lie at a distance of approximately 300 kpc from the Earth, again placing them far beyond the borders of our galaxy. In 1926, using the method of the number count, he also quantitatively showed a homogeneous distribution of galaxies in space, without any observable boundary. This discovery was the experimental verification of Milne’s cosmological principle discussed above, and together with the realization that spiral nebulas are “island Universes” of their own, it can be considered the 20th century version of the Copernican revolution, showing that the Earth occupies no special place at all in the Universe.

However, the contribution to observational cosmology for which Hubble is best known is the discovery of the expansion of the Universe. Building on the studies of Slipher (who in the 1910s already observed the puzzling fact that the spectra of most galaxies are shifted towards the red), and aided by fellow astronomer Milton Humason (1892–1972), Hubble was able to measure the spectral shifts and the distances of a sample of roughly 50 galaxies. The relationship between redshift and distance was first published in a short paper by Hubble alone in 1929 and then in a longer paper authored by Hubble and Humason in 1931. The conclusion was that the redshift of a galaxy is directly proportional to its distance. By interpreting the redshift as a Doppler shift, this relationship, that bears the name of *Hubble law*, can be restated as a proportionality between a galaxy’s distance and its velocity, thus providing evidence for the expansion of the Universe.

Many theorists made several attempts to interpret the expansion on the ground of Einstein’s theory of General Relativity. In fact, an expanding solution of Einstein equations, compatible with the homogeneity and isotropy requirements of the cosmological principle, had already been found by de Sitter in 1917. However, there were two shortcomings associated to the interpretation of Hubble’s findings in terms of the de Sitter solution. The first refers to the requirement of the expansion pattern to be the same as seen from any galaxy, and thus in order to fulfill the cosmological principle,

one has to set particular initial conditions. The second shortcoming is that the de Sitter solution describes an empty Universe, and it is at variance with the observation that galaxies are spread throughout the space. At that time, the only widely known matter-filled cosmological model was the Einstein static model, that of course does not predict any expansion. The solution to the conundrum was in the matter-filled, expanding Universe solution to the General Relativity field equations already found by Friedmann in 1922. His solution was independently rediscovered by Lemaître in 1927, and the latter's work was later brought to more general attention in the early 30s by Eddington and de Sitter. The final step was the demonstration of Robertson (1935) and Walker (1936) that the line element adopted by Friedmann is the more general line element in a spatially homogeneous and isotropic spacetime.

#### 1.4 The Genesis of the Hot Big Bang Model

Hubble's discovery led to dismiss the static model of the Universe, while introducing the idea of an evolving Universe like that described by the Friedmann-Lemaître models. It was soon realized that the extrapolation of the cosmological evolution *backwards* in time implies that matter and radiation, nowadays sparsely scattered through space, were concentrated in a remote period of time. In fact, a feature of the Friedmann models is the existence of an instant of time in the past when the dimension of the Universe extrapolates to zero and the matter and radiation densities correspondingly diverge. Friedmann himself calculated this time for our Universe to be some ten billions years in the past, although it is not clear how much physical significance he attributed to the initial singularity. The first to actually put forward the idea that the Universe expanded from a very dense state was, in 1931, Lemaître, who used to call the initial state the "primeval atom". Remarkably, he also speculated on the possibility that the early hot and dense phase should leave some relic radiation, that he described as "the vanished brilliance of the origin of the worlds". However, the Lemaître theory was mainly developed during the 1940s by Russian-born American physicist George Gamow (born Georgy Antonovich Gamov, 1904–1968), who had briefly been a student of Friedmann in St. Petersburg. Gamow advocated the theory that the chemical elements present in the Universe were synthesized in a very early phase of the Universe, when it was dense and hot enough for thermonuclear reactions to take place. This

was opposed to the theory according to which the chemical elements are produced in stars. We now know that both theories are right, since in the early Universe only the lightest elements (mainly hydrogen and helium) are produced, while the heavier elements are produced in stars. Gamow and his student Ralph Alpher (1921–2007) first exposed this theory in 1948 in the so-called “ $\alpha\beta\gamma$ ” paper, signed by Alpher, Bethe<sup>3</sup> and Gamow. In that paper, the authors argue that “various nuclear species must have originated [...] as a consequence of a continuous building-up process arrested by a rapid expansion and cooling of the primordial matter”. This model was later given the name of *hot Big Bang* model, although this was originally intended as a pejorative monicker, coined by those who opposed the theory. It is also a sort of misnomer, since the term “Big Bang” seems to point to a single event localized in space that, in the past, triggered the expansion. This popular understanding of the model is wrong. The first reason is that the Big Bang did not happen at a specific point in space, as it should be obvious from the cosmological principle. Secondly, and maybe more importantly, the term “Big Bang” *does not refer* to the initial singularity that is present in the Friedmann models. Although usually cosmologists indulge in the habit of calling “Big Bang” the singularity, properly speaking the term should preferably be used to describe the very hot and dense primeval phase of the expansion, regardless of the presence of singularity. The difference is more than semantic since, as we shall see below, we have compelling evidence for the physical reality of the hot phase, while the same certainly cannot be said for the singularity. In fact, the hot Big Bang model would still be true even if the singularity did not occur.

Later in 1948 Alpher, joined by Robert Herman (1914–1997), and Gamow independently realized that their theory implied that, at the time of the synthesis of the chemical elements, the Universe was filled by a black-body radiation with an associated thermal energy of the order of 1 MeV. This radiation was actually providing the main contribution to the energy density of the Universe at that stage of evolution. The present-day temperature of this radiation field can be estimated and it turns out to be around a few Kelvins, so that the maximum of its spectrum should be in the microwave range. This blackbody with  $T \sim \text{few K}$  is what today we call the *Cosmic Microwave Background* (CMB). The presence of the CMB was a definite prediction of the hot Big Bang model, and thus it could be

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<sup>3</sup>Hans Bethe (1906–2005) did not actually contributed to the paper; his name was added humorously by Gamow in order to create a joke with the first three letters of the Greek alphabet.

used to test its validity. Unfortunately, for some time the Gamow-Alpher picture was put aside since it became clear that the elements' primordial build up could not proceed past  ${}^4\text{He}$ , making the stellar synthesis picture more appealing.

In the same years, the Big Bang theory was opposed to the so-called *steady-state* theory, formulated by Hermann Bondi (1919–2005) and Thomas Gold (1920–2004) and further developed by Fred Hoyle (1915–2001). Although nowadays referred to as an “alternative cosmology”, at the time the steady-state theory was much more credited, among scientists, than its hot Big Bang competitor. The steady-state model was based on the perfect cosmological principle, i.e. on the notion that the Universe should be the same at every point in space and at every instant of time. In order to reconcile this assumption with the, by the time accepted, cosmological expansion, the theory postulated the continuous creation of matter, in order to balance for the dilution caused by the expansion and maintain the same average density. A steady-state Universe has no beginning nor end in time. One point in favor of the steady-state model was the fact that the age of some astronomical object seemed to be larger than the age of the Universe as estimated in the framework of the hot Big Bang model. Today, we know that this was due to the severe overestimate of the Hubble constant (to which the age of the Universe is inversely proportional).

The debate, as it should happen in natural sciences, was settled by observations. In the early '60s, many scientists (re-)realized that the observation of a microwave blackbody radiation would have provided compelling evidence for the Big Bang. In particular, both the group of Robert Dicke in Princeton and that of Yakov Zel'dovich in Moscow independently arrived at this conclusion. In 1964, Dicke and collaborators (among them Jim Peebles and David Wilkinson) were actually planning to search for a microwave radiation of cosmological origin, building a dedicated radiometer. However, they were unaware that the CMB radiation had actually already been observed by a radiometer at the Bell Telephone Laboratories in Holmdel, just 50 kilometers away from Princeton.

The radiometer had originally been used for the first experiments on satellite communications, and in 1963 Arno Penzias and Robert Woodrow Wilson started to prepare it for radio astronomy observations.

After removing known sources of noise (like for example radio broadcasting), they were left with an apparently inexplicable residual, isotropic noise, corresponding to a 3.5 K excess antenna temperature.

After learning of the ongoing efforts of the Princeton group, they started

to guess the possible cosmological implications of their findings. They contacted Dicke and a joint meeting with the Princeton group was organized, where the conclusion was reached that the origin of the 3.5 K signal was extraterrestrial. The two groups decided to publish their results independently but at the same time. Thus, in the same volume of the *Astrophysical Journal* two papers appeared, one authored by Penzias and Wilson, the other by Dicke, Peebles, Roll and Dickinson. In the first, conservatively titled “*A Measurement of Excess Antenna Temperature at 4080 Megacycles per Second*”, the existence of the isotropic signal was reported. In the second, the signal was interpreted as the relic radiation from the hot Big Bang. The discovery of the CMB dealt the final blow in the steady-state model and led to the definitive acceptance of the hot Big Bang theory.

Later, in 1978, Penzias and Wilson received the Nobel Prize for their discovery.

### 1.4.1 *Recent developments*

#### 1.4.1.1 *Observed isotropy*

After the observation of the CMB, much effort was devoted to the precise determination of its frequency spectrum, in order to confirm the black body shape. This was at least in part due to the claims by the supporters of the steady-state theory that an isotropic microwave background can be generated by the scattered, redshifted light of very distant galaxies. This background would not have a thermal spectrum, however. In the same years, during the 70s, theoretical cosmologists started to realize that small inhomogeneities, of order of one part in  $10^4 - 10^5$ , should have been present in the primordial plasma, that eventually grew (through gravitational instability) and formed the galactic structures that we observe today. These small inhomogeneities should leave an imprint in the CMB radiation, producing anisotropies approximately at the same level. Then, during the 80s the observational efforts converged on the measurements of the angular fluctuations in the CMB, other than to the ultimate determination of its frequency spectrum. Both these goals were reached by the COsmic Background Explorer (COBE), a NASA satellite launched in 1988. COBE carried two instruments, the Far-InfraRed Absolute Spectrometer (FIRAS) and the Differential Microwave Radiometer (DMR). FIRAS provided the definitive measurement of the frequency spectrum of the CMB, showing that it is a nearly perfect black body with a temperature  $T = 2.723$  K (it

is actually the most accurate blackbody that is observed in nature), while DMR observed for the first time the large scale CMB anisotropies, detecting temperature fluctuations of some tens of microkelvins. Two of COBE's principal investigators, John Mather and George Smoot, were awarded the Nobel Prize in 2006 for their work on the experiment.

Following the observations of COBE, in the 90s many ground-based and balloon-borne experiments were designed to measure the CMB anisotropies at smaller angular scales with respect to those that were accessible by COBE. The determination of the exact anisotropy pattern could discriminate between different theories for the origin of the primeval seeds from which cosmic structures originated. In particular, the two most credited theories, cosmic strings and inflation, predicted different anisotropy patterns. In the cosmic strings scenario, the power spectrum of the anisotropies should appear as nearly featureless, while in the inflationary scenario, it should exhibit a characteristic alternation of peaks and dips, caused by the presence of coherent acoustic waves in the early Universe. The existence of at least one peak was hinted by several experiments during the 90s, and finally in 2000 the BOOMERanG experiment, led by Andrew E. Lange and Paolo de Bernardis, detected the presence of multiple peaks and provided a precise determination of the position of the first acoustic peak, at an angular scale of roughly one degree. The BOOMERanG results implied that the spatial geometry of the Universe is nearly flat and confirmed inflation as the leading theory for the origin of primordial fluctuations. BOOMERanG also provided, for the first time, fairly tight constraints on the value of the cosmological parameters, thus marking the birth of *precision cosmology*.

In the last decade, the CMB anisotropy spectrum has been measured with increased precision and down to smaller angular scales. A new observational target is the pattern of the fluctuations in the *polarization* of the CMB photons, that itself encodes many information on the Universe, like those related to the formation of the first stars or to the presence of a relic background of gravitational waves. In 2001, NASA launched another CMB space mission, called Wilkinson Microwave Anisotropy Probe (WMAP) in tribute to David Wilkinson. WMAP has obtained the most precise measurement of the CMB temperature fluctuations to date and provides the tightest constraints on the values of the cosmological parameters. In 2009, the ESA mission Planck was launched, and is expected to provide the ultimate measurement of the CMB temperature fluctuations.

#### 1.4.1.2 *Dark matter*

Another striking feature of the Standard Cosmological Model is the presence in the Universe of a large amount of non-baryonic matter, accounting for roughly 25% of the total matter-energy content of the Universe. The existence of this “dark matter” was suggested by the Swiss astronomer Fritz Zwicky (1898-1974) to explain the motion of galaxies inside the Coma cluster. In particular, he found that the mass of the cluster, estimated on the basis of galactic motions, was 400 times larger than the visible mass inside the cluster. This was known as the “missing mass” problem, and Zwicky – although not taken seriously at that time – suggested that it was due to the presence of a matter component not interacting with light. The existence of dark matter remained a hypothesis for 40 years, until the ’70s, when the American astronomers Vera Rubin and W. Kent Ford Jr. used a new spectrograph to measure the rotation curves of galaxies, i.e. the orbital velocities of stars inside galaxies as a function of their distance from the center, with unprecedented accuracy. They found that the mass required to explain the observed curves was roughly 10 times larger than the visible mass of the galaxy, and that this mass extended far beyond the visible edge of the galaxy. This is considered the first strong observational hint for the existence of dark matter. From the 70s until today, a bulk of evidence has been gathered confirming its presence (for example, the CMB anisotropy spectrum, or the galaxy power spectrum) although a direct detection is still missing. The best evidence to date is provided by the *Bullet cluster* observed by the Chandra X-ray Observatory. The Bullet cluster consists of two merging clusters passing one through the other. The hot (collisional) plasma inside the clusters can be clearly seen in the X-rays to be stuck in the middle of the two colliding objects due to its collisional nature. However, the total mass distribution in the cluster can be estimated by studying the gravitational lensing of background objects, and it shows that the centers-of-mass of the two clusters are separated. This indicates that most of the mass in the clusters is in the form of a dark, collisionless component. The common scientific view is that dark matter is made by Weakly Interacting Massive Particles (WIMPs). Although there is no WIMP candidate in the framework of the Standard Model (SM) of particle physics, nevertheless they are predicted by many extensions of the Standard Model itself. The most popular candidate is the neutralino, appearing in supersymmetric extensions of the SM. Some proposals have also been made trying to explain the missing mass problem not through the presence of dark matter,



but instead through modifications to the theory of gravity. However, these theories have to face the problem that the presence of dark matter can be inferred by observations made at many different scales, from galactic, to cluster, to the largest cosmological scales (probed by the CMB).

#### 1.4.2 *Discovery of the acceleration*

Another important development in contemporary Cosmology took place in the late '90s, when a very puzzling fact emerged from the observations of distant type Ia Supernovae (SNIa). SNIa are *standard candles*, i.e. object of known intrinsic luminosity, so that they can be used to build a Hubble diagram. Since they are very luminous, they can be observed up to very large distances and used to probe the expansion history deeper in the past. In 1998 two groups, the Supernova Cosmology Project and the High- $z$  Supernova Search, measured the supernovae Hubble diagram and independently reported evidence that distant SNIa are less luminous than they would be in a decelerating Universe, implying that the Universe is now accelerating. This fact, albeit strange, can be easily accommodated in the framework of the Standard Cosmological Model. In fact, even if neither matter nor radiation can give rise to an accelerated expansion, a component with negative pressure could. A natural candidate is the cosmological constant that Einstein introduced to obtain a static Universe and that he later regretted as the “greatest blunder” of his life. In fact, when interpreted in the framework of a Friedmann Universe with matter and a cosmological constant, the SNIa data provide compelling evidence for the presence of the latter. Unfortunately, this raises some very problematic issues from the point of view of quantum field theory. The observational evidence for the acceleration has grown in the last years and is now well established. However, its theoretical interpretation is still unclear and can probably be regarded as one of the biggest open problems in cosmology nowadays. One possibility is that, as stated above, the expansion is caused by a component with negative pressure (like a dynamical scalar field), generically dubbed “dark energy”. Another is that the theory of General Relativity fails at the cosmological scales and has to be replaced by a more general theory of the gravitational interaction. The third possibility is that the observed acceleration is just an artefact due to the effect of small-scale inhomogeneities on the propagation of photons.

### 1.4.3 *Generic nature of the cosmological singularity: The Cambridge and the Landau School*

The problem regarding the existence and nature of the singularity was widely studied during the second half of the XX century, providing many interesting features about the possible beginning of the Universe and still leaving many intriguing topics unanswered. The existence of a singularity came out after the works by Roger Penrose and Stephen Hawking, regarding the analysis of geodesics in different conditions: the impossibility, in some cases, of an indefinite continuation suggested the presence of a singularity in the general solution of the Einstein equations (see Sec. 2.7).

The *Landau School* was created by L. D. Landau (1908–1968) at the Science Academy in Moscow. This group of scientists gave an important contribution to the development of Relativistic Cosmology in the '60s and '70s of the last century. Apart from the work of Landau on superfluids (he got the Nobel prize in 1962 for this study) and a few other issues, the Landau School was surprisingly in the excellent capability to extract significance from the implementation of a theory for the synthesis of new physics. The studies in Cosmology we are going to refer to are an excellent example of the technical power that this team of scientists had in implementing General Relativity.

Within the Landau school, at the beginning of the '60s, was pursued a detailed and deep analysis of the general solutions of the Einstein equations when evolving toward the singularity, either considering the instability of density perturbations, either addressing the generality of the properties of the solutions themselves.

The two most important results obtained in the investigations of the early Universe can be recognized in the Lifshitz analysis of the gravitational stability of the isotropic case and in the discovery by Belinskii, Lifshitz and Khalatnikov (BKL) of the chaotic behavior of the generic cosmological solution near the initial singularity.

The Lifshitz results demonstrated the stability of the Friedmann-Robertson-Walker (FRW) Universe when the volume expands, ensuring how this highly symmetric solution can represent the present Universe.

The BKL analysis clarified the existence of a past time-like singularity as a general feature of the Einstein equations under cosmological hypotheses. This result has to be considered in comparison and contrast with the general theorems due to the Hawking School. Such rigorous framework, described in Sec. 2.7, has a powerful nature, but being of topological nature

(i.e. concerning the behavior of world lines), it is not able to characterize the physical properties of the singularity as a space-time pathology. The investigations of the Landau School allowed one to determine a piecewise analytical representation of the generic cosmological solution near the initial singularity, so properly characterized as a real space-time feature for a very wide class of models (the word *generic* can be qualitatively understood as absence of any specific symmetry). The nature of the BKL achievements and the modern developments in this line of research are discussed in great detail in the third part of the book, namely Mathematical Cosmology.

Indeed, an important step in the development of a general point of view on the origin of a non-symmetric Universe is constituted by the work of Lifshitz and Khalatnikov in 1963. In this study, they derived the so-called generalized Kasner solution, extending the intrinsic anisotropic exact solution of the Einstein equations for the Bianchi I model (provided by Kasner in 1921) to the inhomogeneous sector and asymptotically to the singularity.

This is the building block of the generic cosmological solution, being its analytical segment, iterated to the cosmological singularity in a sequence of infinite alternation of equivalent regimes (known as Kasner epochs). Despite this generalized Kasner solution is derived by imposing a condition which limits its generality, Lifshitz and Khalatnikov did not realize the underlying scenario at that stage of understanding and claimed that the generic solution had to be asymptotically Kasner-like.

Only at the end of the '60s, when Belinskii and Khalatnikov investigated deeper the behavior of the homogeneous model, became clear that the instability of a Kasner epoch could result in the transition to a new one with different values for the metric parameters. From this new intuition arose first the BKL study of the Bianchi type VIII and IX model and then the extension of this prototype to the generic case.

The very surprising feature was the discovery that the iteration of equivalent regimes were associated to the appearance of chaotic properties of the time evolution. Much later studies, mainly due to A. A. Kirillov and G. Montani outlined how the stochasticity of the time evolution induces a chaotic morphology of the spatial slices too. This phenomenon associated to the inhomogeneous BKL solution makes the spacetime near the singularity like a foam of statistical nature.

After the work of Charles Misner in 1969, the BKL dynamics was called as Mixmaster Universe (with particular reference to the Bianchi IX cosmological model), in view of its Hamiltonian representation as a particle

randomizing in a closed potential. Over the last three decades the interest for the Mixmaster evolution of the early Universe remained high, both for the attention devoted to such behavior by the dynamical system theory, and because its generality suggests that it trace very well the dynamical conditions under which the Universe was borne, especially in view of a quantum gravity scenario.

Indeed, the Mixmaster Universe properties are still intensely studied providing a deep insight about the Universe and offering a test field for different methods of analysis coming from several fields to a generic problem. Quite often, an evolutionary property derived for a specific topic is extended and tested (sometimes in a speculative way) also in the Mixmaster model.

In particular, a wide interest emerged on the chaos features shown by the Mixmaster Universe, giving rise to several tests from different points of view, such as dynamical systems' approaches, discrete mathematics and numerical calculations.

The possibility to reconcile the exotic nature of the generic cosmological solution with the regular homogeneous and isotropic model of the hot Big Bang is today recognized in the inflationary scenario, formulated at the end of the '70s and the beginning of the '80s of the last century. This paradigm, discussed in details in Chap. 5 was introduced by Guth and Linde to overcome the shortcomings of the Standard Cosmological Model.

#### 1.4.4 *The inflationary paradigm*

After the hot Big Bang scenario became the standard model of cosmology, the theoretical effort was mainly aimed at two different goals. The first, more phenomenological, was to develop the theoretical tools that are necessary to extract meaningful, observable predictions. The second, more philosophical, was to understand some puzzling facts about the standard model which offered a coherent picture, confirmed by the presence of the CMB, of the history of the Universe from the time of nucleosynthesis until today. The supporting evidence in favor of the hot Big Bang model has grown through the years and to date there are no observations that are at variance with it. However, some paradoxes arise that cannot be solved in the framework of the standard model. Simply put, these paradoxes are mainly related to the fact that, in order to evolve into its present state, the Universe should have started from *very peculiar* initial conditions in the early phases. This was first noted by Zel'dovich in the early '70s. In the

following years, it was realized that an early de Sitter phase of exponential expansion (nowadays called an *inflationary era*) would solve these problems. In 1980, Alexei Starobinsky and Alan Guth independently proposed two mechanisms to generate the exponential expansion.<sup>4</sup> In Starobinsky's model, this is caused by quantum corrections to gravity that become important at very high energy. In Guth's model, inflation is caused by the fact that the early Universe is trapped in a metastable, false vacuum state; the expansion is driven by the vacuum energy associated to the false vacuum. Unfortunately, Guth himself realized that this model suffers from the "graceful exit" problem, namely the fact that the phase transition between the false and true vacua never takes place and inflation never ends. The graceful exit problem was solved independently shortly after by Andrei Linde on one hand and Andreas Albrecht and Paul Steinhardt on the other. In their variant (referred to as "new inflation" or "slow-roll inflation") of the original model, inflation is driven by the energy density of a scalar field slowly rolling down its potential. Other than solving the standard model paradoxes, inflation also offers a mechanism for the generation of the primordial density fluctuations. To date, inflation is a successful paradigm that has received its confirmation mainly by the measurements of the CMB anisotropy spectrum. On the other hand, from the theoretical point of view, there are still many open questions: for example, we still do not know what the scalar field allegedly responsible for the inflationary expansion (the *inflaton*) is. The ambition is that the inflationary scenario will some day be embedded in a more general theory, like supersymmetry or string theory. We conclude by stressing that the inflationary expansion can be produced by many different mechanisms so that inflation is more correctly referred to as a paradigm or a scenario; within this scenario, many different models exist, and can be discriminated through the observations.

#### 1.4.5 *The idea of non-singular cosmology: The cyclic Universe and the Big Bounce*

We conclude the historical picture of the development of a modern cosmology by discussing the inflationary scenario and the observation of an accelerating Universe (collocated mainly 30 and ten years ago, respectively)

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<sup>4</sup>However, it should be noted that Starobinsky was probably mostly motivated by the goal of avoiding the initial singularity. Thus, in his work there is no reference to the standard model paradoxes nor to the exponential expansion as a possible solution of the paradoxes themselves.

because they represent the most significant theoretical and observational progresses achieved by cosmologists.

Indeed the great efforts made in the recent years to improve and complete our knowledge of the Early Universe gave rise to very promising new points of view on the nature of the singularity (see for instance the pre-Big Bang scenarios predicted by String cosmology), although still at the center of the contemporary debate about their prediction capability.

However, let us characterize some recent developments in Loop Quantum Cosmology, like the possible existence of a Big Bounce for the Planckian evolution of the isotropic Universe. This attention is not motivated by the conviction that these studies are completely settled down and predictive, but in view of their peculiar features, presented in Chap. 12.

Despite some theoretical shortcomings, like the problem of entropy, the idea of a cyclic (closed) Universe, oscillating between a Big Bounce and a turning point, seemed to Einstein and other theoreticians a very pleasant alternative to the Big Bang singularity. In this respect, the results obtained by Ashtekar and collaborators are a very encouraging issue in favor of this cyclical idea. The isotropic Big Bounce has been derived implementing the ideas and formalism of Loop Quantum Gravity (mainly due to Ashtekar, Smolin and Rovelli). This canonical approach to the quantization of the gravitational field has the great merit of starting from a continuous description of the spacetime manifold, nonetheless recovering the discrete structure of the space, in terms of discrete spectra of the geometrical operators, like areas and volumes. The kinematical sector of Loop Quantum Gravity resembles a non-Abelian gauge theory and it allows the extension to the gravitational field of the so-called Wilson loops approach for strongly coupled Yang-Mills theories. However, the dynamical implementation of the super-Hamiltonian quantum constraint contains a certain level of ambiguity, e.g. the non-unitary equivalence of theories corresponding to different values of the Immirzi parameter, entering the canonical variables definition.

The application of Loop Quantum Gravity to the minisuperspace of a homogeneous cosmological model, expectedly implies a non-singular behavior of the quantum Universe, as a direct consequence of the cut-off scale imposed on the Universe volume, by the minimal (taken Planckian) value of its operator spectrum. Indeed, the Friedmann-Robertson-Walker geometry acquires in Loop Quantum Cosmology a non-singular behavior as described in terms of a free massless scalar field (the kinetic term of the inflaton field) playing the role of a relational time. The semiclassical picture of this non-singular Universe can be restated in the form of a maximal critical energy

density for the asymptotic approach to the initial instant.

As we already stressed in Sec. 1.4, this cut-off on the maximal available temperature of the primordial Universe does not affect the theory of the Hot Big Bounce, because its scale is much greater than the physical regions of interest for the Standard Cosmological Model predictions like inflation, baryogenesis and nucleosynthesis. In this new scenario the idea of a cyclical Universe takes new vigor and is substantiated by a precise quantum and semiclassical scenario.

Although the Big Bounce theory is a promising perspective and deserving many attempts to extend its applicability to more general cosmological models (up to the generic quantum Universe), nevertheless its derivation is affected by some open issues. In fact, the restriction of the Loop Quantum Gravity theory to the minisuperspace has the non-trivial implication to replace the non-Abelian  $SU(2)$  by an Abelian  $U(1)$  symmetry, unable to ensure the discreteness of the volume spectrum. The possibility to recover the Big Bounce from the minisuperspace dynamics relies on the introduction by hands of the space discreteness as a natural, but not direct, consequence of the full theory equipment. These shortcomings of imposing the symmetries of the isotropic model before quantizing its dynamics prevent the Big Bounce to be self-consistently derived, but do not seem able to affect the impact of this issue on the modern idea of a primordial Universe.

## 1.5 Guidelines to the Literature

The history of Cosmology is based on a variety of sources which cannot be compactly summarized for what concerns the most ancient documents. Since the approach to the Cosmos was borne as the vision of the philosophers along the centuries on the basis of the limited physical evidences available, some reference can be searched consulting textbooks on history of philosophy, specialized for the different historical periods, such as Guthrie [213] for the Greek era and Grant for the Middle Ages [204] or covering up to the 19th century [205].

The works of most authors, although reprinted from time to time up to the XX century, are nowadays directly accessible by several online services, such as [books.google.com](http://books.google.com) which is a generic source of original documentation or by other online libraries, such as <http://digital.lib.lehigh.edu/planets/> for some works by Copernicus and Brahe.

A summary of the fundamental works by Galileo Galilei writings is given by [1, 183–185, 192] and by Isaac Newton by [362, 363].

To read the revolutionary ideas of Riemann on a new foundation of geometry, among several translations, one can refer to [390, 391], while the pioneering study on differential geometry obtaining the classification of the non-equivalent Lie groups is [86].

The most relevant papers by Einstein during the second decade of the XX century can be found as [160–166].

In the 70s, Penrose and Hawking introduced the theorems on the singularity [231, 232] in the fertile environment of mathematical cosmology on the path from the first results by Kasner [266] towards the main results of the Landau School, which can be found in the papers [58, 65, 66, 273, 274, 312, 313, 315, 345].

An overall vision over the last century is provided by the book of Longair [325] while the physical Cosmology progress is summarized as follows.

The work of Starobinsky on the possibility of avoiding the initial singularity by taking into account higher-order curvature corrections to the Einstein-Hilbert action can be found in [426]. The idea of an inflationary Universe as a solution to the standard model paradoxes was proposed by Guth in his 1981 paper [212]. The “slow-roll” inflation was invented shortly after by Linde [320] and Albrecht & Steinhardt [2].

A very detailed account of the birth and development of physical cosmology, until the discovery of the CMB, can be found in the first part of the book by Peebles [378].

The first cosmological solution to the equations of General Relativity (with the addition of a cosmological term), describing a static homogeneous and isotropic Universe was found by Einstein in 1917 [165]. The term “Cosmological Principle” that denotes Einstein’s assumption of isotropy and homogeneity was coined nearly 20 years later, in 1935, by Milne in his book [343]. Also in 1917, de Sitter derived the solution for an expanding, empty Universe [143]. The solution describing an expanding Universe filled with matter and radiation was derived by Friedmann first in 1922 for a space with positive curvature [176] and then in 1924 for a space with negative curvature [177]. Lemaître independently re-derived these solutions and was the first to make the very important connection between the expansion of a matter-filled Universe and the observed redshift of the galaxies [305]. The original work is written in French and was later translated into English [307]. The translation came after papers by Eddington [157] and de Sitter [144] drew attention on Lemaître’s work that had gone unnoticed at that



time. A few years later, the line element used by Friedmann and Lemaître was shown independently by Robertson [395] and Walker [458] to be the most general element compatible with the requests of spatial isotropy and homogeneity. This important result is actually geometrical in nature and does not rely at all on the theory of General Relativity.

The spectroscopic observations performed by Slipher during the 1910s showed that the spectra of most spiral nebulae are redshifted towards the red, indicating that they are receding from us [420]. Slipher's results also implied that the recession velocities of the nebulae were much bigger than those of other astronomical objects, hinting to their extragalactic nature. In 1922 Öpik made an estimation of the distance of the Andromeda nebula that placed it outside the Milky Way [367]. This result was confirmed by Hubble's observations of the Andromeda and Triangulum nebulae reported in 1926 [252, 254]. In the same year Hubble also showed that galaxies are distributed homogeneously and without a visible edge [253]. In 1929, Hubble reported the first evidence for a proportionality between the redshift of a galaxy and its distance [250]. Stronger evidence was given in a 1931 paper by Hubble and Humason, where what we know today as Hubble law was formulated for the first time [251].

The notion that the Universe started from a very dense and hot state was first put forward by Lemaître with his theory of the “primeval atom” [306]. In 1946, Gamow proposed the idea that the chemical elements have been synthesized in the early Universe [187]. This idea was developed in the  $\alpha\beta\gamma$  paper [6]. The implication of the presence of a thermal radiation associated with the hot phase was realized independently by Alpher [5] and Gamow [188, 189]. Gamow gave an order-of-magnitude estimate of the present temperature of the radiation; his calculations were refined by Alpher and Herman [7]. The competitor theory of the steady state Universe was proposed by Bondi and Gold [101] and by Hoyle [242], independently. The fact that the presence of a cosmic blackbody radiation with  $T \sim 1-10$  K was potentially detectable and that it would have constituted a confirmation of the hot Big Bang theory was pointed out again nearly 20 years later, in 1964, by Doroshkevich & Novikov [156]. Shortly after, the CMB was observed by Penzias & Wilson [381] and correctly identified as the relic of the hot Big Bang by Dicke and collaborators [152].

The results of COBE-FIRAS ([http://lambda.gsfc.nasa.gov/product/cobe/firas\\_overview.cfm](http://lambda.gsfc.nasa.gov/product/cobe/firas_overview.cfm)) on the CMB frequency spectrum and those of COBE-DMR ([http://lambda.gsfc.nasa.gov/product/cobe/dmr\\_overview.cfm](http://lambda.gsfc.nasa.gov/product/cobe/dmr_overview.cfm)) on the CMB angular anisotropies can be found

in [173, 337, 338] and [423, 469], respectively. The observations of BOOMERanG (<http://cmb.phys.cwru.edu/boomerang/>) and their cosmological interpretation were discussed in [139, 140, 303, 339, 361]. The latest results of the WMAP satellite ([map.gsfc.nasa.gov](http://map.gsfc.nasa.gov)), after several years of observations, can be found in [72, 198, 264, 291, 304, 460]. For more information on the Planck mission, we refer the reader to Planck's homepage.

<http://www.sciops.esa.int/index.php?project=PLANCK>.

The original paper (in German) by Zwicky on the motion of galaxies inside the Coma cluster can be found in Ref. [471]. The results reported there can also be found in a later paper (in English) [472]. The work of Rubin & Ford on the rotation curves of galaxies can be found in [403, 404]. The observations of the Bullet cluster and their dark matter interpretation have been reported in [122].

The first evidence for the acceleration of the Universe from the SNIa observations was reported in 1998 by the High- $z$  Supernova Search ([www.cfa.harvard.edu/supernova/HighZ.html](http://www.cfa.harvard.edu/supernova/HighZ.html)) [392] and the Supernova Cosmology Project ([www.supernova.lbl.gov](http://www.supernova.lbl.gov)) [386] teams. For a recent review on both the observational and theoretical status, see [178].

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## Chapter 2

# Fundamental Tools

This Chapter is devoted to introduce of some basic aspects of General Relativity (GR) and of its formalism, completed by some selected topics which are relevant for later analyses presented in this Volume. The study of this Chapter endows the reader with some fundamental tools, necessary for understanding some technical and conceptual passages in the discussion of cosmological issues.

We start with a very schematic review of the ideas and of the formalism at the ground of GR and then we provide a rather detailed discussion of the type of matter fields, emphasizing the features of impact in the study of primordial Cosmology.

The Hamiltonian formulation for the dynamics of the gravitational field is faced outlining the constrained structure of GR in the phase space. This formulation is at the ground of the canonical quantum gravity in the metric approach, presented in Chap. 10.

We devote Sec. 2.4 to the description of the synchronous reference because of the particularly simple form that the cosmological problem assumes in this special coordinate frame. Then, the tetradic formalism is illustrated to characterize the existence of a local Lorentz gauge symmetry, motivating a first-order formulation of the Einstein-Hilbert action. Such revised framework for GR is addressed starting from the Holst gravitational Lagrangian, introducing the Ashtekar-Barbero-Immirzi variables in the Hamiltonian picture. The paradigm of Loop Quantum Gravity, which will be discussed in detail in Chap. 12, is based on such analysis.

Finally, we provide a schematic discussion of the singularity theorems, developed on a topological setting, to fix the conditions under which a singular space-time point appears. This study has to be regarded as complementary to the behavior of the Universe, asymptotically to the initial

singularity, with reference to the work of the Landau School and presented in Part 3 (entirely devoted to mathematical cosmology).

## 2.1 Einstein Equations

The main issue of the Einstein theory of gravity is the dynamical character of the space-time metric, described within a fully covariant scheme. Assigned a four-dimensional manifold  $\mathcal{M}$ , endowed with space-time coordinates  $x^i$  and a metric tensor  $g_{ij}(x^k)$ , its line element reads as

$$ds^2 = g_{ij}dx^i dx^j. \quad (2.1)$$

This quantity fixes the Lorentzian notion of distances. The motion of a free test particle on  $\mathcal{M}$  corresponds to the solution of the geodesic equation

$$\frac{du^i}{ds} + \Gamma_{jl}^i u^j u^l = 0, \quad (2.2)$$

where  $u^i \equiv dx^i/ds$  is its four-velocity, defined as the vector tangent to the curve  $x^i(s)$ , and  $\Gamma_{jl}^i = g^{im}\Gamma_{jl\ m}$  are the Christoffel symbols given by

$$\Gamma_{jl\ m} = \Gamma_{lj\ m} = \frac{1}{2}(\partial_j g_{lm} + \partial_l g_{jm} - \partial_m g_{lj}). \quad (2.3)$$

The geodesic character of a curve requires to deal with a parallel transported tangent vector  $u^i$ . However, for a Lorentzian manifold this curve is an extremal for the distance functional, i.e. it is provided by the variational principle

$$\delta \int ds = \delta \int ds \sqrt{g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}} = 0. \quad (2.4)$$

If a test particle has zero rest mass, its motion is given by  $ds = 0$  and therefore an affine parameter must be introduced to describe the corresponding trajectory. The equivalence principle is here recognized as the possibility to have vanishing Christoffel symbols at a given point of  $\mathcal{M}$  (or along a whole geodesic curve). The space-time curvature is ensured by a non-vanishing Riemann tensor

$$R^i{}_{jkl} = \partial_k \Gamma_{jl}^i - \partial_l \Gamma_{jk}^i + \Gamma_{jl}^m \Gamma_{mk}^i - \Gamma_{jk}^m \Gamma_{ml}^i, \quad (2.5)$$

with the physical meaning of tidal forces acting on two free-falling observers and whose effect is expressed by the geodesic deviation equation

$$u^l \nabla_l (u^k \nabla_k s^i) = R^i{}_{jlm} u^j u^l s^m, \quad (2.6)$$

$s^i$  being the vector connecting two nearby geodesics. The Riemann tensor obeys the algebraic cyclic relations

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0, \quad (2.7)$$

and the first order differential equations

$$\nabla_m R_{ijkl} + \nabla_l R_{ijmk} + \nabla_k R_{ijlm} = 0. \quad (2.8)$$

The constraint (2.8) is called the Bianchi identity and it is identically satisfied as soon as one expresses the Riemann tensor in terms of the metric  $g_{ij}$ . On the other hand it can be considered as a real equation for the Riemann tensor when the covariant derivatives are only expressed in terms of the metric.

Contracting the Bianchi identity with  $g^{ik}g^{jl}$ , we get the equation

$$\nabla_j G_i^j = 0, \quad (2.9)$$

where the Einstein tensor  $G_{ij}$  is defined as

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}, \quad (2.10)$$

in terms of the Ricci tensor  $R_{ij} = R^l{}_{ilj}$  and of the scalar of curvature  $R = g^{ij}R_{ij}$ .

In order to get the Einstein equations in the presence of a matter field described by a Lagrangian density  $\mathcal{L}_m$ , we must fix a proper action for the gravitational field, i.e. for the metric tensor  $g_{ij}$  of the manifold  $\mathcal{M}$ . A gravity-matter action, satisfying the fundamental requirements of a covariant geometrical physical theory of the space-time, takes the Einstein-Hilbert matter form

$$S = S_g + S_m = -\frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} (R - 2\kappa \mathcal{L}_m), \quad (2.11)$$

where  $g$  is the determinant of the metric tensor  $g_{ij}$  and  $\kappa$  is the Einstein constant. The variation of the action (2.11) with respect to  $g^{ij}$  leads to the field equations in the presence of matter as

$$G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij} = \kappa T_{ij}. \quad (2.12)$$

Here  $T_{ij}$  denotes the energy-momentum tensor of a generic matter field and reads as

$$T_{ij} = \frac{2}{\sqrt{-g}} \left( \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ij}} - \frac{\partial}{\partial x^l} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta(\partial_l g^{ij})} \right). \quad (2.13)$$

As a consequence of the Bianchi identity (2.9), we find the conservation law  $\nabla_j T_i^j = 0$ , which describes the motion of matter and arises from the

Einstein equations. The gravitational equations thus imply the equations of motion for the matter itself. The whole content of the Einstein theory can be summarized as follows.

*The space-time is defined as a four-dimensional manifold  $\mathcal{M}$  on which a Lorentzian metric tensor  $g_{ij}$  is assigned. The space-time (metric) is a dynamical entity which evolves in tandem with matter according to the Einstein equations (2.12). The physical laws are background independent, i.e. they must retain the same tensor form for any assigned reference frame. Such statement is known as the Principle of General Relativity.*

Finally, by comparing the static weak field limit of the Einstein equations (2.12) with the Poisson equation of the Newton theory of gravity, we get the form of the Einstein constant in terms of the Newton constant  $G$  as  $\kappa = 8\pi G$ . We emphasize that the whole analysis discussed so far regards the Einstein geometrodynamics formulation of gravity. In the gravity-matter action, the cosmological constant term is considered as vanishing, although it would be allowed by the paradigm of GR. This choice is based on the idea that such term should come out from the matter contribution itself, expectably on a quantum level.

## 2.2 Matter Fields

In GR, continuous (macroscopic) matter fields are described by the energy-momentum tensor  $T_{ij}$  introduced above in Eq. (2.13). Here, we will mainly focus our attention on tensor fields. In particular, we will discuss the relevant cases of the perfect fluid, the scalar, the electromagnetic and the Yang-Mills fields.

### 2.2.1 Perfect fluid

The energy-momentum tensor of a perfect fluid is given by

$$T_{ij}^{\text{PF}} = (P + \rho)u_i u_j - P g_{ij}, \quad (2.14)$$

where  $u_i$  is a unit time-like vector field representing the four-velocity of the fluid. The scalar functions  $\rho$  and  $P$  denote the energy density and the pressure, respectively, as measured by an observer in a locally inertial frame co-moving with the fluid. These two quantities can be related to each other by an equation of state of the form  $P = P(\rho)$ . Since no term describing heat conduction or viscosity is introduced here, the fluid is considered as

perfect. For the isothermal early Universe, an appropriate equation of state can be cast as

$$P = (\gamma - 1)\rho, \quad (2.15)$$

where  $\gamma$  is the *polytropic index*.

The equations of motion for a perfect fluid on a curved background cannot be in general derived from a Lagrangian formulation. They can be constructed by the conservation law of the corresponding energy-momentum tensor (2.14) expressed as

$$\nabla_k T^{\text{PF}k}_i = \nabla_k [(P + \rho) u_i u^k - P \delta_i^k] = 0, \quad (2.16)$$

which can be restated as

$$u_i \nabla_k [(\rho + P) u^k] + (\rho + P) u^k \nabla_k u_i = \partial_i P. \quad (2.17)$$

Multiplying Eq. (2.17) by  $u^i$  and making use of the relation  $u^i \nabla_k u_i = 0$  (a direct consequence of the normalization  $u^i u_i = 1$ ), one obtains the scalar equation

$$\nabla_k [(\rho + P) u^k] = u^i \partial_i P. \quad (2.18)$$

Substituting this relation into Eq. (2.17), we arrive at the equations of motion for the perfect fluid flow

$$u^k \nabla_k u_i = \frac{1}{(\rho + P)} (\partial_i P - u_i u^k \partial_k P). \quad (2.19)$$

In the particular case when  $P = 0$ , we deal with a dust, whose elements follow geodesic trajectories (this is also true if  $P = \text{const.}$ ). It is worth noting that, from Eq. (2.19), the pressure effects prevent the geodesic motion of a perfect fluid and then the co-moving frame cannot also be a synchronous one, because it would be a geodesic reference (see Sec. 2.4).

We stress that for a homogeneous isotropic space, i.e. where the pressure is time dependent only, the right-hand side of Eq. (2.19) vanishes when  $u_i = (1, 0, 0, 0)$ . In this case, the co-moving system should also be a synchronous reference and in the isotropic case the energy-momentum tensor in the co-moving frame would read as

$$T^{\text{PF}}_{ij} = \text{diag}(\rho, -P, -P, -P). \quad (2.20)$$

We can conclude that the only models admitting a co-moving synchronous reference are the homogeneous spaces.



### 2.2.2 Scalar field

The Lagrangian density for the linear, relativistic, scalar field theory reads as

$$\mathcal{L}_\phi = \frac{1}{2} (\partial^k \phi \partial_k \phi - m^2 \phi^2) , \quad (2.21)$$

and on a curved space-time the dynamics can be implemented by the minimal substitution rule  $\eta_{ij} \rightarrow g_{ij}$  and  $\partial_i \rightarrow \nabla_i$ , i.e. we deal with the Klein-Gordon equation

$$g^{ij} \nabla_i (\partial_j \phi) + m^2 \phi = 0 . \quad (2.22)$$

The corresponding energy-momentum tensor is then expressed as

$$T_{ij}^\phi = \partial_i \phi \partial_j \phi - \frac{1}{2} g_{ij} (\partial^k \phi \partial_k \phi - m^2 \phi^2) . \quad (2.23)$$

Let us now look for a Lagrangian formulation of the perfect fluid dynamics, based on a scalar degree of freedom, able to reproduce the features of the desired energy-momentum tensor. We consider a massless scalar field  $\phi$  whose dynamics is governed by the Lagrangian density

$$\mathcal{L}_\phi = \frac{1}{2} (g^{ik} \partial_i \phi \partial_k \phi)^\zeta , \quad (2.24)$$

$\zeta$  being a free parameter. Using the definition of the energy-momentum tensor (2.13), from Eq. (2.24), we get

$$T_{ij}^\phi = \zeta (g^{kl} \partial_k \phi \partial_l \phi)^{\zeta-1} \partial_i \phi \partial_j \phi - \mathcal{L}_\phi g_{ij} . \quad (2.25)$$

Comparing this expression with the perfect fluid energy-momentum tensor (2.14), we can identify the fundamental quantities  $\rho$ ,  $P$  and  $u_i$  as follows

$$\rho \equiv \left( \zeta - \frac{1}{2} \right) (g^{jl} \partial_j \phi \partial_l \phi)^\zeta , \quad (2.26a)$$

$$P \equiv \frac{1}{2} (g^{jl} \partial_j \phi \partial_l \phi)^\zeta , \quad (2.26b)$$

$$u_i \equiv \frac{\partial_i \phi}{\sqrt{g^{jl} \partial_j \phi \partial_l \phi}} . \quad (2.26c)$$

One can immediately recognize that this scheme allows to reproduce a perfect fluid with an equation of state of the form  $P = \rho/(2\zeta - 1)$ . Therefore the parameter  $\zeta$  is related to the polytropic index  $\gamma$  by the relation

$$\zeta = \frac{\gamma}{2(\gamma - 1)} . \quad (2.27)$$

It is worth noting that, according to the identifications (2.26), the variation of the action  $\mathcal{L}_\phi$  (2.24) for the scalar field provides the equations  $\nabla^j T_{ij}^\phi = 0$ .

The particular case  $\zeta = 1$  corresponds to a massless Klein-Gordon field and it is associated to the equation of state  $P = \rho$ , where the sound speed  $v_s \equiv \sqrt{dP/d\rho}$  equals the speed of light. Since this case has a clear physical interpretation in terms of a fundamentally free field, we can generalize it by considering a self-interacting scalar field  $\phi$ , described by the action

$$\mathcal{L}_\phi = \frac{1}{2} \partial^k \phi \partial_k \phi - V(\phi). \quad (2.28)$$

Applying the same analysis as above to this self-interacting case, we gain the new identifications

$$\rho = \frac{1}{2} (g^{jl} \partial_j \phi \partial_l \phi) + V(\phi), \quad (2.29a)$$

$$P = \frac{1}{2} g^{jl} \partial_j \phi \partial_l \phi - V(\phi), \quad (2.29b)$$

$$u_i = \frac{\partial_i \phi}{\sqrt{g^{jl} \partial_j \phi \partial_l \phi}}. \quad (2.29c)$$

One can realize how the self-interacting scalar field has the characteristic feature of being associated to different equations of state in different dynamical regimes. This follows from the relations

$$P = \frac{KT - V(\phi)}{KT + V(\phi)} \rho, \quad KT = \frac{1}{2} g^{kl} \partial_k \phi \partial_l \phi, \quad (2.30)$$

where the notation  $KT$  stands for a kinetic term-like expression. When the potential term  $V(\phi)$  is negligible with respect to the kinetic one  $KT$  we recover the equation of state  $P \simeq \rho$ . On the other hand, in the opposite regime ( $V(\phi) \gg KT$ ) we deal with the condition  $P \simeq -\rho$ . We will see in Chap. 5 the relevance of this property of the self-interacting scalar field when its dynamics is implemented at the cosmological level.

Let us observe that, if we take  $V(\phi) = \frac{1}{2} m^2 \phi^2$ , we recover the case of the free Klein-Gordon field. In this sense, a generic potential term describes a self-interaction of the field  $\phi$ , as naturally arises in a quantum perturbation theory. In fact, when we can treat the non-quadratic terms as small corrections, we can define asymptotic free states at  $t \rightarrow -\infty$  and study scattering processes among the scalar particles, which generate the out-going free modes at  $t \rightarrow \infty$ . When the theory is fully non-perturbative, the particle interpretation is not feasible and we have to speak of self-interacting scalar

modes. Indeed, around a local minimum, say at  $\phi = \phi_{\min}$ , the potential term admits the expansion

$$V(\phi) \simeq V(\phi_{\min}) + \frac{1}{2} \left( \frac{d^2 V}{d\phi^2} \right)_{\phi=\phi_{\min}} (\phi - \phi_{\min})^2. \quad (2.31)$$

Without any loss of generality, redefining  $\phi_{\min} \equiv 0$  and  $V(\phi_{\min}) \equiv 0$ , we get to the dominant order a Klein-Gordon field, under the identification  $m^2 \equiv (d^2 V/d\phi^2)_{\phi=\phi_{\min}}$ . This is the classical picture underlying the quantum notion of particle mass as the effect of small fluctuations around a vacuum state (classically the lowest minimum) of a given field.

However we emphasize that, when the gravitational interaction is included in this paradigm, the redefinition of the minimum value of the potential describing a zero energy density is no longer allowed. In fact, the background metric tensor is sensitive to such vacuum energy density, unless the whole quantum dynamics of the field can be regarded as a test one over that background. The relevance of this consideration will be discussed in Chap. 5, where we will deal with the inflationary scenario and with the transition of a scalar field from a false to the true vacuum state.

Finally we remark that, as it takes place for an electromagnetic field, also the boson scalar dynamics has a classical character under suitable conditions. In fact, when we deal with a free scalar field having extremely high occupation numbers characterizing its states, we can properly address its evolution as a classical one, retaining the quantum effects just as small perturbations to the background field. This is at the ground of our classical consideration on the self-interacting scalar field and we will see in the study of the inflation paradigm (see Chap. 5), how this quasi-classical picture is relevant on a cosmological level.

### 2.2.3 Electromagnetic field

The electromagnetic Maxwell field is described by the Lagrangian density<sup>1</sup> ( $\mu_0 = \epsilon_0^{-1} = 4\pi$ )

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F_{ij} F^{ij}, \quad (2.32)$$

and the electromagnetic energy-momentum tensor reads as

$$T_{ij}^{\text{EM}} = \frac{1}{4\pi} \left( F_{ik} F^k{}_j + \frac{1}{4} g_{ij} F_{kl} F^{kl} \right). \quad (2.33)$$

---

<sup>1</sup>Let us remember that the universal relation  $c^2 \mu_0 \epsilon_0 = 1$  among the speed of light  $c$ , the electric permittivity of vacuum  $\epsilon_0$  and the magnetic permeability of vacuum  $\mu_0$  holds.

Here  $F = dA$  denotes the curvature 2-form associated to the connection 1-form  $A = A_i dx^i$ , i.e.

$$F_{ij} \equiv \nabla_i A_j - \nabla_j A_i = \partial_i A_j - \partial_j A_i. \quad (2.34)$$

We note that the trace of the energy-momentum tensor defined as in Eq. (2.33) is identically vanishing, i.e.  $g^{ij} T_{ij}^{\text{EM}} = 0$ . From the minimal substitution rule, the Maxwell equations in a curved space-time become

$$\nabla_l F^{kl} = -4\pi J^k, \quad (2.35a)$$

$$\nabla_{[i} F_{jk]} = \partial_{[i} F_{jk]} = 0, \quad (2.35b)$$

where  $J^k$  denotes the current density four-vector of electric charge and the square brackets around the indices are the compact notation for anti-symmetrization. The continuity equation (namely, the conservation of the electric charge)  $\nabla_k J^k = 0$  follows from the antisymmetry of the Faraday tensor  $F_{ij}$ .

We will express  $F_{ij}$  in terms of the electric and magnetic fields. Such decomposition allows a fluid description for the Maxwell field. Given an observer moving with a four-velocity  $u_i$  (such that  $u^i u_i = 1$ ), the quantities

$$E_i = F_{ji} u^j, \quad B_i = \frac{1}{2} \epsilon_{ijkl} F^{jk} u^l \quad (2.36)$$

are the electric and the magnetic fields, respectively, measured by the observer ( $\epsilon_{ijkl}$  denotes the totally antisymmetric pseudo-tensor on curved space-time). Note that  $E_i u^i = B_i u^i = 0$ , so that  $E_i$  and  $B_i$  are space-like vector fields. The electromagnetic tensor can be decomposed as

$$T_{ij}^{\text{EM}} = \begin{pmatrix} W & S_\alpha \\ S_\beta & -\sigma_{\alpha\beta} \end{pmatrix}. \quad (2.37)$$

Here

$$W = \frac{1}{8\pi} (E_i E^i + B_i B^i) \quad (2.38)$$

is the energy density of the field,  $S_i$  is the electromagnetic Poynting vector

$$S_i = \epsilon_{ijkl} E^j B^k u^l, \quad (2.39)$$

while

$$\sigma_{\alpha\beta} = \frac{1}{4\pi} (E_\alpha E_\beta + B_\alpha B_\beta - 4\pi W \delta_{\alpha\beta}), \quad (2.40)$$

is the Maxwell stress tensor. Equation (2.37) provides a fluid description of the electromagnetic field and manifests its intrinsic anisotropic nature. In particular, the Maxwell field corresponds to an imperfect fluid with energy density  $W$ , anisotropic stresses given by  $\sigma_{\alpha\beta}$  and an energy-flux vector represented by  $S_i$ .

Finally, we give the world line for a charged particle moving in the electromagnetic field

$$u^l \nabla_l u^i = \frac{q}{m} F^{ij} u_j, \quad (2.41)$$

where  $q$  and  $m$  denote the charge and the mass of the particle, respectively.

### 2.2.4 Yang-Mills fields and $\Theta$ -sector

We now introduce the concept of non-Abelian gauge fields in view of the later comparison in Sec. 2.6 between the first-order formulation of gravity with such non-linear theories.

Let us consider a set of fields  $\psi^A = \psi^A(x^i)$ , having a generic nature (scalars, spinors, etc., encoded in the generic set of internal indices  $A$ ) whose Lagrangian density, in Minkowski space-time, has the form

$$\mathcal{L}_\psi = \frac{1}{2} \eta^{ij} \partial_i \psi^{A\dagger} \cdot \partial_j \psi^A - V(|\vec{\psi}|), \quad (2.42)$$

where  $\psi^{A\dagger}$  denotes the hermitian conjugated of  $\psi^A$ , while  $\cdot$  is the product on the internal space. It is immediate to recover the invariance of this Lagrangian density under the internal unitary  $SU(N)$  transformations

$$(\psi^A)' = U \psi^A = \exp(i\lambda \Theta^a T^a) \psi^A, \quad a = 1, \dots, N^2 - 1, \quad (2.43)$$

where  $U \in SU(N)$ ,  $\Theta^a$  are constant parameters and  $\lambda$  is a coupling constant. The generators  $T^a$  of the symmetry group  $SU(N)$  are Hermitian matrices satisfying the  $su(N)$  Lie algebra

$$[T^a, T^b] = iC^{abc}T^c, \quad (2.44)$$

in which  $C^{abc}$  are called structure constants. Since the transformation (2.43) describes a global symmetry of the theory, the  $\Theta^a$  are independent of the coordinates  $x^i$ . In the limit of an infinitesimal transformation, i.e.  $\Theta^a \rightarrow \delta\Theta^a \ll 1$ , we get

$$(\psi^A)' = [1 + i\lambda\delta\Theta^a T^a] \psi^A, \quad (\psi^{A\dagger})' = \psi^{A\dagger} [1 - i\lambda\delta\Theta^a T^a]. \quad (2.45)$$

The existence of such internal symmetries is an observed feature of the Lagrangians associated to elementary particle physics. The  $SU(2)$  group describing the isospin was historically discovered from the independence of the nuclear interaction with respect to the electric charge, i.e. by guessing that protons and neutrons were different states associated to the same particles. The generators of the isospin symmetry, today recognized as a fundamental one in the Standard Model of elementary particles, are given by  $T_a \equiv \sigma_a/2$ , where  $\sigma_a$  are the Pauli matrices. The structure constants of this group are given by  $C^{abc} = \epsilon^{abc}$ ,  $\epsilon^{abc}$  denoting the totally antisymmetric tensor on the internal indices.

Let us now promote the parameters  $\Theta^a$  (and thus also the infinitesimal ones  $\delta\Theta^a$ ) to space-time functions, i.e.  $\Theta^a = \Theta^a(x^i)$ . The Lagrangian density of the theory (2.42) is no longer invariant under this local gauge transformation because the term

$$\frac{i\lambda}{2} \eta^{ij} (\partial_i \psi^{A\dagger} T^a \partial_j \delta\Theta^a \psi^A - \partial_i \delta\Theta^a \psi^{A\dagger} T^a \partial_j \psi^A) \quad (2.46)$$

does not cancel out. Local invariance is restored only by introducing a set of vector fields  $A_i^a(x^j)$  and redefining the Lagrangian density as

$$\mathcal{L}_\psi = \frac{1}{2} \eta^{ij} \mathcal{D}_i \psi^{A\dagger} \cdot \mathcal{D}_j \psi^A - V(|\psi^A|). \quad (2.47)$$

Here  $\mathcal{D}_i$  denotes the covariant (gauge) derivative which explicitly acts as

$$\mathcal{D}_i \psi^A \equiv \partial_i \psi^A + i\lambda A_i^a T^a \psi^A, \quad \mathcal{D}_i \psi^{A\dagger} \equiv \partial_i \psi^{A\dagger} - i\lambda A_i^a \psi^{A\dagger} T^a. \quad (2.48)$$

The invariance of the Lagrangian under the local transformation of  $\psi^A$  and  $\psi^{A\dagger}$ , called a gauge transformation, is ensured by the corresponding variation of the gauge vector field  $A_i^a$ , as

$$(A_i^a)' = A_i^a - \partial_i \delta \Theta^a + i\lambda C^a_{bc} \delta \Theta^b A_i^c. \quad (2.49)$$

This transformation rule corresponds to the electromagnetic gauge prescription plus an additional term containing the structure constants. This reflects the non-Abelian character of the Yang-Mills fields here introduced (i.e.  $C_{abc} \neq 0$ ). The EM case is associated to the  $U(1)$  symmetry group, for which  $C_{abc} = 0$  (i.e. it is an Abelian field).

The picture traced above needs to be completed by specifying the dynamical properties of the gauge vector fields. In analogy to the electromagnetic case, we can define a gauge tensor by means of repeated applications of the covariant derivative. For instance, we get

$$[\mathcal{D}_i, \mathcal{D}_j] = i\lambda T^a (\partial_i A_j^a - \partial_j A_i^a - \lambda C^a_{bc} A_i^b A_j^c) \equiv i\lambda T^a F_{ij}^a. \quad (2.50)$$

The antisymmetric tensor  $F_{ij}^a$ , which takes values in the  $SU(N)$  group, is known as the field strength and transforms according to

$$(F_{ij})' = U F_{ij} U^{-1}. \quad (2.51)$$

An important difference with the electromagnetic case is the quadratic nature of the gauge tensor in the Yang-Mills potential vector fields. Furthermore, it can be checked that such gauge tensor is not invariant under the transformation (2.49) and thus it is not a physical observable, differently from the linear case of a Faraday tensor, which is gauge invariant.

A natural choice for the Yang-Mills Lagrangian density is the quadratic gauge-invariant term<sup>2</sup>  $\delta_{ab} F_{ij}^a F^{bij} = -2 \text{Tr}[F_{ij} F^{ij}]$ . The complete Lagrangian density takes the form

$$\mathcal{L}_{\psi+\text{YM}} = \frac{1}{2} \eta^{ij} \mathcal{D}_i \psi^{A\dagger} \cdot \mathcal{D}_j \psi^A - V(|\psi^A|) - \frac{1}{4} \delta_{ab} F_{ij}^a F^{bij}. \quad (2.52)$$

---

<sup>2</sup>Such expression is gauge-invariant because  $\text{Tr}[U F_{ij} F^{ij} U^{-1}] = \text{Tr}[F_{ij} F^{ij}]$ .

We see how a local symmetry of the matter dynamics implies the presence of gauge vector fields (bosons) carrying the interaction related to that specific symmetry. In fact, the Lagrangian density (2.52) contains the free evolution of the matter and boson fields, but also their reciprocal interaction, emerging from the covariant gauge derivative.

In the Hamiltonian formulation of a Yang-Mills field, the components  $A_0^a$  behave as Lagrangian multipliers, whose variation yields the Gauss constraints

$$G_a = \partial_\alpha E_a^\alpha + iC_{abc}A_\alpha^b E^{\alpha c} = 0, \quad (2.53)$$

$E_a^\alpha$  denoting the canonically conjugate momenta to the variables  $A_\alpha^a$ .

The analysis above is referred to a Minkowski space-time, but it can almost straightforwardly be extended to a curvilinear coordinate system or to a real curved space-time. In fact, the metric  $\eta_{ij}$  can be replaced by a tensor  $g_{ij}$ , while the covariant structure of the theory is restored by means of the covariant derivative. We remark that the antisymmetry of  $F_{ij}^a$  implies the cancellation of the Christoffel symbol from its expression which, therefore, retains again the form (2.50).

We conclude with a brief discussion about the structure of the vacuum in non-Abelian gauge theories. Let us introduce the dual field strength tensor  $\star F_{ij}$  defined as

$$\star F_{ij} = \frac{1}{2} \epsilon_{ij}{}^{kl} F_{kl} \quad (2.54)$$

and satisfying the Bianchi identity

$$\mathcal{D}_i \star F^{ij} = 0. \quad (2.55)$$

We can construct the topological charge  $Q$ , that is

$$Q \propto \int d^4x \operatorname{Tr}[\star F_{ij} F^{ij}] \quad (2.56)$$

a topological invariant closely related to the physical vacuum of a Yang-Mills theory.  $Q$  is invariant with respect to any local variation  $\delta A_i$  whether the equations of motion are satisfied or not. In fact,

$$\begin{aligned} \delta Q &\propto \frac{1}{2} \int d^4x \operatorname{Tr}[\star F_{ij} (\mathcal{D}^i \delta A^j - \mathcal{D}^j \delta A^i)] = \int d^4x \operatorname{Tr}[\star F_{ij} \mathcal{D}^i \delta A^j] \\ &= \int d^4x \partial^i \operatorname{Tr}[\star F_{ij} \delta A^j] - \int d^4x \operatorname{Tr}[(\mathcal{D}^i \star F_{ij}) \delta A^j] = 0. \end{aligned} \quad (2.57)$$

The first term in Eq. (2.57) is zero because there are not surface contributions while the second one vanishes because of the Bianchi identity (2.55). A

quantity which satisfies this property is called a topological invariant. The topological invariant charge  $Q$  is closely related to the physical vacuum of a Yang-Mills theory.

In gauge theories the vacuum is usually defined by the conditions  $F_{ij} = 0$ . However, it turns out that there are infinite topologically distinct vacua in the  $SU(N)$  gauge theories (assuming that all vector potentials decrease faster than  $1/|x|$  at large distances). Furthermore, distinct vacua are inequivalent because of the Gauss constraint (2.53). This result is mainly due to the (topological) equivalence of  $SU(2)$  with the three-sphere  $S^3$ . The gauge mapping  $S^3 \rightarrow SU(2)$  is characterized by an integer  $W$ , known as *winding number*. The different states  $|W\rangle$  are related to each other by a unitary transformation corresponding to the generators of the Gauss constraint. Since it is unitary, its eigenvalues are given by  $\exp(i\Theta)$ ,  $\Theta \in [0, 2\pi)$ . It is possible to show that the *physical* vacuum of non-Abelian gauge theories is given by

$$|\Theta\rangle = \sum_{W=-\infty}^{\infty} \exp(iW\Theta) |W\rangle. \quad (2.58)$$

The charge (2.56) is related to the winding number  $W$  which in turn can be obtained by a spatial integration of the 3-form  $A \wedge A \wedge A$ . Requiring that gauge potentials tend to a pure gauge at large distances, and fixing the gauge by  $A_0 = 0$ , it is possible to show that

$$Q = W|_{\infty}, \quad (2.59)$$

where  $\infty$  stands for spatial infinity. The topological charge (2.56) can thus be interpreted as the winding number of the pure gauge configuration to which  $A_\alpha$  tends.

Summarizing, it is possible to add to the Yang-Mills Lagrangian density a term  $\text{Tr}[\star F_{ij} F^{ij}]$ . This new term does not influence the classical equations of motion and does not contribute to the energy-momentum tensor. However, it modifies the quantum dynamics which depends on the action. For example, the vacuum-to-vacuum transition  $\langle\Theta| \exp(-i\mathcal{H}t)|\Theta\rangle$  in QCD is sensitive to the topological charge  $Q$ . These different physical scenarios are known as the  $\Theta$ -sectors of QCD.

## 2.3 Hamiltonian Formulation of the Dynamics

The aim of this Section is to analyze the Hamiltonian formulation of GR in the metric formalism. This program, that culminates in the formulation



given by Arnowitt, Deser and Misner in 1962, allows to identify some peculiar features of the Einstein theory. The corresponding Hamilton-Jacobi theory and the so-called ADM reduction of the dynamics will also be discussed.

### 2.3.1 Canonical General Relativity

The generally covariant system *par excellence* is the gravitational field in GR, being an invariant under arbitrary changes of the space-time coordinates (four-dimensional diffeomorphisms). The canonical formulation of GR assumes a global hyperbolic topology for  $\mathcal{M}$  (the physical space-time), allowing the splitting

$$\mathcal{M} = \mathbb{R} \times \Sigma, \quad (2.60)$$

$\Sigma$  (the three-space) being a compact three-dimensional manifold. As follows from standard theorems, all physical space-times possess such a topology. This way,  $\mathcal{M}$  can be foliated by a one-parameter family of embeddings  $X_t : \Sigma \rightarrow \mathcal{M}$ ,  $t \in \mathbb{R}$ , of  $\Sigma$  in  $\mathcal{M}$ . As a consequence, the mapping  $X : \mathbb{R} \times \Sigma \rightarrow \mathcal{M}$ , defined by  $(x, t) \rightarrow X_t$ , is a diffeomorphism of  $\mathbb{R} \times \Sigma$  to  $\mathcal{M}$ . A useful parametrization of the embedding is given by the deformation vector field

$$Y^i(X) \equiv \frac{\partial X^i(x, t)}{\partial t} = N(X)n^i(X) + N^i(X), \quad (2.61)$$

where  $N^i(X) \equiv N^\alpha \partial_\alpha X^i$ . As soon as the vector field  $Y^i$  is everywhere time-like, it can be interpreted as the “flow of time” throughout the space-time. In Eq. (2.61),  $n^i$  is the unit vector field normal to  $\Sigma_t$ , i.e. the relations

$$g_{ij}n^i n^j = 1 \quad (2.62a)$$

$$g_{ij}n^i \partial_\alpha X^j = 0, \quad (2.62b)$$

hold. The quantities  $N$  and  $N^\alpha$  are known as the *lapse function* and the *shift vector*, respectively. The space-time metric  $g_{ij}$  induces a spatial metric, i.e. a three-dimensional Riemannian metric tensor  $h_{\alpha\beta}$  on each  $\Sigma_t$ , by

$$h_{\alpha\beta} = -g_{ij} \partial_\alpha X^i \partial_\beta X^j. \quad (2.63)$$

The space-time line-element adapted to this foliation thus reads as

$$ds^2 = N^2 dt^2 - h_{\alpha\beta} (dx^\alpha + N^\alpha dt) (dx^\beta + N^\beta dt). \quad (2.64)$$

This formalism is known as the ADM procedure. The geometrical meaning of  $N$  and  $N^\alpha$  is the following: the lapse function  $N$  specifies the proper

time separation between the hypersurfaces  $X_t(\Sigma)$  and  $X_{t+dt}(\Sigma)$  measured in the direction  $n^i$  normal to the first hypersurface. On the other hand, the shift vector  $N^\alpha$  measures the displacement of the point  $X_{t+dt}(x^\alpha)$  from the intersection of the hypersurface  $X_{t+dt}(\Sigma)$  with the normal geodesic drawn from the point  $X_t(x^\alpha)$  (see Fig. 2.1). In order to have a future directed foliation of the space-time, the lapse function  $N$  must be positive everywhere in the domain of definition.

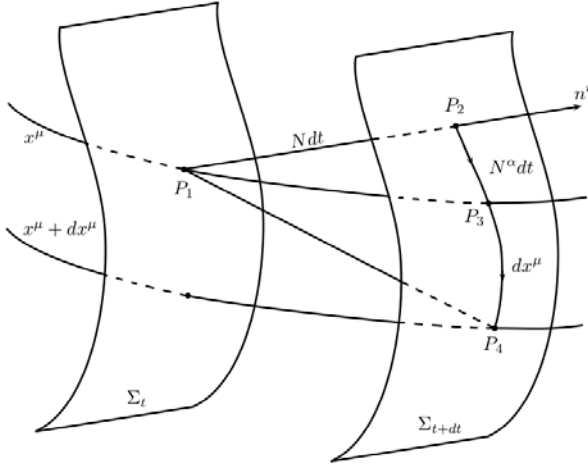


Figure 2.1 Geometric interpretation of the lapse function and of the shift vector:  $n^i$  is the unit vector field normal to  $\Sigma_t$ . From this emerges the link between the lapse function  $N$  and time diffeomorphisms, and between  $N^\alpha$  and spatial diffeomorphisms.

In the canonical analysis of GR, the Riemannian metric  $h_{\alpha\beta}$  on  $\Sigma_t$  plays the role of the fundamental configuration variable. The rate of change of  $h_{\alpha\beta}$  with respect to the time label  $t$  is related to the extrinsic curvature of the hypersurface  $\Sigma_t$  by the relation

$$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n h_{\alpha\beta}, \quad (2.65)$$

where  $\mathcal{L}_a$  denotes the Lie derivative along the vector field  $a$ . In the case of a splitting as in Eq. (2.60), Eq. (2.65) explicitly reads as

$$\begin{aligned} K_{\alpha\beta}(x, t) &= -\frac{1}{2N} \left( \partial_t h_{\alpha\beta} - (\mathcal{L}_N h)_{\alpha\beta} \right) \\ &= -\frac{1}{2N} \left( \partial_t h_{\alpha\beta} - \nabla_\alpha N_\beta - \nabla_\beta N_\alpha \right). \end{aligned} \quad (2.66)$$

Let us pull-back the Einstein Lagrangian density by the adopted foliation  $X : \mathbb{R} \times \Sigma \rightarrow \mathcal{M}$  and express the result  $X^* : \mathcal{M} \rightarrow \mathbb{R} \times \Sigma$  in terms of the extrinsic curvature  $K_{\alpha\beta}$ , the three-metric  $h_{\alpha\beta}$ ,  $N$  and  $N^\alpha$ . This procedure yields the so-called *Gauss-Codazzi relation*, which relates the four-dimensional Ricci scalar  ${}^4R$  to three-dimensional one  ${}^3R$ ; it explicitly stands as

$$\begin{aligned} X^*(\sqrt{-g}{}^4R) &= N\sqrt{h}\left((K_\gamma^\gamma)^2 - K_{\alpha\beta}K^{\alpha\beta} - {}^3R\right) \\ &\quad + 2\frac{d}{dt}\left(\sqrt{h}K_\alpha^\alpha\right) + \partial_\beta\left(K_\alpha^\alpha N^\beta - h^{\alpha\beta}\partial_\alpha N\right), \end{aligned} \quad (2.67)$$

where  $\sqrt{-g} = N\sqrt{h}$ ,  $h = \det h_{\alpha\beta}$ .

We are able to re-cast the original Hilbert action into a 3+1 form simply by dropping the total differential expressed by the last two terms on the r.h.s. of Eq. (2.67) as

$$\begin{aligned} S_g(h, N, N^\alpha) &= \int_{\mathbb{R} \times \Sigma} \mathcal{L}_{3+1} dt d^3x \\ &= -\frac{1}{2\kappa} \int_{\mathbb{R} \times \Sigma} N\sqrt{h}\left((K_\gamma^\gamma)^2 - K_{\alpha\beta}K^{\alpha\beta} - {}^3R\right) dt d^3x. \end{aligned} \quad (2.68)$$

By performing a Legendre transformation of the Lagrangian density  $\mathcal{L}_{3+1}$  appearing in Eq. (2.68), we obtain the corresponding Hamiltonian density. Let us note that the action (2.68) does not depend on the time derivatives of  $N$  and  $N^\alpha$ ; therefore, using the definition (2.66) and the fact that  ${}^3R$  does not contain time derivatives, we obtain that the conjugate momenta are given by

$$\Pi^{\alpha\beta}(x, t) \equiv \frac{\delta \mathcal{L}_{3+1}}{\delta \dot{h}_{\alpha\beta}(x, t)} = \frac{\sqrt{h}}{2\kappa} (h^{\alpha\beta} K_\gamma^\gamma - K^{\alpha\beta}), \quad (2.69a)$$

$$\Pi(x, t) \equiv \frac{\delta \mathcal{L}_{3+1}}{\delta \dot{N}(x, t)} = 0, \quad (2.69b)$$

$$\Pi_\alpha(x, t) \equiv \frac{\delta \mathcal{L}_{3+1}}{\delta \dot{N}^\alpha(x, t)} = 0. \quad (2.69c)$$

From Eqs. (2.69), it follows that not all conjugate momenta are independent, i.e. one cannot solve for all velocities as functions of coordinates and momenta: one can express  $\dot{h}_{\alpha\beta}$  in terms of  $h_{\alpha\beta}$ ,  $N$ ,  $N^\alpha$  and  $\Pi^{\alpha\beta}$ , but the same is not possible for  $\dot{N}$  and  $\dot{N}^\alpha$ . In other words, we deal with the so-called *primary constraints*

$$C(x, t) \equiv \Pi(x, t) = 0, \quad C^\alpha(x, t) \equiv \Pi^\alpha(x, t) = 0, \quad (2.70)$$

where “primary” emphasizes that the equations of motion have not been used to obtain these relations.

According to the theory of constrained Hamiltonian systems, let us introduce the new fields  $\lambda(x, t)$  and  $\lambda^\alpha(x, t)$  as the Lagrange multipliers for the primary constraints, making the Legendre transformation invertible. The corresponding action is thus given by

$$S_g = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \left[ \dot{h}_{\alpha\beta} \Pi^{\alpha\beta} + \dot{N} \Pi + \dot{N}^\alpha \Pi_\alpha - (\lambda C + \lambda^\alpha C_\alpha + N^\alpha \mathcal{H}_\alpha + N \mathcal{H}) \right]. \quad (2.71)$$

Here

$$\mathcal{H} \equiv \mathcal{G}_{\alpha\beta\gamma\delta} \Pi^{\alpha\beta} \Pi^{\gamma\delta} - \frac{\sqrt{h}}{2\kappa} {}^3R, \quad (2.72a)$$

$$\mathcal{H}_\alpha \equiv -2h_{\alpha\gamma} \nabla_\beta \Pi^{\gamma\beta}, \quad (2.72b)$$

$$\mathcal{G}_{\alpha\beta\gamma\delta} \equiv \frac{\kappa}{\sqrt{h}} (h_{\alpha\gamma} h_{\beta\delta} + h_{\beta\gamma} h_{\alpha\delta} - h_{\alpha\beta} h_{\gamma\delta}). \quad (2.72c)$$

Equation (2.72a) defines the *super-Hamiltonian*, Eq. (2.72b) the *super-momentum*, while Eq. (2.72c) defines the *super-metric*  $\mathcal{G}_{\alpha\beta\gamma\delta}$  on the space of the three-metrics. Varying action (2.71) with respect to the two conjugate momenta  $\Pi$  and  $\Pi_\alpha$  we obtain

$$\dot{N}(x, t) = \lambda(x, t), \quad \dot{N}^\alpha(x, t) = \lambda^\alpha(x, t), \quad (2.73)$$

ensuring that the trajectories of the lapse function and of the shift vector in the phase space are completely arbitrary.

The classical canonical algebra of the system can be expressed in terms of the standard Poisson brackets as

$$\{h_{\alpha\beta}(x, t), h_{\gamma\delta}(x', t)\} = 0 \quad (2.74a)$$

$$\{\Pi^{\alpha\beta}(x, t), \Pi^{\gamma\delta}(x', t)\} = 0 \quad (2.74b)$$

$$\{h_{\gamma\delta}(x, t), \Pi^{\alpha\beta}(x', t)\} = \delta_{(\gamma}^\alpha \delta_{\delta)}^\beta \delta^3(x - x'). \quad (2.74c)$$

From Eq. (2.71), we can define the Hamiltonian of the system as follows

$$\begin{aligned} \mathbf{H} &\equiv \int_{\Sigma} d^3x (\lambda C + \lambda^\alpha C_\alpha + N^\alpha \mathcal{H}_\alpha + N \mathcal{H}) \\ &\equiv \left[ C(\lambda) + \vec{C}(\vec{\lambda}) + \vec{H}(\vec{N}) + H(N) \right]. \end{aligned} \quad (2.75)$$

The variations of the action (2.71) with respect to the Lagrange multipliers  $\lambda$  and  $\lambda^\alpha$  reproduce the primary constraints (2.70). The consistency of

the dynamics is ensured preserving  $C$  and  $C_\alpha$  during the evolution of the system, i.e. by requiring

$$\dot{C}(x, t) \equiv \{C(x, t), \mathbf{H}\} = 0, \quad \dot{C}^\alpha(x, t) \equiv \{C^\alpha(x, t), \mathbf{H}\} = 0. \quad (2.76)$$

However, the Poisson brackets in Eq. (2.76) do not vanish but are equal to  $\mathcal{H}(x, t)$  and  $\mathcal{H}^\alpha(x, t)$ , respectively, and therefore the consistency of the motion leads to the *secondary constraints* by means of the equations of motion

$$\mathcal{H}(x, t) = 0, \quad \mathcal{H}^\alpha(x, t) = 0. \quad (2.77)$$

Let us observe that the Hamiltonian of the Einstein theory is constrained as  $\mathbf{H} \approx 0$ , being weakly zero, i.e. vanishing on the constraint surface (defined as the surface where the constraints hold): this is not surprising since we are dealing with a generally covariant system.

A problem that in general can arise is that the constraint surface could not be preserved under the motion generated by the constraints themselves, but this is not the case for the Einstein theory: the Poisson algebra of the super-momentum  $\mathcal{H}_\alpha$  and of the super-Hamiltonian  $\mathcal{H}$ , computed using the relations (2.74), is in fact closed. In other words, the set of constraints is a *first class* set, i.e. the Poisson brackets of the Hamiltonian  $\mathbf{H}$  with any of the constraints weakly vanish. This can be gained from the relations

$$\{\mathbf{H}, \vec{H}(\vec{f})\} = \vec{H}(\mathbb{L}_{\vec{N}}\vec{f}) - H(\mathbb{L}_{\vec{f}}N), \quad (2.78)$$

$$\{\mathbf{H}, H(f)\} = H(\mathbb{L}_{\vec{N}}f) + \vec{H}(\vec{N}(N, f, h)), \quad (2.79)$$

$f$  being a smooth test function, and  $\vec{N}^\alpha(N, f, h) = h^{\alpha\beta}(N\partial_\beta f - f\partial_\beta N)$ . These equations are equivalent to the Dirac algebra

$$\{\vec{H}(\vec{f}), \vec{H}(\vec{f}')\} = \vec{H}(\mathbb{L}_{\vec{f}}\vec{f}'), \quad (2.80a)$$

$$\{\vec{H}(\vec{f}), H(f)\} = H(\mathbb{L}_{\vec{f}}f), \quad (2.80b)$$

$$\{H(f), H(f')\} = \vec{H}(\vec{N}(f, f', h)), \quad (2.80c)$$

which underlines the canonical formulation of any field theory based on a diffeomorphism ( $\text{Diff}(\mathcal{M})$ ) invariant action, like the Einstein theory.

Three remarks on this algebra are in order.

- (i) Because of Eq. (2.80a),  $\vec{H}(\vec{f})$  generates a sub-algebra which can be identified with the Lie algebra  $\text{diff}(\Sigma)$  of the spatial diffeomorphism group  $\text{Diff}(\Sigma)$  of the Cauchy surface  $\Sigma$ . This way the super-momentum constraint  $\mathcal{H}^\alpha = 0$  is also called the spatial diffeomorphism constraint.

- (ii) Equation (2.80b) states that the super-Hamiltonian constraint (which is also called the scalar constraint)  $\mathcal{H} = 0$  is not  $\text{Diff}(\Sigma)$ -invariant. This constraint, or more precisely its Hamiltonian flow, generates a gauge motion which can be identified with the evolution generated by vector fields orthogonal to the spatial surfaces  $\Sigma_t$ .
- (iii) The relation (2.80c) implies that the Dirac algebra (2.80) is not a Lie algebra in the strict sense. Although the right-hand side of this equation is proportional to the diffeomorphism constraint, the coefficients are not constants but have a highly non-trivial phase-space dependence through the metric tensor  $h_{\alpha\beta}(x, t)$ . This feature is not a problem in the classical framework, but is one of the key difficulties in constructing a quantum theory of the gravitational field in the canonical framework (see Secs. 10.1 and 12.1).

Let us make some considerations on the formulation described above. The Hamiltonian of the theory in Eq. (2.75) is not a standard Hamiltonian but a linear combination of constraints. From Eqs. (2.78) and (2.79), it is possible to show that rather than generating time translations, the Hamiltonian generates space-time diffeomorphisms, whose parameters are the completely arbitrary functions  $N$  and  $N^\alpha$ , and the corresponding motions on the phase-space have to be regarded as gauge transformations.

An *observable* is defined as a function on the constraint surface that is gauge invariant; more precisely, in a system with first class constraints an observable can be described as a phase-space function that has weakly vanishing Poisson brackets with the constraints. In our case,  $\mathcal{O}$  is an observable if and only if

$$\{\mathcal{O}, \mathbf{H}(\lambda, \lambda^\alpha, N^\alpha, N)\} \approx 0, \quad (2.81)$$

for generic  $\lambda$ ,  $\lambda^\alpha$ ,  $N^\alpha$  and  $N$ . By this definition, one treats on the same footing the ordinary gauge invariant quantities and the constants of motion with respect to the evolution along the foliation associated to  $N$  and  $N^\alpha$ . The basic variables of the theory,  $h_{\alpha\beta}$  and  $\Pi^{\alpha\beta}$ , are not observables as they are not gauge invariant. In particular, no observables for GR are known, except for the particular situations characterized by asymptotically flat boundary conditions. Let us remark that the equations of motion

$$\dot{h}_{\alpha\beta}(x, t) = \frac{\delta \mathbf{H}}{\delta \Pi^{\alpha\beta}(x, t)}, \quad \dot{\Pi}^{\alpha\beta}(x, t) = -\frac{\delta \mathbf{H}}{\delta h_{\alpha\beta}(x, t)}, \quad (2.82)$$

together with the eight constraints (2.70) and (2.77) are completely equivalent to the Einstein equations in vacuum given by  $R_{ij} = 0$ .

Let us finally remark that the canonical framework is manifestly generally covariant since it is faithfully represented in terms of the Dirac algebra (2.80). One never uses, in the Hamiltonian construction, the notion of a background metric, and the space-time diffeomorphisms invariance is not violated at any point. Although a splitting between space and time is performed, all the splittings are simultaneously considered (this feature is reflected by the presence of constraints) and the diffeomorphisms invariance is preserved. Such invariance should not be confused with the Poincaré one. The Poincaré invariance is not a gauge symmetry of GR and refers only to a special solution (the flat solution) of the vacuum Einstein equations. The gauge group of the theory is  $\text{Diff}(\mathcal{M})$ , which is background independent since it needs a differential manifold  $\mathcal{M}$  rather than a metric one  $(\mathcal{M}, g_{ij})$ . Only when the space-time manifold is equipped with asymptotically flat boundary conditions, the Poincaré group (and its generators) can be properly defined.

### 2.3.2 Hamilton-Jacobi equations for gravitational field

The formulation of the Hamilton-Jacobi theory for a covariant system is simpler than the conventional non-relativistic version. In fact, in such case the Hamilton-Jacobi equations are expressed as

$$\mathcal{H}\left(q_a, \frac{\partial S}{\partial q_a}\right) = 0, \quad (2.83)$$

where  $\mathcal{H}$  and  $S(q_a)$  denote the Hamiltonian and the Hamilton function, respectively. For GR, the Hamilton-Jacobi equations arising from the super-Hamiltonian and super-momentum read as

$$\widehat{HJ} S \equiv \mathcal{G}_{\alpha\beta\gamma\delta} \frac{\delta S}{\delta h_{\alpha\beta}} \frac{\delta S}{\delta h_{\gamma\delta}} - \frac{\sqrt{h}}{2\kappa} {}^3R = 0 \quad (2.84a)$$

$$\widehat{HJ}_\alpha S \equiv -2h_{\alpha\gamma} \nabla_\beta \frac{\delta S}{\delta h_{\gamma\beta}} = 0. \quad (2.84b)$$

These four equations, together with the primary constraints (2.70), completely define the classical dynamics of the theory.

Let us point out how, through a change of variable, we can define an internal time-like coordinate. Writing  $h_{\alpha\beta} \equiv \tau^{4/3} u_{\alpha\beta}$ , with  $\tau \equiv h^{1/4}$  and  $\det u_{\alpha\beta} = 1$ , from the scalar Hamilton-Jacobi Eq. (2.84a), the following relation stands

$$-\frac{3}{16} \left( \frac{\delta S}{\delta \tau} \right)^2 + \frac{2}{\tau^2} u_{\alpha\gamma} u_{\beta\delta} \frac{\delta S}{\delta u_{\alpha\beta}} \frac{\delta S}{\delta u_{\gamma\delta}} - \tau^{2/3} V = 0, \quad (2.85)$$

where the potential term  $V = V(u_{\alpha\beta}, \nabla\tau, \nabla u_{\alpha\beta})$  comes out from the spatial Ricci scalar and  $\nabla$  refers to spatial gradients only. As we can see from Eq. (2.85),  $\tau$  has the correct signature for an internal time-like variable candidate; when dealing with cosmological settings, this variable turns out to be a power of the isotropic volume of the Universe (see Chap. 3).

### 2.3.3 The ADM reduction of the dynamics

The ADM reduction of the dynamics relies on the possibility of identifying a temporal parameter as a functional of the geometric canonical variables.

Let us enumerate the degrees of freedom of the gravitational field. There are 20 phase-space functions, given in the 3+1 formalism by the set  $(N, \Pi)$ ,  $(N^\alpha, \Pi_\alpha)$  and  $(h_{\alpha\beta}, \Pi^{\alpha\beta})$ , subjected to eight first-class constraints ( $\Pi = 0, \Pi_\alpha = 0, \mathcal{H} = 0, \mathcal{H}_\alpha = 0$ ). Since each first-class constraint eliminates two phase-space variables, we remain with four of them, corresponding to the two physical degrees of freedom of the gravitational field, i.e. to the two independent polarizations of a gravitational wave in the weak field limit. Apart from  $N$  and  $N^\alpha$  (and their vanishing momenta  $\Pi$  and  $\Pi_\alpha$ ), we deal with  $12 \times \infty^3$  variables  $(h_{\alpha\beta}(x, t), \Pi^{\alpha\beta}(x, t))$ . We can remove  $4 \times \infty^3$  variables by means of the secondary constraints (2.77). The remaining  $4 \times \infty^3$  non-physical degrees of freedom, in analogy with the Yang-Mills theory, must be eliminated by imposing some sort of gauge on the lapse function and on the shift vector.

This procedure can be implemented in three steps.

- (i) Perform a canonical transformation

$$(h_{\alpha\beta}, \Pi^{\alpha\beta}) \rightarrow (\chi^A, P_A; \phi^r, \pi_r), \quad (2.86)$$

where  $A = 1, 2, 3, 4$  and  $r = 1, 2$ . Here  $\chi^A$  defines a particular choice of the space and time coordinates,<sup>3</sup>  $P_A$  are the corresponding canonically conjugate momenta and the four phase-space variables  $(\phi^r, \pi_r)$  represent the physical degrees of freedom of the system. We emphasize that these “physical” fields are however not Dirac observables, in the sense defined above. Consequently, the symplectic structure of the theory is determined by

$$\{\chi^A(x, t), P_B(x', t)\} = \delta_B^A \delta^3(x - x'), \quad (2.87a)$$

$$\{\phi^r(x, t), \pi_s(x', t)\} = \delta_s^r \delta^3(x - x') \quad (2.87b)$$

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<sup>3</sup>These fields can be interpreted as defining an embedding of  $\Sigma$  in  $\mathcal{M}$  via some parametric equations.



while all other Poisson brackets do vanish.

- (ii) Express the super-momentum and super-Hamiltonian in terms of the new fields, and then write the Lagrangian density as

$$\mathcal{L}'_{3+1}(N, N^\alpha, \chi^A, P_A, \phi^r, \pi_r) = P_A \partial_t \chi^A + \pi_r \partial_t \phi^r - N \mathcal{H}' - N^\alpha \mathcal{H}'_\alpha. \quad (2.88)$$

- (iii) Remove  $4 \times \infty^3$  variables arising from the constraints  $\mathcal{H} = 0$  and  $\mathcal{H}_\alpha = 0$  by solving the equations

$$P_A(x, t) + h_A(x, t; \chi, \phi, \pi) = 0 \quad (2.89)$$

with respect to  $P_A$  and by inserting them back in Eq. (2.88). After removing the remaining  $4 \times \infty^3$  non-dynamical variables we obtain the so-called *reduced Lagrangian density*

$$\mathcal{L}_{\text{red}} = \pi_r \partial_t \phi^r - h_A \partial_t \chi^A, \quad (2.90)$$

where the lapse function and the shift vector do not play any role, but only specify the form of the functions  $\partial_t \chi^A$ . Once the constraints are solved, the evolution of  $\chi^A$  is not related anymore to the parametric time  $t$ . Thus, in Eq. (2.90) we can choose the conditions  $\chi^A(x, t) = \chi_t^A(x)$ , obtaining reduced Hamiltonian, i.e.

$$\mathbf{H}_{\text{red}} = \int \partial_t \chi^A h_A(\chi_t, \phi, \pi) d^3x. \quad (2.91)$$

From Eq. (2.91) one can derive the equations of motion as

$$\partial_t \phi^r = \{\phi^r, \mathcal{H}_{\text{red}}\}_{\phi, \pi}, \quad (2.92a)$$

$$\partial_t \pi_s = \{\pi_s, \mathcal{H}_{\text{red}}\}_{\phi, \pi} \quad (2.92b)$$

where the notation  $\{.. \}_{\phi, \pi}$  refers to the Poisson brackets evaluated in the reduced phase space with coordinates given by the physical fields  $\phi^r$  and  $\pi_s$  only.

This is an operative prescription for solving the constraints on a classical level, pulling out all the gauges, and obtaining a canonical description for the physical degrees of freedom only. It is worth noting that such procedure violates the geometrical structure of GR, since it removes part of the metric tensor. Even if this is not a problem at a classical level, it poses several questions when implemented at the quantum one (especially in the reduced phase-space quantization).

## 2.4 Synchronous Reference System

In this Section we will focus our attention on one of the most interesting reference systems, i.e. the synchronous one. This reference is defined by the following choices for the metric tensor  $g_{ij}$

$$g_{00} = 1, \quad (2.93a)$$

$$g_{0\alpha} = 0, \quad (2.93b)$$

and thus, in the canonical framework, the conditions  $N = 1$  and  $N^\alpha = 0$  in Eq. (2.64) have to be taken into account. The condition in Eq. (2.93a) is allowed by the freedom to rescale the variable  $t$  with the transformation  $\sqrt{g_{00}}dt$ , in order to reduce  $g_{00}$  to unity and setting the time coordinate  $x^0 = t$  as the proper time at each point of space. The condition (2.93b) is possible because of the non-vanishing  $h = \det(h_{\alpha\beta})$  and allows the synchronization of clocks at different points of space. The corresponding line element is then provided by the expression

$$ds^2 = dt^2 - h_{\alpha\beta}(x, t) dx^\alpha dx^\beta \quad (2.94)$$

and in such reference system the time-like curves along the  $t$ -direction result to be geodesics of the space time. Indeed the four-vector  $u^i = dx^i/ds$ , which is tangent to the  $t$ -lines, has components  $u^0 = 1$ ,  $u^\alpha = 0$  and automatically satisfies the geodesic equation

$$\frac{du^i}{ds} + \Gamma_{kl}^i u^k u^l = \Gamma_{00}^i = 0. \quad (2.95)$$

The choice of such reference is always possible and is not unique. Let us consider a generic infinitesimal displacement, i.e.

$$t' = t + \xi(x, t), \quad x^{\alpha'} = x^\alpha + \xi^\alpha(x, t). \quad (2.96)$$

It is easy to show that, if

$$\partial_t \xi = 0 \Rightarrow t' = t + \xi(x^\alpha) \quad (2.97a)$$

$$\partial_\alpha \xi = 0 \Rightarrow x'^\alpha = x^\alpha + \partial_\beta \xi \int h^{\alpha\beta} dt, \quad (2.97b)$$

then the new four-metric tensor  $g'_{ij} = g_{ij} - 2\nabla_{(i}\xi_{j)}$ , satisfies Eq. (2.94) (the round brackets around indices imply a symmetric linear combination).

In this reference frame, the Einstein equations in mixed components read as

$$R_0^0 = \frac{\partial}{\partial t} K_\gamma^\gamma - K_\sigma^\delta K_\delta^\sigma = \kappa \left( T_0^0 - \frac{1}{2} T \right) \quad (2.98a)$$

$$R_\alpha^0 = \nabla_\alpha K_\delta^\delta - \nabla_\gamma K_\alpha^\gamma = \kappa T_\alpha^0 \quad (2.98b)$$

$$R_\alpha^\beta = -{}^3R_\alpha^\beta + \frac{1}{\sqrt{h}} \frac{\partial}{\partial t} \left( \sqrt{h} K_\alpha^\beta \right) = \kappa \left( T_\alpha^\beta - \frac{1}{2} \delta_\alpha^\beta T \right). \quad (2.98c)$$

The extrinsic curvature  $K_{\alpha\beta}$  (2.66) explicitly reads as  $K_{\alpha\beta} = -\partial_t h_{\alpha\beta}/2$ , while  ${}^3R_{\beta}^{\alpha}$  is the three-dimensional Ricci tensor obtained from the metric  $h_{\alpha\beta}$  and stands, in terms of the spatial Christoffel symbols  $\bar{\Gamma}_{\alpha\beta}^{\gamma}$ , as

$${}^3R_{\alpha\beta} = \partial_{\gamma} \bar{\Gamma}_{\alpha\beta}^{\gamma} - \partial_{\alpha} \bar{\Gamma}_{\beta\delta}^{\delta} + \bar{\Gamma}_{\alpha\beta}^{\sigma} \bar{\Gamma}_{\sigma\lambda}^{\lambda} - \bar{\Gamma}_{\alpha\iota}^v \bar{\Gamma}_{\beta v}^{\iota} \quad (2.99)$$

$$\bar{\Gamma}_{\alpha\beta}^{\gamma} = \frac{1}{2} h^{\gamma\delta} (\partial_{\alpha} h_{\delta\beta} + \partial_{\beta} h_{\alpha\delta} - \partial_{\delta} h_{\alpha\beta}). \quad (2.100)$$

In this analysis the spatial metric is used to raise and lower indices within the spatial sections. From Eq. (2.98a) it is straightforward to derive, even in the isotropic case, the Landau-Raychaudhuri theorem, stating that the metric determinant  $h$  must monotonically vanish in a finite instant of time. However, we want to stress that the singularity in this reference system is not physical and can be removed by a coordinate transformation.

## 2.5 Tetradic Formalism

The tetradic formalism consists in replacing the metric tensor  $g_{ij}$  with four linearly independent covariant vector fields  $e_i^I = e_i^I(x^k)$ . Let  $(\mathcal{M}, g_{ij})$  be the space-time four-dimensional manifold and  $\hat{e}$  a one-to-one correspondence on it, i.e.  $\hat{e} : \mathcal{M} \rightarrow T\mathcal{M}$ . More precisely,  $\hat{e}$  maps tensor fields on  $\mathcal{M}$  to tensor fields on the Minkowski tangent space  $T\mathcal{M}$ . The four linearly independent fields  $e_i^I$  (tetrads or vierbein<sup>4</sup>) are an orthonormal basis for the local Minkowski space-time and satisfy the only condition

$$e_I^i e_{ij} = \eta_{IJ}, \quad (2.101)$$

$\eta_{IJ}$  being a symmetric, constant matrix with signature  $(+, -, -, -)$ . Let us introduce the reciprocal (dual) vectors  $e^{iI}$ , such that  $e_i^I e_J^i = \delta_J^I$ . By definition of  $e_i^I$  and by Eq. (2.101), the condition  $e_i^I e_I^j = \delta_i^j$  is also verified. This way, Lorentzian indices are lowered and raised by the matrix  $\eta_{IJ}$ , and the vector fields  $e_i^I$  are related to the metric tensor  $g_{ij}$  by the relation

$$g_{ij}(x^k) = \eta_{IJ} e_i^I(x^k) e_j^J(x^k). \quad (2.102)$$

From this perspective, the gravitational field is a 1-form  $e^I = e_i^I dx^i$  with values in the Minkowski space-time. From a physical point of view, the tetrad  $e_i^I$  describes the departure of a space-time manifold from being flat.

The projections of a vector field  $A^i$  along the four  $e_i^I$  are denoted as “vierbein components” and read as  $A_I = e_i^I A_i$  and  $A^I = e_i^I A^i = \eta^{IJ} A_J$ . In

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<sup>4</sup>Capital latin letters denote Lorentzian indices.

particular, for the partial differential operator we have  $\partial_I = e_I^i \partial_i$ . The generalization to a tensor of any number of covariant or contravariant indices is straightforward.

Using the tetradic fields  $e_i^I$  we can rewrite the Lagrangian of the Einstein theory in a more elegant and compact form. Firstly, we notice that the tetradic fields  $e_i^I$  define a connection  $\omega^{IJ} = -\omega^{JI}$ . This is a 1-form, known as spin connection, with values in the Lie algebra of the Lorentz group  $SO(3, 1)$  and uniquely determined by the II Cartan structure equation. In the torsion-free case, this equation reads as

$$\mathcal{T}_{ij}^I = \partial_{[i} e_{j]}^I + \omega_{[iJ}^I e_{j]}^J = 0, \quad \Rightarrow \quad \omega = \omega(e), \quad (2.103)$$

where the (Lorentz algebra valued) 2-form  $\mathcal{T}^I = \mathcal{T}_{ij}^I dx^i dx^j$  denotes the torsion field. Equation (2.103) admits the following solution

$$\omega_i^{IJ} = e^{IJ} \nabla_i e_j^J. \quad (2.104)$$

Let us introduce the curvature of the spin connection  $R^{IJ}(\omega)$ , a Lorentz valued 2-form defined by the I Cartan structure equation, i.e.

$$R^{IJ}_{ij}(\omega) = \partial_{[i} \omega_{j]}^{IJ} + \omega_{[iK}^I \omega_{j]}^{JK}, \quad (2.105)$$

where the anti-symmetrization regards only the spatial indices and  $R^{IJ}(\omega)$  is related to the Riemann curvature tensor by

$$R^i_{jkl}(g) = e_I^i e_{Jj} R^{IJ}_{kl}(\omega). \quad (2.106)$$

The action (2.11) of GR, in the absence of matter fields, can be recast in the form

$$S_g(e) = -\frac{1}{2\kappa} \int_{\mathcal{M}} e e_I^i e_J^j R^{IJ}_{ij}(e) d^4x, \quad (2.107)$$

where  $e = \sqrt{-g}$  denotes the determinant of  $e_i^I$ . When a theory depends only on the metric  $g_{ij}$  or on tetrads  $e_i^I$ , as in this case, we deal with the so-called *second-order formalism*.

As is well known, GR also admits a first-order formulation (*à la* Palatini) in which the tetrads  $e_i^I$  and the spin connection  $\omega_i^{IJ}$  are considered as independent variables. The Einstein-Hilbert action then reads as

$$S_P(e, \omega) = -\frac{1}{2\kappa} \int_{\mathcal{M}} e e_I^i e_J^j R^{IJ}_{ij}(\omega) d^4x. \quad (2.108)$$

The variation of Eq. (2.108) with respect to the connection  $\omega_i^{IJ}$  gives the II Cartan structure equation (2.103), and thus the second-order formalism is recovered. It is worth noting that in the presence of matter, however, the two formalisms are not equivalent if the Lagrangian of the matter fields

contains connections (for instance, fermion fields). Notably, the action (2.108) is invariant under space-time diffeomorphisms of  $\mathcal{M}$  as well as local  $SO(3, 1)$  (Lorentz) transformations.

Variation of the action (2.108) with respect to the gravitational field  $e^I$  leads to the Einstein equations in vacuum

$$R_i^I - \frac{1}{2}R e_i^I = 0, \quad (2.109)$$

where the Ricci tensor  $R_i^I$  is defined as  $R_i^I = R^{IJ}_{ij} e_j^I$ , while  $R = R_i^I e_i^I$  denotes the Ricci scalar. In the presence of matter, the Einstein equations in the tetradic formalism read as

$$R_i^I - \frac{1}{2}R e_i^I = \kappa T_i^I, \quad (2.110)$$

$T_i^I = T_{ij} e^{Ij}$  being the tetradic projection of the energy-momentum tensor (2.13). From a geometric point of view, we are dealing with a Lorentz vector bundle over the space-time manifold where the spin connection  $\omega_i^{IJ}$  is the connection on the bundle.

The analysis of the tetradic formalism is completed by introducing the Ricci coefficients  $\gamma_{IJK} = -\gamma_{JIK}$  and their linear combinations  $\lambda_{IJK} = -\lambda_{IKJ}$ , i.e.

$$\gamma_{IJK} = \nabla_k e_{Ii} e_j^i e_K^k, \quad (2.111a)$$

$$\lambda_{IJK} = \gamma_{IJK} - \gamma_{IKJ}. \quad (2.111b)$$

The Riemann and the Ricci tensors can be expressed in terms of  $\gamma_{IJK}$  and of  $\lambda_{IJK}$  as

$$R_{IJKL} = \partial_L \gamma_{IJK} - \partial_K \gamma_{IJL} + \gamma_{IJM} (\gamma^M_{KL} - \gamma^M_{LK}) + \gamma_{IMK} \gamma^M_{JL} - \gamma_{IML} \gamma^M_{JK}, \quad (2.112a)$$

$$R_{IJ} = -\frac{1}{2} \left( \partial_K \lambda_{IJ}^K + \partial_K \lambda_{JI}^K + \partial_J \lambda_{KI}^K + \partial_I \lambda_{KJ}^K + \lambda^{KL}_{JL} \lambda_{KLI} + \lambda^{KL}_{JL} \lambda_{KLI} + \right. \\ \left. - \frac{1}{2} \lambda_J^{KL} \lambda_{IKL} + \lambda^K_{KL} \lambda_{IJ}^L + \lambda^K_{KL} \lambda_{JI}^L \right). \quad (2.112b)$$

Finally, the relation between the spin connection  $\omega_i^{IJ}$  and the Ricci coefficients is given by

$$\omega_i^{IJ} = \gamma^{IJ}_K e_i^K. \quad (2.113)$$

## 2.6 Gauge-like Formulation of GR

This Section is devoted to the analysis of the more recent formulation of the Einstein theory, due to Ashtekar (and generalized by Barbero and Immirzi). Such formulation reveals a structural identity between GR and Yang-Mills theories and finds the most important application to quantum gravity, since it opens the possibility of using the Wilson loops technique to quantize the gauge fields also in the case of gravity. As a matter of fact, this new formalism leads to the Loop Quantum Gravity theory, which can be considered as the most advanced implementation of the canonical approach to quantum gravity and will be discussed in Sec. 12.1.

Both the Lagrangian and the Hamiltonian formulations are restated in details through this Section also paying attention to recent debates on this approach.

### 2.6.1 Lagrangian formulation

As we have seen in Sec. 2.2.4, in Yang-Mills theories it is possible to add a *topological term* to the action which does not change the classical equations of motion because its integrand can be expressed as a total derivative of a 3-form and this property holds also in the gravitational case. Let us consider the integral

$$\begin{aligned} S_{\text{TT}}(e, \omega) &= \frac{1}{2\kappa\gamma} \int_{\mathcal{M}} d^4x \, e \, e_I^i \, e_J^j \, \star R^{IJ}{}_{ij}(\omega) \\ &= \frac{1}{4\kappa\gamma} \int_{\mathcal{M}} d^4x \, e \, e_I^i \, e_J^j \, \epsilon^{IJ}{}_{KL} R^{KL}{}_{ij}(\omega), \end{aligned} \quad (2.114)$$

where  $\epsilon_{IJKL}$  is the Levi-Civita tensor on the tangent space and  $\star$  is the Hodge dual operator defined in Eq. (2.54).  $S_{\text{TT}}$  is a topological term, but in a sense weaker than the Yang-Mills case, as it identically vanishes on the histories (trajectories) where the II Cartan structure equation (2.103) holds. In fact, its integrand is equal to zero because of the Bianchi cyclic identity (2.7)  $R_{i[jkl]} = 0$  and then

$$e \, e_I^i \, e_J^j \, \epsilon^{IJ}{}_{KL} R^{KL}{}_{ij}(\omega) = e_k^K \, e_l^L \, \epsilon^{ijkl} R_{ijKL}(\omega) = \epsilon^{ijkl} R_{ijkl}(e). \quad (2.115)$$

In the last equality of Eq. (2.115) the spin connection (2.104) is taken into account, and therefore this is only true when the II Cartan structure equation (2.103) holds, and therefore the second-order formalism is restored. This way,  $S_{\text{TT}}$  (2.114) can be added to the Palatini action (2.108) without

affecting the equations of motion, obtaining the Holst action

$$S_H(e, \omega) = S_P + S_{TT} = -\frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \, e \, e_I^i e_J^j \left( R^{IJ}{}_{ij} - \frac{1}{\gamma} \star R^{IJ}{}_{ij} \right). \quad (2.116)$$

The coupling constant  $\gamma \neq 0$  is called the Immirzi parameter and does not affect the classical theory. Indeed, the Yang-Mills and the Holst gravitational theories present, in some respects, the same features. In both cases it is possible to add a term that does not change the equations of motion but induces a canonical transformation in the classical phase-space that cannot be unitarily implemented at a quantum level (see Chap. 12). The quantum theory built on the Holst action (2.116), i.e. the Loop Quantum Gravity theory, has inequivalent  $\gamma$ -sectors resembling the inequivalent  $\Theta$ -sectors in the Yang-Mills one (see Sec. 2.2.4). The Immirzi parameter can be considered, in this respect, as the analogous of the  $\Theta$ -angle in QCD.

Before analyzing the canonical formulation of this theory, let us discuss the relevant cases  $\gamma = \pm i$  leading to the original formulation of GR proposed by Ashtekar in 1986. A generic tensor  $T^{IJ}$  is called *self-dual* (respectively *anti-self-dual*) if it satisfies

$$T^{IJ} = \mp i \star T^{IJ} = \mp \frac{i}{2} \epsilon^{IJ}{}_{KL} T^{KL}. \quad (2.117)$$

When the Immirzi parameter is fixed to  $\gamma = -i$ , we are naturally led to consider as basic connections the complex quantities

$$A_i^{IJ}(\omega) = \omega_i^{IJ} - i \star \omega_i^{IJ}, \quad (2.118)$$

instead of the spin connections  $\omega_i^{IJ}$ . The new variables  $A_i^{IJ}(\omega)$ , called *self-dual spin connection*, are the Ashtekar connections. If  $F^{IJ}{}_{ij}(A)$  is the curvature 2-form (2.50) of the self-dual spin connection  $A_i^{IJ}$ , i.e.

$$F^{IJ}{}_{ij}(A) = \partial_{[i} A_{j]}^{IJ} + A_{[iK}^I A_{j]}^{JK}, \quad (2.119)$$

it is not difficult to show that  $F^{IJ}{}_{ij}(A)$  is related to the curvature  $R^{IJ}{}_{ij}(\omega)$  of the spin connection  $\omega_i^{IJ}$  by

$$F^{IJ}{}_{ij}(A) = R^{IJ}{}_{ij} - i \star R^{IJ}{}_{ij}. \quad (2.120)$$

The curvature of the Ashtekar connections, i.e. the Yang-Mills field strength of  $A_i^{IJ}$ , is just the self-dual part of the spin-connection curvature. This quantity is exactly the term in brackets in Eq. (2.116) in the  $\gamma = -i$  case.

Considering the change of coordinates from  $(e_i^I, \omega_i^{IJ})$  to  $(e_i^I, A_i^{IJ})$ , the action in Eq. (2.116) rewrites as

$$S_H(e, A) = -\frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \, e \, e_I^i e_J^j F^{IJ}{}_{ij}(A). \quad (2.121)$$

We can consider  $S_H(e, A)$  as the starting point for the Ashtekar gravitational theory. The equations of motion which follow are given by

$$\epsilon^{ijkl} e_{jJ} F^{IJ}_{kl} = 0, \quad (2.122a)$$

$$\left( \delta^{IK} \delta^{JL} + \frac{i}{2} \epsilon^{KLIJ} \right) \epsilon^{ijkl} \mathcal{D}_k (e_{iI} e_{jJ}) = 0, \quad (2.122b)$$

where  $\mathcal{D}_i$  denotes the covariant derivative (2.48) defined by the connection in Eq. (2.118). Let us stress once again that this formalism is completely equivalent to the Einstein formulation of GR. Indeed, if the couple  $(e_i^I(x^k), A_i^{IJ}(x^k))$  satisfies the equations of motion (2.122), then the metric tensor (2.102) is a solution of the vacuum Einstein equations. The inverse statement is also true.

By means of Eqs. (2.122), the Ashtekar connection  $A_i^{IJ}(x^k)$  results in the self-dual part of the spin connection defined in Eq. (2.104), i.e.

$$A_i^{IJ}(e) = e^{Ij} \nabla_i e_j^J - \frac{i}{2} \epsilon^{IJ}_{KL} e^{Kj} \nabla_i e_j^L. \quad (2.123)$$

The geometric interpretation of this framework is the following: the complex Lorentz group (and also its algebra) splits into two complex  $SO(3; \mathbb{C})$  groups, the self and anti-self dual ones. More precisely, there exists an isomorphism between the direct sum of these two reduced algebras and the original complex Lorentz algebra, i.e.

$$so(1, 3; \mathbb{C}) = so(3; \mathbb{C}) \oplus so(3; \mathbb{C}). \quad (2.124)$$

The connection on the  $SO(1, 3; \mathbb{C})$  vector bundle over the space-time manifold splits into two independent components, the self dual and the anti-self-dual. These are independent since the self-dual part of the curvature is the curvature of the self-dual connection, i.e. Eq. (2.120) holds. It is worth stressing that the difference between the self-dual curvature and the real one is nothing but the topological term  $S_{TT}$  (2.114). Since the complexification of the Lorentz algebra decomposes as in relation (2.124), not all the components of the Ashtekar connection (2.118) are independent. In order to deal with real rather than complex GR, one has to impose the reality condition

$$A_i^{IJ} + (A_i^\dagger)^{IJ} = 2\omega_i^{IJ}. \quad (2.125)$$

Although the original Ashtekar connection has a clear geometric interpretation (see below in Sec. 2.6.3), it takes values in the Lie algebra of a non-compact group (namely the Lorentz group).



The most relevant result in this field has been obtained adopting real, rather than complex, connections, the so-called Barbero-Immirzi connections

$$A_i^{IJ}(\omega) = \omega_i^{IJ} - \frac{1}{\gamma} \star \omega_i^{IJ} \quad (2.126)$$

defined for real values of  $\gamma$ . As we said, the  $\gamma$  parameter induces a canonical transformation of the form (2.126) which, in general, is a vector space isomorphism on the Lorentz group. In the particular cases of  $\gamma = \pm i$ , this map is a Lie algebra homomorphism.

### 2.6.2 Hamiltonian formulation

As in the metric case analyzed before, the starting point of the Hamiltonian analysis of the Holst theory is the 3+1 splitting of the space-time manifold  $\mathcal{M}$ . As usual, we assume that the space-time is globally hyperbolic and that, from the Geroch theorem, can be foliated as  $\mathcal{M} = \mathbb{R} \times \Sigma$ . Here it is convenient to carry out a partial gauge fixing. Let the internal vector field  $n^I$  orthogonal to the spatial Cauchy surfaces  $\Sigma$  be defined by  $n^I n_I = 1$  and  $n^I e_{\alpha I} = 0$ . The time component of the tetrad field  $e_i^I$  can be written as

$$e_0^I = N n^I + N^\alpha e_\alpha^I. \quad (2.127)$$

The gauge which is normally adopted is the *time-gauge*, i.e. the tetrad is chosen such that

$$n^I = (1, 0, 0, 0). \quad (2.128)$$

This choice implies that the spatial components of the tetrad  $e_i^a$  (denoted with small latin letters) span the space tangent to the Cauchy surfaces and that  $e_\alpha^0 = 0$ . The splitting reduces the tetrad fields  $e_i^I$  and their inverse  $e_J^i$  to the following form

$$e_i^I = \begin{pmatrix} N & N^\alpha e_\alpha^a \\ 0 & e_\alpha^a \end{pmatrix}, \quad e_J^i = \begin{pmatrix} N^{-1} & 0 \\ -N^{-1} N^\beta & e_\beta^b \end{pmatrix}. \quad (2.129)$$

Because of the adopted time-gauge, the boost sector of the (local) Lorentz group is frozen out and the Lorentz invariance reduces to a local  $SO(3) \simeq SU(2)$  invariance. The resulting implications will be discussed below.

Because of the splitting, the spin connection  $\omega_i^{IJ}$  defines two  $so(3)$ -valued 1-forms on the spatial surfaces (whose metric  $h_{\alpha\beta}$  in Eq. (2.64) is given by  $h_{\alpha\beta} = \delta_{ab} e_\alpha^a e_\beta^b$ ) which read as

$$\Gamma_\alpha^a = e^{ai} e_{\alpha I} (\star \omega_i^{IJ}) n_J, \quad (2.130a)$$

$$K_\alpha^a = e^{ai} e_{\alpha I} \omega_i^{IJ} n_J. \quad (2.130b)$$

These quantities have a natural geometric interpretation.  $\Gamma_\alpha^a$  denotes a  $so(3)$ -connection on  $\Sigma$  and furthermore, if the spin connection is a solution of Eq. (2.103),  $\Gamma_\alpha^a$  satisfies the II Cartan structure equation induced on the spatial surface: in this case,  $\Gamma_\alpha^a$  is said to be *compatible* with  $e_\alpha^a$ . On the other hand,  $K_\alpha^a = e^{a\beta} K_{\alpha\beta}$  stands for the extrinsic curvature 1-form. It is the Lie derivative of  $h_{\alpha\beta}$  with respect to the normal vector to the spatial slice and thus can be written as

$$K_\alpha^a = (L_{\vec{n}} h_{\alpha\beta}) \delta^{ab} e_b^\beta. \quad (2.131)$$

The next step is to consider the linear combination of the 1-forms in Eqs. (2.130) as

$$A_\alpha^a = \Gamma_\alpha^a + \gamma K_\alpha^a, \quad (2.132)$$

which is again a connection on  $\Sigma$  taking values in the Lie algebra  $so(3)$  (namely,  $su(2)$ ).

Let us now take into account the densitized triads  $E_\alpha^a$  related to the three-metric  $h_{\alpha\beta}$  by

$$E_\alpha^a = \frac{1}{2} \epsilon^{\alpha\beta\gamma} \epsilon_{abc} e_\beta^b e_\gamma^c = e e_\alpha^a = \sqrt{|h|} e_\alpha^a. \quad (2.133)$$

This is a vector density of weight 1 on  $\Sigma$  which takes values in the dual of  $so(3)$ . The densitized triads (2.133) carry information about the spatial geometry (encoded in the three-metric), while the connections (2.132) describe the spatial curvature through the spin connections and the extrinsic curvature. The most peculiar feature of this framework is that the two quantities in Eqs. (2.132) and (2.133) span the phase space of GR. In fact, they are canonically conjugate fields whose Poisson brackets read as

$$\{A_\alpha^a(x, t), E_b^\beta(x', t)\} = \kappa \delta_b^a \delta_\alpha^\beta \delta^3(x - x'). \quad (2.134)$$

The phase-space exactly resembles the one of a Yang-Mills theory, with  $SU(2)$  as gauge group. Following the conventions of gauge theory, we can call  $E_\alpha^a$  as the gravitational electric field since it is the momentum canonically conjugate to the connection  $A_\alpha^a$ , which is the configuration field of the theory.

Given all this apparatus, it is now possible to rewrite the Holst action (2.116) in the appropriate canonical form, although we will skip the explicit computation which is straightforward but rather tedious (the interested reader is referred to the literature). The result is given by the 3 + 1 action

$$S_H = \frac{1}{\kappa} \int dt \int_\Sigma d^3x \left[ \dot{A}_\alpha^a E_\alpha^a - (A_0^a G_a + N^\alpha \mathcal{H}_\alpha + N \mathcal{H}) \right], \quad (2.135)$$

where  $A_0^a$ ,  $N^\alpha$  and  $N$  are the Lagrange multipliers. Let us note that  $A_0^a$  results to behave as a multiplier according to gauge theories (see Sec. 2.2.4). The term in brackets denotes the Hamiltonian (density) of GR which, as in the metric case, is a linear combination of constraints. The diffeomorphisms ( $\mathcal{H}_\alpha = 0$ ) and scalar ( $\mathcal{H} = 0$ ) constraints rewrite, in the connection formalism, as

$$\mathcal{H}_\alpha = E_a^\beta F_{\alpha\beta}^a - (1 + \gamma^2) K_\alpha^a G_a, \quad (2.136a)$$

$$\begin{aligned} \mathcal{H} = \frac{1}{2\sqrt{|h|}} E_a^\alpha E_b^\beta \left( \epsilon^{ab}{}_c F_{\alpha\beta}^c - 2(1 + \gamma^2) K_{[\alpha}^a K_{\beta]}^b \right) \\ + (1 + \gamma^2) \partial_\alpha \left( \frac{E^{\alpha a}}{\sqrt{|h|}} \right) G_a, \end{aligned} \quad (2.136b)$$

respectively, and

$$F_{\alpha\beta}^a = 2\partial_{[\alpha} A_{\beta]}^a + \gamma \epsilon^a{}_{bc} A_\alpha^b A_\beta^c \quad (2.137)$$

denotes the components of the curvature 2-form associated to the connection  $A_\alpha^a$ . In this formalism, a new constraint  $G_a = 0$  arises with respect to the metric approach and explicitly reads as

$$G_a = \mathcal{D}_\alpha E_a^\alpha = \partial_\alpha E_a^\alpha - \gamma \epsilon_{ab}{}^c A_\alpha^b E_c^\alpha = 0, \quad (2.138)$$

and it is the analogous of the Gauss constraint (2.53) of Yang-Mills theories which gets rid of the  $SU(2)$  degrees of freedom. If we ignore the two constraints (2.136) we deal with a  $SU(2)$  gauge theory. However, differently from the Yang-Mills case, the Hamiltonian of GR is a linear combination of constraints. In such new reformulation, the Einstein theory can be regarded, as a *background independent*  $SU(2)$  gauge theory.

In order to illustrate the meaning of the two constraints (2.136) and their consequences, the terms proportional to  $G_a$  generating internal rotations, will be removed. As a matter of fact, since  $G_a$  generates a sub-algebra of the constraint algebra, the system described by Eqs. (2.136) without the  $G_a$  term, defines the same constraint surface in the phase space. Thus, it is completely equivalent to work with the set of constraints

$$\mathcal{H}_\alpha = E_a^\beta F_{\alpha\beta}^a = 0, \quad (2.139a)$$

$$\mathcal{H} = \frac{1}{2\sqrt{|h|}} E_a^\alpha E_b^\beta \left( \epsilon^{ab}{}_c F_{\alpha\beta}^c - 2(1 + \gamma^2) K_{[\alpha}^a K_{\beta]}^b \right) = 0, \quad (2.139b)$$

together with Eq. (2.138). As in the metric case, Eq. (2.139a) generates spatial diffeomorphisms along the vector field  $N^\alpha$  on the Cauchy surface  $\Sigma$ , while Eq. (2.139b) generates the time evolution off  $\Sigma$ . These constraints

exactly reflect the gauge freedom of the physical theory, in particular the internal automorphism of the  $SU(2)$  gauge bundle and the diffeomorphism invariance of the space-time, so constraining the system on a restricted region of the phase space. Furthermore, a direct calculation of the Poisson algebra between the constraints shows that Eqs. (2.139) and (2.138) are of first class. Because we are dealing with a canonical transformation, such constraint algebra coincides with the Dirac one (2.80) on the sub-manifold  $G_a = 0$  of the phase space.

Two remarks are in order.

- (i) The Hamiltonian (scalar) constraint Eq. (2.139b) has an important peculiarity: considering the  $\gamma = \pm i$  case, it reads as

$$\mathcal{H} = \epsilon^{ab} {}_c E_a^\alpha E_b^\beta F_{\alpha\beta}^c = 0, \quad (2.140)$$

after a rescaling of the factor  $1/2\sqrt{|\hbar|}$ . In other words, it becomes polynomial and a huge simplification occurs as soon as the original Ashtekar variables (defined on the slicing surface  $\Sigma$ )

$$A_\alpha^a = \Gamma_\alpha^a \pm i K_\alpha^a, \quad E_b^\beta = e e_b^\beta, \quad (2.141)$$

are taken into account. Most of the initial excitement over the Ashtekar discovery was exactly due to such feature. The price one has to pay using these variables is that they are complex valued. When the  $\gamma$  parameter is real,  $A_\alpha^a$  and  $E_b^\beta$  are both real valued and can be directly interpreted as the canonical pair for the phase space of a  $SU(2)$  gauge theory. On the other hand, when  $\gamma$  is complex, some reality conditions have to be imposed (see Eq. (2.125)) and read as

$$A_\alpha^a + (A^\dagger)_\alpha^a = 2\Gamma_\alpha^a, \quad E_b^\beta + (E^\dagger)_b^\beta = 0 \quad (2.142)$$

and guarantee that there is no doubling of the number of degrees of freedom. This way, only  $SU(2)$  gauge transformations are allowed but not general<sup>5</sup>  $SL(2, \mathbb{C})$  transformations. However, the reality conditions (2.142) are non-polynomial and thus difficult to implement in the quantum theory. Of course, these two reality conditions are trivially satisfied when  $\gamma$  is real, i.e. when we deal with the real Barbero-Immirzi connection.

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<sup>5</sup>The complexification of the Lorentz group can be identified with its universal cover  $SL(2, \mathbb{C})$ .

- (ii) All complex values of the Immirzi parameter  $\gamma$  lead to Hamiltonian formulations completely equivalent to the ADM formulation. In fact, the framework described above can also be obtained from the metric one by the use of a canonical transformation. Such construction consists of two steps: an extension of the ADM phase-space passing through the tetradic formalism and a canonical transformation on such extended phase space. The parameter  $\gamma$  enters in this second step as a rescaling of the conjugate variables  $K_\alpha^a$  and  $E_b^\beta$ . This way, as far as  $SU(2)$  invariant observables are concerned (i.e. considering the symplectic reduction with respect to the Gauss constraint (2.138)), both the ADM and the gauge formulations are completely equivalent to each other.

### 2.6.3 On the gauge group of GR

Let us now focus our attention on the (internal) gauge group of GR. Because of the nature of the Lorentz and Poincaré groups, it is generally argued that the (local) gauge group of GR must be a non-compact one. The puzzle is that, as we have seen, the Barbero-Immirzi Hamiltonian formulation uses a real and compact gauge group, i.e. the  $SU(2)$  one. In order to investigate this issue, the geometrical meaning of the connection defined in Eq. (2.132) has to be analyzed.

We want to discuss the conceptual differences between the real and the complex valued connections. Both are  $su(2)$ -valued connections and the relation with the metric variables has the same form in both cases. Nonetheless, an important difference occurs: only the Ashtekar connection (2.141) is the (anti)self-dual piece of the pull-back to  $\Sigma$  of the four-dimensional spin connection  $\omega_i^{IJ}$  and then has a covariant interpretation. In all other cases ( $\gamma$  real), this is not true. The manifestly covariant origin of the phase-space spanned by the Barbero-Immirzi connection is lost due to the (partial) gauge fixing of the Lorentz group provided by the time gauge. Unless  $\gamma = \pm i$ , the Holst action (2.116) leads to constraints of second class, i.e. constraints which are not generators of gauge transformations. These constraints are solved by imposing the time-gauge which eliminates the boost component of the Lorentz group leading to the  $SO(3)$  (or  $SU(2)$ ) sub-group. However, this reduction does not pose any difficulty. This criticism is only of aesthetic nature since we are not interested in non-gauge-invariant objects; there will be no lack of it at quantum level. A space-time geometry is the analogous of a trajectory in particle mechanics and trajectories do

not play any essential role in quantum mechanics.

## 2.7 Singularity Theorems

In this Section we investigate the space-time singularities through the celebrated theorems given by Hawking and Penrose at the end of the '60s. We will show that singularities are true, generic features of the Einstein theory of gravity and how they arise under certain, quite general, assumptions. These theorems are of fundamental importance since they state that GR has a limited range of validity out of which quantum gravity effects could be required. In this respect, the initial cosmological singularity at the beginning of our Universe is expected to be tamed by quantum properties, similarly to the instability problem of a classical hydrogen atom which is solved by the existence of a finite energy ground-state of the electron. The prediction of space-time singularities in GR implies the necessity to work out a quantum theory of gravity able to solve such unphysical predictions. As we will see in Sec. 12.2, Loop Quantum Cosmology faces exactly this problem replacing the Big Bang of the Universe by a non-singular Big Bounce.

After the definition of a space-time singularity, we will present some basic techniques and finally we will discuss the singularity theorems; we enter the main aspects only without giving rigorous proofs, for which we refer the reader to the original works. In this Section, we adopt the signature  $(-, +, +, +)$  for coherence with the standard literature on this subject.

### 2.7.1 Definition of a space-time singularity

Let us clarify the meaning of *singularity* of a space-time. In analogy with field theory, we can represent such a singularity as the “place” of the space-time where the curvature diverges, or where some similar pathological behavior of the geometric invariants takes place. The characterization as place, however, poses several problems: since in GR the space-time consists of a manifold  $\mathcal{M}$  and a metric  $g_{ij}$  defined everywhere on  $\mathcal{M}$ , a singularity (as the Big Bang singularity of the isotropic cosmological solution or the  $r = 0$  singularity in the Schwarzschild space-time) cannot be considered as a part of the manifold itself. We can speak of a physical event only when a manifold and a metric structure are defined around it. *A priori*, it is possible to add points to the manifold in order to describe the singularity as a

real place (as the boundary of the manifold), but apart from very peculiar cases, no general notion or definition of a singular boundary exists. Another problem is that singularities in gravity are not always accompanied by unbounded curvature as in the best known cases. Several examples of singularities without diverging curvature can be given. In fact, as we will see, this feature is not the basic mechanism behind singularity theorems.

The best way to clarify what a singularity means is the geodesic incompleteness, i.e. the existence of geodesics which are inextensible at least in one direction and thus have only a finite range for the affine parameter. We can then define a singular space-time as the one possessing at least one incomplete (time-like or null) geodesic curve.

### 2.7.2 Fluid kinematics

Now we will provide the reader with some basic notions of fluid kinematics.

We initially define the notion of *congruence*.

**Definition 2.1 (Congruence).** *Let  $O$  be an open set of a space-time manifold  $\mathcal{M}$ . A congruence in  $O$  is defined as a family of curves such that only one curve of this family passes through each point  $p \in O$ .*

There exist different kinds of congruences, resembling the properties of the tangent vector field  $\xi^i$  associated to the family of curves. In particular, time-like, null, or space-like congruences are generated by nowhere vanishing time-like, null, or space-like vector fields  $\xi^i$ , respectively; geodesic congruences are generated by vector fields which have vanishing covariant derivative  $\xi^j \nabla_j \xi^i = 0$ . Now we will be interested in time-like congruences, while in the next subsection we will be dealing with geodesic congruences. The treatment of the remaining congruences is conceptually similar and not discussed here.

Given a unit time-like vector  $\xi^i$  (i.e.  $\xi^j \xi_j = -1$ ), we can construct a tensor  $\tilde{h}_{ij}$  that projects the other tensors onto their orthogonal components as

$$\tilde{h}_{ij} = g_{ij} + \xi_i \xi_j. \quad (2.143)$$

Let us analyze the covariant derivative of  $\xi^i$ . From the relation

$$(\xi^j \nabla_j \xi^i \xi_l + \nabla_l \xi^i) \xi^l = 0 \quad (2.144)$$

it follows that the term in parentheses is orthogonal to  $\xi^i$ , i.e.

$$\xi^k \nabla_k \xi_i \xi_j + \nabla_j \xi_i = \tilde{h}^k_i \tilde{h}^l_j \nabla_k \xi_l = \theta_{ij} + \omega_{ij}. \quad (2.145)$$

In the last equality, we have introduced the so-called *expansion tensor*  $\theta_{ij}$ , corresponding to the symmetric part, and the *vorticity tensor*  $\omega_{ij}$ , corresponding to the antisymmetric one. We can also define the *acceleration vector*  $\dot{\xi}_i \equiv \xi^j \nabla_j \xi_i$ . If we further decompose  $\theta_{ij}$  in a term proportional to its trace  $\theta$  (the *expansion scalar*) plus a trace-less part (the *shear tensor*  $\sigma_{ij}$ ) as

$$\theta_{ij} = \frac{1}{3}\theta\tilde{h}_{ij} + \sigma_{ij}, \quad (2.146)$$

we arrive at the so-called *kinematical decomposition*, that explicitly reads as

$$\nabla_i \xi_j = \frac{1}{3}\theta\tilde{h}_{ij} + \sigma_{ij} + \omega_{ij} - \dot{\xi}_i \xi_j. \quad (2.147)$$

It is worth noting that all the tensors defined above are orthogonal to the vector field  $\xi^i$  because they are constructed from the projection tensor  $\tilde{h}_{ij}$ .

Because of the physical meaning of  $\theta$ , an average length scale  $L(t)$  can be defined along the fluid flow lines by the equation  $\dot{L}/L = \theta/3$ , so that the volume  $\delta V$  of any small fluid element evolves like  $L^3$  along any flow line and it is easy to obtain the equation

$$\dot{\theta} + \frac{1}{3}\theta^2 = 3\frac{\ddot{L}}{L}. \quad (2.148)$$

Finally, the magnitude of  $\sigma_{ij}$  and  $\omega_{ij}$  are given respectively by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} \geq 0, \quad \omega^2 = \frac{1}{2}\omega_{ij}\omega^{ij} \geq 0. \quad (2.149)$$

These quantities vanish if and only if the corresponding tensors vanish, i.e.  $\sigma^2 = 0 \Leftrightarrow \sigma_{ij} = 0$  and  $\omega^2 = 0 \Leftrightarrow \omega_{ij} = 0 \Leftrightarrow \xi_{[j}\partial_{f]}\xi_l = 0$ . The last equivalence implies that the vorticity  $\omega_{ij}$  vanishes if and only if the flow vector field  $\xi^i$  is orthogonal to a family of hypersurfaces of the space-time. The evolution equations for  $\theta$ ,  $\sigma_{ij}$  and  $\omega_{ij}$  follow from the geodesic equation. If we define  $B_{ij} \equiv \nabla_i \xi_j$ , then we have that ( $\dot{y} \equiv \xi^k \nabla_k y$ )

$$\begin{aligned} \xi^k \nabla_k B_{ij} &\equiv \dot{B}_{ij} = \xi^k \nabla_k \nabla_i \xi_j = \xi^k (\nabla_i \nabla_k \xi_j + R_{kij}{}^l \xi_l) \\ &= \nabla_i (\xi^k \nabla_k \xi_j) - (\nabla_i \xi^k) (\nabla_k \xi_j) + R_{kij}{}^l \xi^k \xi_l \\ &= \nabla_i \dot{\xi}_j - B_i{}^k B_{kj} + R_{kij}{}^l \xi^k \xi_l. \end{aligned} \quad (2.150)$$

In the simpler case of vanishing acceleration vector  $\dot{\xi}^i$ , we have that taking the trace, the symmetric trace-free part, and the antisymmetric one of Eq. (2.150), we obtain

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2(\sigma^2 - \omega^2) - R_{kl}\xi^k \xi^l, \quad (2.151a)$$



$$\begin{aligned} \dot{\sigma}_{ij} = & -\frac{2\theta}{3}\sigma_{ij} - (\sigma_{il}\sigma^l_j + \sigma_{il}\sigma^l_j) - R_{imjn}\xi^m\xi^n \\ & + \frac{2}{3}(\sigma^2 + \omega^2 + R_{kl}\xi^k\xi^l)\tilde{h}_{ij}, \end{aligned} \quad (2.151b)$$

$$\dot{\omega}_{ij} = -\frac{2\theta}{3}\omega_{ij} - (\sigma_{im}\omega^m_j\omega_{im}\sigma^m_j). \quad (2.151c)$$

Equation (2.151a), known as the Raychaudhuri equation, is of fundamental importance in proving the singularity theorems and it will be analyzed in the following subsection.

### 2.7.3 The Raychaudhuri equation

Let us specialize the treatment discussed in the previous subsection to the case of a geodesic congruence, and let us focus our attention on the right-hand side of Eq. (2.151a): using Einstein equations, the last term can be written as

$$R_{kl}\xi^k\xi^l = \kappa \left( T_{ij} - \frac{1}{2}Tg_{ij} \right) \xi^i\xi^j = \kappa \left( T_{ij}\xi^i\xi^j + \frac{1}{2}T \right). \quad (2.152)$$

Let us assume a physical criterion in order to prevent the stresses of matter from becoming so large to make the right-hand side of Eq. (2.152) negative, i.e.

$$T_{ij}\xi^i\xi^j \geq -\frac{1}{2}T. \quad (2.153)$$

This condition is known as the *strong energy condition* and it is commonly expected that every reasonable kind of matter should satisfy such condition. From the Raychaudhuri equation (2.151a) one can see that, if the congruence is non-rotating ( $\omega_{ij} = 0$ ) and the strong energy condition holds,  $\theta$  always decreases along the geodesics. More precisely, we get

$$\dot{\theta} + \frac{1}{3}\theta^2 \leq 0, \quad (2.154)$$

whose integral implies

$$\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau, \quad (2.155)$$

where  $\theta_0$  is the initial value of  $\theta$ . For negative values of  $\theta_0$  (i.e. the congruence is initially converging),  $\theta$  will diverge after a proper time not larger than  $\tau \leq 3/|\theta_0|$ . In other words, the geodesics must intersect before such instant and form a *caustic* (a focal point). Of course, a singularity of  $\theta$  is

nothing but a singularity in the congruence and not a space-time one, since the smooth manifold is well-defined on caustics.

To get insight into the strong energy conditions, consider the simple case of the perfect fluid as in Eq. (2.20). Thus Eq. (2.153) reads as

$$\rho + 3P \geq 0, \quad \rho + P \geq 0, \quad (2.156)$$

and it is satisfied for  $\rho \geq 0$  and for a negative pressure component smaller than  $\rho$  in magnitude.

To translate the occurrence of caustics into space-time singularities, we need to introduce some notions of differential geometry and topology. Let  $\gamma$  be a geodesic with tangent  $v_i$  defined on a manifold  $\mathcal{M}$ . We call  $\eta^i$  a solution of the geodesic deviation equation<sup>6</sup> (2.6), and this constitutes a Jacobi field on  $\gamma$ . If  $\eta^i$  is non-vanishing along  $\gamma$ , but  $\eta^i(p) = \eta^i(q) = 0$  ( $p, q \in \gamma$ ), then the two points  $p, q$  are said to be conjugate. It is possible to show that a point  $q \in \gamma$  lying in the future of  $p \in \gamma$  is conjugate to  $p$  if and only if the expansion of all the time-like geodesics congruence passing through  $p$  approaches  $-\infty$  at  $q$ . A point is then conjugate if and only if it is a caustic of such congruence. A necessary hypothesis in this statement is that the space-time manifold  $(\mathcal{M}, g_{ij})$  satisfy  $R_{ij}\xi^i\xi^j \geq 0$ , for all the time-like  $\xi^i$ . Moreover, a necessary and sufficient condition for a time-like curve  $\gamma$ , connecting  $p, q \in \mathcal{M}$ , to locally maximize the proper time between  $p$  and  $q$ , is that  $\gamma$  is a geodesic without any point conjugate to  $p$  between  $p$  and  $q$ .

An analogous analysis can be made for time-like geodesic and a smooth space-like hypersurface  $\Sigma$ . In particular, let  $\theta$  be the expansion of the geodesic congruence orthogonal to  $\Sigma$ . Then, for  $\theta < 0$  and within a proper time  $\tau \leq 3/|\theta|$ , there will be a point  $p$  conjugate to  $\Sigma$  along the geodesic orthogonal to  $\Sigma$  (for a space-time  $(\mathcal{M}, g_{ij})$  satisfying  $R_{ij}\xi^i\xi^j \geq 0$ ). As above, a time-like curve that locally maximizes the proper time between  $p$  and  $\Sigma$  has to be a geodesic orthogonal to  $\Sigma$  without conjugate point to  $\Sigma$ .

The last step toward the singularity theorems is to prove the existence of maximum length curves in globally hyperbolic space-times. We recall that this is the case because they possess Cauchy surfaces in accordance with the determinism of classical physics. Without entering the details, in such a case a curve  $\gamma$  for which  $\tau$  attains its maximum value exists, and a necessary condition is that  $\gamma$  be a geodesic without conjugate points.

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<sup>6</sup>In this case, a minus sign appears on the right-hand side of (2.6) due to the different signature of the metric.

### 2.7.4 Singularity Theorems

**Theorem 2.1.** *Let a space-time manifold  $(\mathcal{M}, g_{ij})$  be globally hyperbolic satisfying the condition*

$$R_{ij}\xi^i\xi^j \geq 0 \quad (2.157)$$

*for all the time-like vectors  $\xi^i$ . Suppose that the expansion  $\theta$  of a Cauchy surface everywhere satisfies  $\theta \leq C < 0$ , for a constant  $C$ .*

*Then, no past-directed time-like curves  $\lambda$  from  $\Sigma$  can have a length greater than  $3/|C|$ .*

**Proof.** If there is a past-directed time-like curve, then a maximum length curve would also exist; this curve should be a geodesic, thus contradicting the property that no conjugate point exists between  $\Sigma$  and  $p \in \lambda$ . Therefore such curve cannot exist. In particular, all past-directed time-like geodesics are incomplete.  $\square$

This theorem is valid in a cosmological context and expresses that, if the Universe is expanding everywhere at a certain instant of time, then it must have begun with a singular state at a finite time in the past. It is also possible to show that the previous theorem remains valid also relaxing the hypothesis that the Universe is globally hyperbolic. The price to be paid is that  $\Sigma$  has to be assumed as a compact manifold (dealing with a closed Universe) and, especially, that only one incomplete geodesic is predicted.

We will now discuss the most general theorem, which completely eliminates the assumptions of a Universe expanding everywhere and the global hyperbolicity of the space-time manifold  $(\mathcal{M}, g_{ij})$ . On the other hand, we lose any information about the nature of the incomplete geodesic. Such a theorem, in fact, implies the existence of only one incomplete geodesic, i.e. it does not distinguish between a time-like and a null geodesic.

**Theorem 2.2.** *A space-time  $(\mathcal{M}, g_{ij})$  is singular under the following three hypotheses:*

- (i) *the condition  $R_{ij}v^iv^j \geq 0$  holds for all time-like or null vectors  $v^i$*
- (ii) *no closed time-like curve exists*
- (iii) *at least one of the following properties holds:*

- a)  *$(\mathcal{M}, g_{ij})$  is a closed Universe*
- b)  *$(\mathcal{M}, g_{ij})$  possesses a trapped surface<sup>7</sup>*

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<sup>7</sup>A trapped surface is a compact smooth space-like manifold, such that the expansion  $\theta$  of either outgoing either incoming future directed null geodesics is everywhere negative.

- c) there exists a point  $p \in \mathcal{M}$  such that the expansion  $\theta$  of the future or past directed null geodesics emanating from  $p$  becomes negative along each geodesic in this congruence.*

This theorem states that our Universe, as classically described, must be singular. In fact, conditions (i)-(ii) hold and  $\theta$ , for the past-directed null geodesics emanating from us at the present time, becomes negative before the decoupling time, i.e. the time up to when the Universe is well described by the Friedmann-Robertson-Walker model.

The occurrence of a space-time singularity undoubtedly represents a breakdown of the classical theory of gravity. The removal of such singularities is a prerequisite for any fundamental theory, as expected in the quantum formulation of the gravitational field. The singularity theorems are very powerful instruments, although do not provide any information about the nature of the predicted singularity. Unfortunately, we do not have a general classification of singularities, i.e. many different types exist and the unbounded curvature is not the basic mechanism behind such theorems.

Summarizing, we have shown that a space-time singularity in GR can be defined following two criteria. The first one is the causal geodesic incompleteness (global criterion) and the second one is the divergence of the scalars built up from the Riemann tensor (local criterion). Although the latter is useful to characterize a singularity, it is unsatisfactory since a space-time can be singular without any pathological character of these scalars. Not all singularities have large curvature but, most importantly, diverging curvature is not the assumption of the singularity theorems. Singularity theorems, which demonstrate the geodesic incompleteness, are based on general properties of differential geometry and topology, in addition with positive curvature. The Einstein theory enters only in replacing positive curvature with positive energy conditions affecting the Raychaudhuri equation. It is worth noting that no general mechanism able to demonstrate a non-singular behavior for a space-time is available. Although violating the energy conditions is an immediate way to avoid the singularity theorems, this is not an enough general criterion.

## 2.8 Guidelines to the Literature

The Einstein theory of gravity described in Sec. 2.1 can be found in many classical textbooks. We recommend Landau & Lifshitz [301] and Misner, Thorne & Wheeler [347] for a introductory exposition while Hawking & Ellis [228], Wald [456] or Weinberg [462] for a rigorous analysis.

The inclusion of macroscopic matter fields in GR as in Sec. 2.2 is discussed in the above textbooks. For what concerns the energy-momentum tensor in GR, see the review [428], while regarding the Yang-Mills fields (Sec. 2.2.4) we suggest the textbooks of Pokorski [388], Weinberg [463], and the reviews [289, 368].

The canonical formulation of GR, as well as the reduction of the dynamics, developed in Sec. 2.3 was formulated by Arnowitt, Deser & Misner in [17–19] (for reviews see [197, 262, 263, 438, 456]). The general theory of constrained systems can be found in the book of Dirac [153] and Henneux & Teitelboim [237]. The Hamilton-Jacobi formalism for GR (Sec. 2.3.2) is discussed in [193, 241].

An exposition on the synchronous reference frame (Sec. 2.4) can be found in Landau & Lifshitz [301].

The tetradic formalism presented in Sec. 2.5 is analyzed in the standard textbooks [301, 456] and in that by Chandrasekhar [116].

The reformulation of GR in terms of self-dual connection variables, presented in Sec. 2.6, has been proposed by Ashtekar in [21, 22] (for reviews see [396]). The real Ashtekar variables have been introduced in [37] and generalized in [255, 256]. The action for this new formulation of GR has been proposed by Holst in [240] and generalized in the presence of fermions in [340, 385]. There are several reviews and books on such topics. In particular, [28] starts from the Lagrangian formulation, while in [384, 438] the starting point is the Hamiltonian framework. The gauge group of GR, discussed in Sec. 2.6.3, has been addressed in [3, 407, 408] and later clarified in various works (see for example [4, 121, 437]). For a comparison between the geometrodynamics and the connection formalism see [298].

A complete discussion on the space-time singularity theorems (Sec. 2.7) is given in the textbooks by Hawking & Ellis [228] and Wald [456] while for a recent review see [415]. A presentation of fluid kinematics can be found for example in [125].

## PART 2

# Physical Cosmology

In these Chapters, written in collaboration with Dr. Massimiliano Lattanzi (Dipartimento di Fisica, Università di Roma "Sapienza", Italy), we give a wide description of the Universe evolution as provided by the Standard Cosmological Model, including the inflationary paradigm and the dynamics of small inhomogeneities.

Chapter 3 is dedicated to the analysis of the isotropic Universe evolution, tracing the kinematical and dynamical features of the Friedmann-Robertson-Walker cosmology.

Chapter 4 provides a complete picture of the most relevant observational facts at the ground of the present knowledge of our Universe. Significant links between observations and theoretical predictions are marked.

Chapter 5 concerns the illustration of the inflationary scenario in its most general (model independent) form. Starting from the shortcomings of the Standard Cosmological Model, we describe the hypotheses and the predictions characterizing the inflation.

Chapter 6 discusses the so-called quasi-isotropic solution, which represents a natural inhomogeneous extension of the Friedmann-Robertson-Walker cosmology. In this model the inhomogeneities remain dynamically weak and we study their evolution in presence of the main cosmological sources.

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## Chapter 3

# The Structure and Dynamics of the Isotropic Universe

In this Chapter we will present the main features of the Standard Cosmological Model (SCM). The SCM is built upon the geometrical framework of the homogeneous and isotropic Robertson-Walker (RW) geometry and is able to explain the phenomenology that emerges by the direct observation of the Universe. Our discussion allows the reader to get a synthetic but complete view of the most relevant kinematical and dynamical properties of the expanding Universe, together with a description of the thermal history associated to the evolution of the primordial thermal bath.

We start by analyzing the kinematical properties of the RW Universe and showing how the main signature of an expanding Universe can be derived without the need of implementing the Einstein dynamics. In particular, we describe the motion of free particles on an expanding background, inferring the redshift of light, the Hubble law and the recession of galaxies and, eventually, we will discuss the two fundamental causal scales governing the propagation of signals and of physical interactions across the Universe. Finally, we will introduce the Boltzmann equation on the expanding Universe, outlining how the macroscopic properties of the cosmological fluid can be properly recovered from the microphysics of the elementary species constituting the thermal bath.

The next step in our analysis of the SCM is the implementation of the Einstein equations in correspondence to the highly symmetric RW geometry, i.e. the study of the Friedmann dynamics describing the behavior of the isotropic Universe FRW. Let us remark how we make reference to the cosmological model as FRW and to the underlying geometry as RW. By investigating the structure of the equations and via the derivation of asymptotic solutions, we will characterize the nature of the *Hot Big Bang*, with particular attention on the radiation-dominated era. Then, we will



devote some space to the discussion of the de Sitter solution, describing the evolution of the Universe when it is dominated by a constant energy density term (this regime will have a crucial role in the study of the inflationary paradigm, addressed in Chap. 5). In view of the implementation of a quantum cosmology framework, as it will in Chap. 10 and in Chap. 12, we reformulate the Friedmann dynamics in the Hamiltonian formalism, by stressing the emergence of a super-Hamiltonian constraint in place of the original Friedmann equation.

The exact homogeneous dynamics is completed by discussing two relevant examples of dissipative isotropic cosmologies, i.e. the evolution of the cosmological fluid when the bulk viscosity effect or the possibility of matter creation cannot be neglected.

The study of the RW Universe is eventually enriched by a rather detailed description of the role played by the inhomogeneous perturbations as seeds for the later structure formation. We will introduce the concept of Jeans scale in the cases of a stationary and then of an expanding background. Then we pursue the fully relativistic perturbation theory, coupling the perturbed dynamical system to the inhomogeneous component of the Boltzmann equation.

The Chapter ends with a brief description of the inhomogeneous, spherically symmetric Tolmann-Bondi cosmology. We outline its main dynamical features in the synchronous reference and provide a Lagrangian picture of its geometrodynamics. The relevance of this class of Universes relies on their ability to describe local inhomogeneous structures, properly matched at the large scale with the RW metric.

### **3.1 The RW Geometry**

In this section we will analyze the properties of the homogeneous and isotropic Universe, whose geometry is properly described by the RW line element. Our aim is to extract cosmological information from the structure of the line element describing the space-time, without imposing the corresponding Einstein dynamics. In particular, we are interested in characterizing the particle motion on an expanding background, in order to fix its kinematic properties and to provide a physical insight on some phenomenological issues of the observed Universe. Among the possible kinematic effects, we focus our attention on the motion of nearby galaxies (i.e. the Hubble law), on the non-stationary dynamics of elementary particles

(i.e. the redshift of the wavelengths), and on the causal structure characterizing the propagation of physical signals (i.e. the Hubble length and the cosmological horizon).

### 3.1.1 Definition of isotropy

Qualitatively, isotropy refers to the absence of preferred directions in space. Before formulating in a precise way this notion, we have to stress that, at each point, at most one observer can see the Universe as isotropic. In fact, given a matter field filling the Universe, any observer in relative motion with the matter will measure anisotropies in the expansion of matter. Let us give a precise definition of isotropy.

**Definition 3.1.** *A space-time is spatially isotropic at each point if there exists a congruence of time-like curves (namely observers), with tangents denoted by  $u^i$ , such that: for any point  $p$  and for any two unit spatial tangent vectors  $w_1^i$  and  $w_2^i$  at  $p$ , there exists an isometry of  $g_{ij}$  which leaves  $p$  and  $u^i$  at  $p$  fixed but rotates  $w_1^i$  in  $w_2^i$ .*

In an isotropic Universe it is thus impossible to construct a (geometrically) preferred tangent vector orthogonal to  $u^i$ . In a homogeneous and isotropic space-time (for the definition of homogeneity see Sec. 7.1) the spatial surfaces of homogeneity must be orthogonal to the tangents  $u^i$  to the world lines of the isotropic observers.

We can define the isotropy group  $I_p$  of a point  $p$  as the set of all isometries leaving  $p$  fixed. Suppose now that at  $p$  we choose coordinates such that  $g_{ij}(p) = \eta_{ij}$ . The group  $I_p$  then leaves the Minkowski metric invariant, i.e. the isotropy group must be a subgroup of the (homogeneous) Lorentz group  $SO(3,1)$ . The dimension  $m$  of  $I_p$  is thus  $m \leq 6 = \dim SO(3,1)$ .  $I_p$  is a subgroup of the (full) symmetry group of the manifold (see Sec. 7.1). The Friedmann-RW (FRW) cosmological models are characterized by a three-dimensional isotropy subgroup.

It is worth noting that a space isotropic around any point is necessarily also homogeneous. The contrary is not true, i.e. homogeneous and anisotropic spaces do exist. When referring to the isotropic Universe, it is always also homogeneous.

### 3.1.2 Kinematics of the isotropic Universe

In agreement with the Cosmological Principle, stating that each observer looks at the same Universe (i.e. both privileged space points and preferred space directions are forbidden), we base the description of the Universe kinematics on a non-stationary homogeneous and isotropic three-geometry.

The hypothesis of isotropy imposes that the three spatial directions evolve with the same time law, while the space-time must be characterized by vanishing  $g_{0i}$  (if non-zero, this component would fix a preferred direction because it transforms as a three-vector under spatial coordinate transformations). Thus, in any synchronous reference, we deal with the RW line element

$$ds^2 = dt^2 - a^2(t)dl_{\text{RW}}^2. \quad (3.1)$$

The cosmic scale factor  $a(t)$  is the only degree of freedom available to the dynamical problem and  $dl_{\text{RW}}^2$  denotes the spatial line element of a three-space with constant zero, positive or negative three-curvature, i.e.

$$dl_{\text{RW}}^2 = h_{\alpha\beta}^{\text{RW}} dx^\alpha dx^\beta = \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.2)$$

with  $r$ ,  $\theta$  and  $\phi$  being the usual spherical coordinates, while  $K$  denotes the spatial curvature. When  $K \neq 0$ , it is always possible to set  $|K| = 1$ , by means of the redefinitions  $a \rightarrow a/\sqrt{K} \equiv a_{\text{curv}}$  and  $r \rightarrow \bar{r} \equiv Kr$ . Unless differently specified, we will refer to the cosmic scale factor as the curvature radius of the Universe  $a_{\text{curv}}$ , that for  $K \neq 0$  is a measurable quantity. In the case  $K = 0$ , the curvature radius is infinite, so that the normalization of the scale factor is completely arbitrary and has no physical meaning (i.e. observable quantities like the redshift depend only on the ratio of the scale factor measured at different times).

The three spatial line elements can be respectively interpreted as a hyper-plane ( $K = 0$ ), a hyper-sphere ( $K = 1$ ), and a hyper-saddle ( $K = -1$ ), although the line element does not fix the global topological properties of the three-space. Different choices for the topology are possible: for instance, either the hyper-plane (that is an open space) or the closed torus are characterized by  $K = 0$ .

The RW geometries are often described in terms of an angle-like coordinate  $\chi$ , defined as  $r = \chi$  for  $K = 0$ ,  $r = \sin \chi$  for  $K = 1$ , or  $r = \sinh \chi$  for  $K = -1$  respectively, and defining the co-moving time coordinate  $\eta$  by

means of  $dt = a(\eta)d\eta$ . In this picture, the line element rewrites as

$$\begin{aligned} ds^2 &= a^2(\eta) (d\eta^2 - dl_{\text{RW}}^2) , \\ dl_{\text{RW}}^2 &= d\chi^2 + \alpha_K^2(\chi) (d\theta^2 + \sin^2 \theta d\phi^2) , \\ \alpha_K(\chi) &= \begin{cases} \sinh \chi & 0 < \chi < \infty \quad \text{for } K = -1 \\ \chi & 0 < \chi < \infty \quad \text{for } K = 0 \\ \sin \chi & 0 < \chi < \pi \quad \text{for } K = +1 . \end{cases} \end{aligned} \quad (3.3)$$

This way, we deal with a conformal expression of the space-time metric and the  $(\theta - \chi)$  light-cone for  $K = 0$  is  $\pi/4$  wide.

The time variation of the cosmic scale factor provides the evolution of the Universe (expansion or contraction) and when  $a(t_0) = 0$ , also the metric determinant is zero. However, at this level, such an instant  $t_0$  cannot yet be recognized as a real space-time singularity (i.e. the determinant is not an invariant quantity).

Because of the homogeneity hypothesis, the Ricci scalar is independent of the spatial coordinates, and in terms of the non-zero components of the Ricci tensor it is given by

$$R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \right) \quad (3.4)$$

$$R_{00} = -3 \frac{\ddot{a}}{a} \quad (3.5a)$$

$$R_{\alpha\beta} = - \left( \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{K}{a^2} \right) h_{\alpha\beta}^{\text{RW}} . \quad (3.5b)$$

Since the cosmological implementation of this model has to take place in the presence of a matter source describing the present or the primordial Universe, the above quantities are never globally vanishing and their behavior can reveal the presence of a physical singularity.

### 3.1.3 The particle motion

In this Section, we derive an important feature concerning the behavior of the momentum of a particle moving in a RW space-time.

The trajectory of a test particle, moving across the Universe, is described by the geodesic Eq. (2.2) associated to the RW metric (3.1). For our purposes, we can limit our attention to the zero component only

$$\frac{du^0}{ds} + \frac{\dot{a}}{a} u^2 = 0 , \quad (3.6)$$

where  $u^2 = a^2 h_{\alpha\beta}^{\text{RW}} u^\alpha u^\beta$  is the square of the modulus of the spatial velocity. In a synchronous reference frame, the normalization condition for the four-velocity can be stated as  $(u^0)^2 = u^2 + 1$ , and hence we get  $u^0 du^0 = u du$ . Thus, remembering that  $u^0 ds = dt$ , Eq. (3.6) rewrites as

$$\frac{du}{dt} + \frac{\dot{a}}{a}u = 0. \quad (3.7)$$

This expression admits the relevant solution  $u \propto 1/a$ . If we denote as  $m_0$  the rest mass of the particle, the modulus of its three-momentum is  $p \equiv m_0 u \propto 1/a$ . The momentum of a particle is a time-dependent quantity and, for the particular case of an expanding Universe, it is redshifted by the underlying dynamics. We have to stress that the above derivation does *not* rely on the notion of non-vanishing element of proper time. Indeed, since the differential  $ds$  does not appear in Eq. (3.7), the result does not depend on the particular choice of the affine parameter. Thus this result holds even in the case of a zero mass particle, like a photon. Indeed, for a massless particle we get the relation

$$\mathcal{E} = p = \frac{2\pi}{\lambda} \propto \frac{1}{a}, \quad (3.8)$$

$\mathcal{E}$  denoting the energy and  $\lambda$  the corresponding wavelength. If we consider a photon emitted at a given time  $t_e$  in the past and observed today at  $t_0$ , the ratio of the corresponding wavelengths takes the expression

$$\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} \equiv 1 + z. \quad (3.9)$$

In the case of an expanding Universe  $a(t_0) > a(t_e)$  and the observed wavelength is larger than the wavelength at the emission, i.e. it is shifted towards the red. The quantity  $z$  represents the amount of this redshift and it is measurable, playing a role equivalent to the scale factor whose variability follows the Universe kinematics.

The physical distance between a pair of co-moving observers scales with the cosmic scale factor, exactly like the wavelength of photons. Thus any intrinsic or co-moving length  $l$  becomes a measurable quantity of the expanding Universe only if redefined as  $l_{\text{phys}} = a(t)l$ . This non-stationary feature reflects an intrinsic property of any system living over an expanding geometry.

### 3.1.4 The Hubble law

We will now derive how the RW kinematics can explain the recession of galaxies. Indeed, the Universe expansion accounts for the galaxy recession

via the geodesic motion on the isotropic and homogeneous background. Apart from small proper motions (random physical velocities) and local gravitational interactions (which are able to form bounded systems), the galaxy flow (the Hubble flow) can be properly described as the motion of pressure-less particles (a dust system) that are freely falling on the expanding geometry. However, the specific form of the Hubble law can be reproduced only in the limit of galaxies close to our own (by convention placed at  $r = 0$  and  $t = t_0$ ), i.e. for  $z \ll 1$ .

Before entering into the details of the proof, let us stress that the galaxy expansion, being a pure geometrical effect, does not provide any physical motion. In other words, the co-moving spatial coordinates of any single galaxy remain fixed. Moreover, when a gravitational system (for instance a group of galaxies) has enough binding energy, it detaches from the Hubble flow and forms a non-expanding substructure. This is exactly the reason why within any galaxy, the space-time is essentially flat and inertial frames are allowed. When a density perturbation, corresponding to the galactic scale, reaches a certain critical value, it decouples from the cosmological fluid and starts an independent evolution as a bound system, retaining a geodesic motion only as a whole.

Let us observe how looking far in the Universe (say, at a generic coordinate  $r$ ), we are looking backward in time (say, at a generic instant  $t$ ), i.e. when the scale factor of the Universe was  $a(t) < a(t_0) \equiv a_0$ . Assuming that  $t$  is sufficiently close to  $t_0$ , the Taylor expansion

$$a(t) = a_0 + \dot{a}|_{t=t_0}(t - t_0) + \dots, \quad (3.10)$$

leads to

$$\frac{a}{a_0} \equiv \frac{1}{1+z} = 1 - H_0(t_0 - t) + \dots, \quad (3.11)$$

where the definition of redshift Eq. (3.9) was used. In Eq. (3.11), the *Hubble parameter*  $H(t) \equiv \frac{\dot{a}}{a}$  remains defined together with its present value, the *Hubble constant*  $H_0 \equiv H(t = t_0)$ . The Hubble parameter measures the (logarithmic) expansion rate of the Universe at a given time.

To obtain the Hubble law, we need to express  $(t_0 - t)$ , i.e. the time for the light to go from the source to us, in terms of the same distance. Since for a photon  $ds^2 = 0$ , we can write

$$\int_t^{t_0} \frac{dt}{a(t)} = \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}} = \begin{cases} r & \text{if } K = 0 \\ \sin r & \text{if } K = 1 \\ \sinh r & \text{if } K = -1 \end{cases} \quad (3.12)$$

which, in the limit  $r \ll 1$ , this is simply equal to  $r$ . The spatial curvature can thus be neglected when dealing with small values of  $(t - t_0)$ , i.e. with distances much smaller than the curvature radius of the Universe. At this level, the space can be assumed to be flat. Inserting Eq. (3.10) on the left-hand side of Eq. (3.12) we get, to first order,

$$t_0 - t = d + \dots, \quad (3.13)$$

where  $d = a_0 r$  is the present distance to the source. The physical content of this equality is straightforward: when an object is close enough (so that we can neglect the spatial curvature along with the space expansion), the time for the light signal to reach us is proportional to the distance from the source. This allows us to rewrite Eq. (3.11) in the form (we use  $1/(1+z) \simeq (1-z)$  for  $z \ll 1$ )

$$z = H_0 d + \dots \quad (3.14)$$

Interpreting the geometrical redshift of the photons emitted by a galaxy as the Doppler effect due to a physical velocity  $v$ , we get the well-known expression for the Hubble law

$$v = H_0 d. \quad (3.15)$$

Equation (3.15) can be generalized to account for higher order terms in the scale factor expansion. However, this involves some subtleties related to the notion of distance in cosmology, and to how distances are measured. We have seen that the proper distance at the time  $t$  between us ( $r = 0$ ) and an object at coordinate position  $r$  is  $d(t) = a(t)r$ . Unfortunately, the proper distance is not what is directly measured through observations. For example, distances are often measured recurring to *standard candles*, i.e. objects with known intrinsic luminosity (for example, supernovae Ia). If the luminosity  $L$  is known and the flux  $F$  at the Earth is measured, one can introduce a *luminosity distance*  $d_L$  as

$$d_L \equiv \sqrt{\frac{L}{4\pi F}}. \quad (3.16)$$

This expression again is a *definition* of  $d_L$ . Its operative meaning is directly related to an observable quantity (the flux) and thus it is an observable quantity itself. The reason for the definition is that in a Minkowski space  $d_L$  coincides with  $d$ . However, in general this is not the case, once the effects of the spatial curvature and of the expansion are taken into account. It can be shown, using the conservation of the energy momentum tensor, that the

luminosity distance  $d_L$  and the proper one  $d$  are related by (assuming for simplicity  $K = 0$ )

$$d_L = d(1 + z). \quad (3.17)$$

This formula can also be interpreted as follows:  $L/4\pi a_0^2 r^2$  is the flux that would be measured in the absence of expansion; the flux is reduced by a factor  $1+z$  for the redshift of the energy of the single photon (see Sec. 3.1.3), and by another factor of  $1+z$  for the time dilation effect between the source and the observer.

Repeating the steps above, one finds that

$$H_0 d_L = z + \frac{1}{2} (1 - q_0) z^2 + \dots, \quad (3.18)$$

where the deceleration parameter of the Universe  $q_0$  is defined as

$$q_0 \equiv - \frac{\ddot{a}}{aH^2} \Big|_{t=t_0}. \quad (3.19)$$

We stress that the higher order terms in Eq. (3.18) depend on the distance indicator written on the left-hand side of the equation.

Equation (3.18) is particularly useful when dealing with standard candles since, in that case,  $d_L$  is the directly observable quantity. On the other hand, if one deals with *standard rulers*, i.e. objects of known linear size, the natural distance indicator is the angular diameter distance  $d_A$ , defined as the linear size of the object divided by the angle it subtends in the sky. The Hubble law for  $d_A$  has a form similar to Eq. (3.18) but with a different second (and higher) order term. For the dominant term, in the limit of small  $z$ , all the forms of the Hubble law reduce to  $H_0 d = z$ . This traces the fact that for  $z \ll 1$ , all the distance indicators like  $d_L$  and  $d_A$  coincide with the proper distance  $d$  (like it should be in a static Euclidean geometry). We finally note that the exact knowledge of  $a(t)$  is required to derive the exact relation between  $d_L$  (or  $d_A$ ) and  $z$ , which in turn requires to solve the Einstein equations.

The capability of the RW kinematics to reproduce the observed Hubble law stands as a significant confirmation of the isotropy and homogeneity of the observed Universe.

### 3.1.5 The Hubble length and the cosmological horizon

In the continuation of the Book, the description of the physical scenario living on an isotropic, non-stationary background will often require to fix



the characteristic time and length scales of the system under study, in our case the whole Universe. The fundamental cosmological time scale is given by the inverse of the expansion rate, i.e. by the *Hubble time*  $H^{-1} \equiv a/\dot{a}$ . The Hubble time roughly gives the time in which the scale factor doubles. If the scale factors scales like  $a(t) \propto t^\alpha$ , it follows that  $H^{-1} \propto t$ . In fact, in a Friedmann Universe (see Sec. 3.2) the Hubble time gives, apart from numerical factors of order unity, the age of Universe. This relation can however be deeply altered in more general cosmological models (for example in the inflationary scenario, see Chap. 5).

The length associated to the Hubble time is the *Hubble length*  $L_H(t)$

$$L_H(t) \equiv H(t)^{-1}, \quad (3.20)$$

also called the Hubble radius. This is roughly the distance that a photon can travel in an expansion time around  $t$  and is the scale to which the characteristic length of any physical phenomenon has to be compared in order to understand if it can coherently operate on cosmological scales. For example, the comparison of the characteristic mean free path  $\bar{l}$  of a given particle species with the Hubble length (3.20) determines whether such a species participates in the thermodynamical equilibrium ( $\bar{l} \ll L_H$ ) or is decoupled from it ( $\bar{l} \gg L_H$ ). In other words, for a given process to be able to maintain the equilibrium, its rate must be much faster than the geometrical rate of curvature change or, equivalently, its time scale  $\tau$  must be much smaller than the Hubble time  $H^{-1}$ . In this respect, the Hubble length represents a real horizon for the microphysics of the expanding Universe. The Hubble length, being associated to the space-time curvature of the Universe, is in itself a physical scale and can be measured today by direct and indirect observation on the large scales (see Sec. 4.3).

Another relevant length is the one characterizing the maximal causal distance at which physical signals can propagate in an expanding Universe, starting at a finite initial instant of time, say  $t = 0$ . Such a distance, called the *particle horizon* (often simply “the horizon”) corresponds to the path traveled by a photon emitted at  $t = 0$  and is calculated from the condition for the propagation of a wave front, i.e.  $ds^2 = 0$ . In terms of the RW line element (3.1), this condition can be stated as

$$dt = a(t)dl_{\text{RW}} \quad \Rightarrow \quad l_{\text{RW}} = \int_0^t \frac{dt'}{a(t')}. \quad (3.21)$$

To get from such co-moving length the physical and measurable horizon, we have to rescale it by the cosmic scale factor, obtaining

$$d_H = a(t) \int_0^t \frac{dt'}{a(t')}. \quad (3.22)$$

Objects separated by a distance larger than  $d_H$  have never been in causal contact and they cannot have been affected each other. In particular, this implies that spatial regions which are separated more than one cosmological horizon cannot be in thermal equilibrium. Moreover, for a power-law expression of the scale factor  $a(t) \propto t^\alpha$ , the physical horizon is a finite quantity for  $\alpha < 1$ . As we shall see in Sec. 3.2, this is the case for an expanding Universe filled with ordinary matter and radiation.

Also the co-moving horizon  $d_H/a$  is always increasing, being the integral of a positive-defined quantity. Since, by definition, co-moving distances are, constants, the ratio between the horizon and any given distance decreases backwards in time. In other words, things that are in causal contact to-day were not necessarily so in the past. When studying how a causally connected region at the present time should have looked in the past, one cannot think of it as a unique causal region but, most likely, of a collection of a large number of independent causal patches. This is at the origin of the horizon paradox that affects the SCM and that will be addressed in Chap. 5.

In the case of the power-law example considered above, the Hubble length and the physical horizon are comparable quantities, i.e.  $d_H = \alpha L_H/(1 - \alpha)$ . They coincide when  $\alpha = 1/2$ , corresponding to the cosmological scenario of a radiation dominated Universe.

We remark again that the equivalence of these two spatial scales is not a general feature, as shown by the counterexample of a de Sitter phase of expansion. In fact, even if the numerical values of such two quantities roughly coincide in a Friedmann Universe, their physical meaning is deeply different. The Hubble length gives the distance traveled by a photon in one Hubble time, while the particle horizon is the distance traveled by a photon during the whole life of the Universe. Points that are distant more than one Hubble length have not been in causal contact for the last Hubble time or so, while points that are distant more than one particle horizon have *never* been in causal contact. Thus, the Hubble length is a strictly *local quantity*, depending on the expansion rate at the time  $t$  only, while the particle horizon is an *integral quantity* that receives contributions from all the past expansion history. The value of the horizon can be dramatically altered by contributions coming from a non-standard (with respect to Friedmann models) behavior of the scale factor at  $t \simeq 0$ . We can anticipate that the horizon paradox is resolved in the inflationary scenario through an early phase of de Sitter expansion that makes the particle horizon many orders of magnitude larger than its Friedmann value.

### 3.1.6 *Kinetic theory and thermodynamics in the expanding Universe: The hot Big Bang*

The early Universe, as described in the hot Big Bang theory (firstly formulated by Gamow in the '40s), is characterized by a thermal bath in which all fundamental particle species are embedded and are maintained at equilibrium by interactions with other species. The cosmological expansion implies that the thermodynamical parameters of the macroscopic cosmological fluid depend only on time. Thus, we can assume that the Universe expansion proceeds through equilibrium stages and that it is characterized by a global temperature  $T(t)$ , as far as non-ideal fluid effects due to out-of-equilibrium features (for instance dissipative mechanisms) can be neglected. Indeed, the main part of the thermal history of the Universe can be well represented as equilibrium phases and the cosmological fluid is well-modeled by a perfect one, even if some steps of the early cosmology are associated to phase transitions or species decays and decoupling, which require an appropriate out-of-equilibrium treatment.

The kinetic theory of particles on the RW background is described by the relativistic Boltzmann equation, with the usual Liouville operator is generalized to curved space-time as

$$\hat{L}[f] \equiv \frac{df}{ds} = \frac{dx^i}{ds} \frac{\partial f}{\partial x^i} + \frac{dp^i}{ds} \frac{\partial f}{\partial p^i} = \hat{C}[f], \quad (3.23)$$

where  $f = f(x^i, p^i)$  denotes the distribution function on the (relativistic) phase-space and the collision operator  $\hat{C}[f]$  is the collision integral that characterizes the change in the distribution function, in a unit of proper time, due the particle interactions. Making use of the geodesic equation to describe the particle acceleration<sup>1</sup>  $dp^i/ds$ , Eq. (3.23) for a RW background can be restated. The homogeneity of the space prevents any spatial dependence in the distribution function, while isotropy requires that it can depend on the three-momentum only through its magnitude or, equivalently, on the energy  $E \equiv p^0$ , i.e.  $f = f(t, E)$ . Thus, Eq. (3.23) for the isotropic Universe can be rewritten as

$$E \frac{\partial f}{\partial t} - \frac{\dot{a}}{a} p^2 \frac{\partial f}{\partial E} = \hat{C}[f]. \quad (3.24)$$

Equation (3.24) provides a complete microscopical description of the matter filling the Universe, once a specific form of the collision integral is given.

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<sup>1</sup>Here we adopt the notion of a free-falling scalar particle, assuming that any other effect (like, for example, the spin) averages to zero over the cosmological fluid.

In order to switch to a macroscopic description, one has to integrate the Boltzmann equation over the momentum space.

Let us analyze the case when the collision integral is negligible, which is certainly appropriate when the mean free path of the particles is much larger than the Hubble length. However, as far as we are dealing with species in thermal equilibrium, the collision integral can also be neglected because interacting particles admit the same distribution function and the matrix elements, governing the microphysics, are in general invariant under time reversal.<sup>2</sup>

Dividing Eq. (3.24) by  $E$  and observing that the relativistic dispersion relation implies  $dE/dp = p/E$ , we get

$$\frac{\partial f}{\partial t} - \frac{\dot{a}}{a} p \frac{\partial f}{\partial p} = 0. \quad (3.25)$$

Recalling the definition of the particle number density

$$n \equiv \frac{g^{\text{dof}}}{(2\pi)^3} \int d^3p f(t, E), \quad (3.26)$$

where  $g^{\text{dof}}$  denotes the number of degrees of freedom of the particle, we can derive a macroscopic law from the Boltzmann equation. By integrating Eq. (3.25) over momentum space ( $d^3p = 4\pi p^2 dp$  because of the isotropy assumption) and integrating the second term by parts (the distribution function has to vanish for diverging  $p$ ), we obtain an equation for the number density of the form

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = 0 \quad \Rightarrow \quad n(t) \propto \frac{1}{a^3}. \quad (3.27)$$

Thus the Universe expands and the particle number density decays according to the increase of the spatial volume ( $V \sim a^3$ ). In other words, the number of particles contained inside a coordinate domain is conserved during the expansion.

Even if the result in Eq. (3.27) has been derived assuming a vanishing collision term, it is indeed more general. In particular, it holds as long as the collisions conserve the particle number, ensuring that, even if  $\hat{C}[f] \neq 0$ , one has  $\int (\hat{C}[f]/E) d^3p = 0$ , i.e. the collision term is zero once integrated over momentum space. For what concerns the evolution of the number density, the presence of a collision integral has to be taken into account in those processes which do not preserve the number of interacting particles, as particle decays or annihilations. For definiteness, let us consider a species

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<sup>2</sup>Indeed, under such conditions, the explicit form of the collision integral identically vanishes.

$X$  decaying to species  $Y$ , plus some other particle species  $Z$  whose evolution we are not interested in. Let  $\tau_d$  be the mean lifetime of species  $X$ . The effect of the decay  $X \rightarrow Y + Z$  can be schematically described modeling the collision term in Eq. (3.24) as  $\hat{C}[f] = \pm(E/\tau_d)f_X$ , where the (+) and (−) sign holds for the  $Y$ 's and  $X$ 's respectively. The two Boltzmann equations for the  $X$ 's and  $Y$ 's write as

$$\frac{\partial f_X}{\partial t} - \frac{\dot{a}}{a} p \frac{\partial f_X}{\partial p} = -\frac{f_X}{\tau_d}, \quad (3.28a)$$

$$\frac{\partial f_Y}{\partial t} - \frac{\dot{a}}{a} p \frac{\partial f_Y}{\partial p} = +\frac{f_X}{\tau_d}. \quad (3.28b)$$

Repeating the same steps leading to Eq. (3.27), we get the following equations for the number densities

$$\frac{dn_X}{dt} + 3(H + H_d)n_X = 0 \quad (3.29a)$$

$$\frac{dn_Y}{dt} + 3Hn_Y - 3H_d n_X = 0 \quad (3.29b)$$

where we have defined  $H_d \equiv 1/3\tau_d$  and the number densities finally evolve as

$$n_X(t) \propto \frac{1}{a^3} e^{-3H_d t} \quad (3.30a)$$

$$n_Y(t) \propto \frac{1}{a^3} (1 - e^{-3H_d t}). \quad (3.30b)$$

Let us also note how the Boltzmann equations for the  $X$ 's and  $Y$ 's, either in their unintegrated or integrated form, can be summed to obtain a single equation for the total density  $n_{X+Y} = n_X + n_Y$  perfectly identical to Eq. (3.27), i.e. with vanishing collision term. This implies that  $n_{X+Y} \propto a^{-3}$  (as it can be directly verified from the solutions Eq. (3.30a) and Eq. (3.30b)). This is not surprising, because in each decay process a particle  $X$  is destroyed and a  $Y$  is created, so that there is no net change in the total number of  $X$  and  $Y$ .

Similarly to number density, we can obtain a macroscopic relation involving the energy density and the pressure of the cosmological fluid by multiplying Eq. (3.25) by  $E$ . Again, we consider a vanishing collision term although the final result still holds as long as the collisions conserve the energy. Noting the conjugate character of the variables  $t$  and  $E$ , we get

$$\frac{\partial}{\partial t} (Ef) - \frac{\dot{a}}{a} E p \frac{\partial f}{\partial p} = 0. \quad (3.31)$$

As before, we will apply the integral operator  $\int d^3p = \int_0^\infty 4\pi p^2 dp$  (the angular integration gives  $4\pi$  because of the isotropy condition) to this equation. Multiplying Eq. (3.31) by  $p^2$ , after some manipulation we rewrite it as ( $dE/dp = p/E$ )

$$\frac{\partial}{\partial t} (p^2 E f) - \frac{\dot{a}}{a} \left[ \frac{\partial}{\partial p} (E p^3 f) - 3p^2 E f - \frac{p^4}{E} f \right] = 0. \quad (3.32)$$

Bearing in mind the definitions of the energy density  $\rho$  and of the pressure  $P$  as

$$\rho \equiv \frac{g^{\text{dof}}}{(2\pi)^3} \int d^3p E f(t, E), \quad (3.33)$$

$$P \equiv \frac{g^{\text{dof}}}{(2\pi)^3} \int d^3p \frac{p^2}{3E} f(t, E) \quad (3.34)$$

and integrating Eq. (3.32) over the momenta, we get the following macroscopic equation, called the *continuity equation*

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \quad (3.35)$$

The thermodynamical interpretation of Eq. (3.35) is straightforward in terms of the first law of thermodynamics applied to the Universe. Since the Universe, by definition, cannot exchange heat with an external source,  $\delta Q = 0$  (i.e. the expansion is adiabatic) and the first law reads as  $dU = -PdV$ . The internal energy  $U$  inside a co-moving volume satisfies  $U = \rho V$ , so that the first law can be restated as  $Vd\rho + (\rho + P)dV = 0$ . Since the volume  $V \propto a^3$ , we have that  $dV = 3Vda/a$  and then

$$d\rho + 3(\rho + P)\frac{da}{a} = 0. \quad (3.36)$$

The assumption to deal with an adiabatic expansion follows from the absence of external heat sources (by definition of Universe). On the other hand, on a local level, the adiabatic character of the Universe follows from the impossibility to exchange heat between spatial points being at the same temperature (this in turn is due to the homogeneity of the Universe).

Comparing the kinetic definitions (3.33) and (3.34), an equation of state  $P = P(\rho)$  can be defined in two limiting cases of cosmological interest.

- (i) When the temperature of the Universe is much smaller than the rest mass  $m$  of the particles, we deal with the limit  $E \simeq m$ ,  $p \ll m$ , so that we get the dust-like relations  $\rho \simeq mn$ ,  $P \simeq 0$ . Such approximation well describes the present Universe, where galaxies, apart from proper motions and local interactions, resemble a free falling dust fluid. In this case, Eq. (3.35) implies  $\rho \propto 1/a^3$ , as expected by the behavior of the number density.

- (ii) When the temperature of the Universe is much larger than the rest mass of the particles we get the relations  $E \simeq p$  and hence  $P \simeq \rho/3$ . Such ultrarelativistic equation of state properly describes the behavior of the very early Universe, when only highly energetic particles were present. In this case, Eq. (3.35) provides the expression for  $\rho$  as  $\rho \propto a^{-4}$ .

The two limits discussed here concern the matter dominated and radiation dominated Universe, respectively.

In general, one can assign a generic equation of state of the form (2.15) for the isothermal Universe. Another common way to write the equation of state is  $P = w\rho$ , where  $w = \gamma - 1$  is called the equation of state parameter. Henceforth, we require the polytropic index  $\gamma$  to fulfil the condition  $\gamma \leq 2$ , to ensure a non-superluminal sound velocity  $v_s \equiv \sqrt{dP/d\rho} = \sqrt{\gamma - 1}$  for the cosmological fluid. For such choice of the equation of state, the relation between the energy density  $\rho$  and the cosmic scale factor  $a$ , as obtained from Eq. (3.35), is  $\rho \propto a^{-3\gamma}$ . The matter and the radiation-like behaviors discussed above correspond to  $\gamma = 1$  ( $w = 0$ ) and  $\gamma = 4/3$  ( $w = 1/3$ ), respectively. Equation (3.35) can also be derived by the conservation law of the energy-momentum tensor (2.14) on the RW background, clarifying the assumption that the cosmological fluid is properly represented by a perfect one.

The energy density of the very early Universe scales as  $1/a^4$  and implies that the limit  $a \rightarrow 0$  corresponds to a physical singularity of the RW space-time, associated to the instant (say  $t = 0$ ) when the Universe was born. A peculiar case would correspond to  $P = -\rho$  ( $\gamma = 0$ ), when the Universe energy density is not diverging and, from (3.35), remains constant as  $\rho = \rho_\Lambda = \text{const}$ . From Eq. (2.14), such equation of state corresponds to a matter source described by an energy-momentum tensor of the form

$$T_{ik} = \rho_\Lambda g_{ik}, \quad (3.37)$$

that properly mimics a cosmological constant term.

Let us now link the energy density of the Universe to its temperature and hence fix the expression of the latter in terms of the cosmic scale factor. For a species in kinetic equilibrium, the distribution function takes the form

$$f = \frac{g^{\text{dof}}}{2\pi^3} \frac{1}{\exp \left[ \frac{E - \mu}{K_B T} \right] \pm 1}, \quad (3.38)$$

where  $\mu$  denotes the chemical potential, while the signs (+) and (−) pertain to fermions or bosons, respectively. In the radiation (photon-like) approx-

imation, i.e.  $K_B T \gg m$  and  $K_B T \gg \mu$ , the Universe energy density reads as

$$\rho_{\text{rad}} = \frac{\pi^2}{30} g_*(T) T^4, \quad (3.39)$$

$T$  being the photon temperature. The quantity  $g_*(T)$  is a measure of the effective number of degrees of freedom contributing to the radiation energy density and is given by

$$g_*(T) \equiv \sum_{i_B} g_{i_B}^{\text{dof}} \left( \frac{T_{i_B}}{T} \right)^4 + \sum_{i_F} \frac{7}{8} g_{i_F}^{\text{dof}} \left( \frac{T_{i_F}}{T} \right)^4, \quad (3.40)$$

where the letters B and F indicates Bose-Einstein and Fermi-Dirac species, respectively and the factor  $7/8$  arises from the difference between the fermion and the boson statistics. At sufficiently high temperature, when all the fundamental particles are in thermal equilibrium, the function  $g_*(T)$  is weakly depending on temperature and can be approximated by a constant value  $g_*^{\text{rad}}$  (the same situation takes place even in later phases, far from the energy thresholds that correspond to the annihilation and disappearance of a given particle species). Remembering that for a radiation-dominated Universe the energy density behaves as  $\rho \propto 1/a^4$ , we can infer the inverse proportionality relation between the temperature and the scale factor, i.e.  $T \propto 1/a$ . The cosmological singularity emerging for  $a \rightarrow 0$  is associated to a diverging temperature, as a consequence of the radiation nature of the very early Universe. This consideration is at the ground of the concept that *the Universe was born in a hot Big Bang*.

Let us now briefly discuss the behavior of entropy on a RW background. The relation between entropy and the other thermodynamic quantities is

$$TdS = dU + pdV. \quad (3.41)$$

Introducing the entropy density  $s \equiv S/V$  and recalling that  $U = \rho V$ , this equation can be rewritten as

$$d\rho = (Ts - \rho - P)dV + Tds. \quad (3.42)$$

Since  $\rho$  does not depend on volume but only on temperature, the coefficient in front of  $dV$  must vanish and thus

$$s = \frac{\rho + P}{T} \quad (3.43)$$

so that the entropy inside a co-moving volume is  $S = a^3(\rho + P)/T$ . Since heat transfer between a co-moving region and its surroundings is not possible because of the assumption of homogeneity, it follows that the entropy



inside a co-moving volume is conserved and then that the entropy density scales like  $s \propto a^{-3}$ . This quantity is dominated by the contribution of relativistic particles for which  $P = \rho/3$ , so that Eq. (3.43) gives

$$s = \frac{2\pi^2}{45} g_{*s} T^3, \quad (3.44)$$

where  $g_{*s}$  is defined similarly to  $g_*$ , namely

$$g_{*s}(T) \equiv \sum_{i_B} g_{i_B}^{\text{dof}} \left( \frac{T_{i_B}}{T} \right)^3 + \sum_{i_F} \frac{7}{8} g_{i_F}^{\text{dof}} \left( \frac{T_{i_F}}{T} \right)^3. \quad (3.45)$$

The conservation of  $S$  implies that  $g_{*s} T^3 a^3 = \text{const.}$  Since  $g_{*s}$  is nearly always constant (it varies when a given particle species disappears from the thermal bath),  $T \propto 1/a$ .

We conclude by noting that  $g_*$  and  $g_{*s}$  coincide when all the species have a common temperature  $T$  and this is nearly always the case during the history of the Universe so that one can usually take  $g_* = g_{*s}$ . The only notable exception to this is given by cosmological neutrinos, that have a present temperature  $T_\nu \simeq 1.9$  K, different from the photon temperature  $T_\gamma \simeq 2.7$  K.

## 3.2 The FRW Cosmology

### 3.2.1 Field equations for the isotropic Universe

Let us now analyze the form that the Einstein equations acquire when the hypotheses of homogeneity and isotropy are retained, as for the line element (3.1)-(3.2). In order to specify the Einstein equations (2.12) we take the cosmological fluid as co-moving with the synchronous reference. This choice is possible, even in the presence of pressure, because of the high symmetry of the RW geometry, i.e. the spatial gradients of pressure are identically zero and its time derivative terms cancel out of the fluid equations of motion (see Sec. 2.2.1). Thus,  $u^i = \delta_0^i$  and the components of  $T_i^j$  take the diagonal form presented in Eq. (2.20).

The Einstein equations are obtained using Eqs. (3.4), (3.5) and (2.20). In particular, the 00 component of the field equations takes the form

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa}{3} \rho - \frac{K}{a^2}, \quad (3.46)$$

which is usually called the *Friedmann equation*. The  $\alpha\alpha$  components reduce to three identical equations by virtue of the Universe isotropy, i.e.

$$2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{K}{a^2} = -\kappa P. \quad (3.47)$$

Substituting Eq. (3.46) into Eq. (3.47), we get the equation for the Universe acceleration

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6} (\rho + 3P) . \quad (3.48)$$

Equations (3.46) and (3.48) can be accompanied by the continuity equation (3.35), that however is dependent on the other two. In fact, each of the three equations (3.46), (3.48) and (3.35) can be obtained combining the other two. In general, to describe the dynamics of the Universe it is convenient to choose the Friedmann equation and the continuity equation. The former provides a link between matter and geometry, while the latter closes the dynamical problem, fixing the behavior of the energy density in terms of the scale factor, once an equation of state for the cosmological fluid is assigned.

We observe that Eq. (3.46) and Eq. (3.48), when calculated at the present instant of time, provide simple expressions for the Hubble constant  $H_0$  and for the deceleration parameter  $q_0$  (3.19) in terms of the actual value for the Universe radius of curvature, energy density and pressure. For instance, for  $K = 0$ , we have

$$H_0 = \sqrt{\frac{\kappa}{3} \rho_0} \quad (3.49a)$$

$$q_0 = \frac{1}{2} (1 + 3w) , \quad (3.49b)$$

where we have used the equation of state (2.15).

Let us infer some properties of the isotropic Universe dynamics, from a qualitative analysis of this fixed set of equations. First of all, from (3.48) it comes out that, as far as the Universe evolution is dominated by matter described by an equation of state  $P > -\rho/3$ , the expansion has to decelerate. Thus, the evidences that the Universe is presently accelerating, as discussed in Sec. 4.3, lead to a serious revision of our understanding about the nature of the matter filling the present Universe or, alternatively, of the notion of Friedmann dynamics.

Equation (3.46) establishes a relation between the square of the Hubble function, the energy density and the spatial curvature. While the former two are positive by definition, the latter is fixed by the curvature sign. For  $K = 0, -1$ , there is no time where  $H$ , i.e.  $\dot{a}$ , vanishes and no turning point in the Universe expansion arises. The radiation dominated Universe emerges from the hot Big Bang, then passes through an equilibrium era and finally ends its life in a decelerating matter dominated phase: no re-collapse of the space can take place. When  $K = +1$ , the Hubble function vanishes in

correspondence to a given instant  $t_{\text{tp}}$ , such that  $a_{\text{tp}} \equiv a(t_{\text{tp}}) = \sqrt{(3/\kappa\rho_{\text{tp}})}$ ,  $\rho_{\text{tp}} \equiv \rho(t_{\text{tp}})$ . Both in a radiation and matter dominated Universe, the second time-derivative of the scale factor, evaluated at  $t = t_{\text{tp}}$ , is negative in view of Eq. (3.48). Thus, the above value  $a_{\text{tp}}$  is a maximum for the Universe expansion. The subsequent evolution of the Universe is described by a recollapse phase to a singularity where  $a = 0$ . We can conclude that the flat and negatively curved spaces are characterized by an indefinite expansion from the Big Bang, ending in a decelerating rarefied Universe. On the other hand, the closed RW geometry expands from a Big Bang, reaches a maximum value and then recollapses to Big Crunch, opening fascinating, although debated, perspectives for a cyclic Universe. Furthermore, let us note that, in correspondence to  $a_{\text{tp}}$  and  $\rho_{\text{tp}}$ , it is possible to establish the existence of a static Universe with such structural parameters. However, this configuration, which would have been appropriate for the XIX Century notion of cosmology, has a purely mathematical meaning, in view of its well-known instability.

Finally, if we divide Eq. (3.46) by  $H^2$  and define the quantities

$$\rho_{\text{crit}} \equiv \frac{3H^2}{\kappa} \quad (3.50)$$

$$\Omega \equiv \frac{\rho}{\rho_{\text{crit}}}, \quad (3.51)$$

then it rewrites as

$$\Omega - 1 = \frac{1}{H^2 a_{\text{curv}}^2}. \quad (3.52)$$

Here  $\rho_{\text{crit}}$  denotes the Universe critical density, i.e. the density it would have for  $K = 0$ . The quantity  $\Omega$  is called the *density parameter* of the Universe and it is larger, equal and smaller than unity, for the closed, flat and negatively curved RW models, respectively.

The present value of the critical density is  $\rho_{\text{crit}}^0 = 1.03 \times 10^{-29} \text{ g/cm}^3 \simeq 5.8 \times 10^{-6} \text{ GeV/cm}^3$ , while  $\Omega_0$  is equal to unity within a few percent; its sign has still to be determined.

### 3.2.2 Asymptotic solution toward the Big Bang

Once the relation  $\rho = \frac{\bar{\rho}}{a^{3\gamma}}$  ( $\bar{\rho} = \text{const.}$ ) is assigned, the Friedmann Eq. (3.46) admits analytical solutions that reproduce the qualitative behaviors described above. Since we are mainly interested in the asymptotic

behavior of the Universe towards the Big Bang ( $a \rightarrow 0$ ), the spatial curvature term is negligible with respect to the energy density, as long as  $\gamma > 2/3$  and therefore one can approximate the Friedmann equation as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa\bar{\rho}}{3a^{3\gamma}}. \quad (3.53)$$

The solution of this equation reads

$$a(t) = \left(\frac{t}{\bar{t}}\right)^{\frac{2}{3\gamma}}, \quad \bar{t} = \frac{2}{\sqrt{3}\gamma\sqrt{\kappa\bar{\rho}}}, \quad (3.54)$$

where  $\bar{t}$  is an integration constant, fixed arbitrarily by the generic value  $\bar{\rho}$ . Often  $\bar{t}$  is conveniently taken as the age of the Universe, so that today the scale factor remains fixed to unity, i.e.  $a_0 = 1$ . This solution shows that the singularity appears in correspondence to any positive value of the parameter  $\gamma$  and is characterized by a diverging energy density of the form

$$\rho(t) = \frac{\bar{\rho}\bar{t}^2}{t^2} = \frac{4}{3t^2\gamma^2\kappa}. \quad (3.55)$$

Let us stress how the energy density, being an observable quantity, is independent of the (arbitrarily chosen) value of  $\bar{\rho}$ .

We now introduce the *Planck length*  $l_P \equiv \sqrt{\kappa/8\pi} = \mathcal{O}(10^{-33}\text{cm})$  and the associated *Planck time*  $t_P \equiv l_P = \mathcal{O}(10^{-44}\text{s})$ . The Planck length is the only combination of the three fundamental constants  $G$ ,  $c$  and  $\hbar$  with the dimensions of a length, and represents the length scale where both quantum physics and GR are relevant. Since we do not have yet a settled-down theory of quantum gravity, the Planck length should be regarded as the limit where our understanding of physics starts to be deeply speculative.

Equation (10.30) can be rewritten as

$$\rho(t) = \frac{1}{6\pi\gamma^2(t_P t)^2} = \frac{\rho_P}{6\pi\gamma^2} \left(\frac{t_P}{t}\right)^2, \quad (3.56)$$

where we have further introduced the Planck energy density  $\rho_P$  defined as

$$\rho_P \equiv \frac{1}{t_P^4} = \mathcal{O}(10^{93}\text{g/cm}^3) = \mathcal{O}(10^{117}\text{GeV/cm}^3). \quad (3.57)$$

The era between the Big Bang and the Planck time, called the Planck era, is expected to correspond to a quantum evolution of the Universe (see Chap. 10). In this temporal region, the predictivity of the Friedmann equation is lost in favor of non-deterministic concepts, like the Universe wave function. Nevertheless, the very small value of the Planck time (in

comparison to the Universe age) allows us to extrapolate backward the classical dynamics, disregarding here the finite nature of this value.

From the solution (3.54), we get the Hubble length and the cosmological horizon as

$$L_H = H^{-1} = \frac{a}{\dot{a}} = \frac{3\gamma}{2}t \quad (3.58)$$

$$d_H = a(t) \int_0^t \frac{dt'}{a(t')} = \frac{3\gamma}{3\gamma - 2}t. \quad (3.59)$$

These two lengths are of the same order for a Friedmann Universe and the Hubble time  $H^{-1}$  provides a good estimate for the age of the Universe. The cosmological horizon would become diverging when  $\gamma \leq 2/3$ , i.e. the same range of equation of state where the Universe would accelerate, according to Eq. (3.48). We will discuss later the possible physical interest of such peculiar matter behavior.

**The radiation dominated Universe** When addressing the Universe evolution near the singularity, the thermal energy overcomes the rest mass energy of any species (apart from Planck mass particles) and that the condition  $K_B T \gg m$  holds; in agreement with the discussion of Sec. 3.1.6, the energy content of the Universe is dominated by ultrarelativistic particles, with equation of state  $P = \rho/3$ , i.e.  $\gamma = 4/3$ . The scale factor takes the explicit form

$$a(t) = \sqrt{\frac{t}{\bar{t}}}, \quad (3.60)$$

that provides a coincidence of the Hubble length with the cosmological horizon, i.e.

$$L_H = d_H = 2t. \quad (3.61)$$

This expression allows us to rewrite the energy density (3.56) in the simple form

$$\rho_{\text{rad}} = \frac{3\rho_P}{32\pi} \left( \frac{2l_P}{d_H} \right)^2. \quad (3.62)$$

Finally, the temperature of the Universe is related by Eq. (3.39) to the radiation energy density and reads as

$$T(t) = \left( \frac{45\rho_P l_P^2}{4\pi^3 g_*(T_{\text{lim}})} \right)^{1/4} \frac{1}{\sqrt{d_H}}, \quad (3.63)$$

where  $T_{\text{lim}}$  denotes the temperature above which all fundamental particle species are present and in thermal equilibrium. For the Standard Model of

elementary particles, such temperature is  $T_{\text{lim}}^{\text{ST}} \sim 300 \text{ GeV}$  and the corresponding value of  $g_*$  is  $g_*^{\text{ST}} = 106.75$ .

The radiation dominated Universe is characterized by the birth of the Universe in the form of a hot Big Bang and the expansion from this initial state decelerates with time. In fact the Universe has a diverging geometrical velocity when it emerges from Big Bang (on a physical point of view, having a Planck value), accordingly to the law  $\dot{a} \propto 1/a$ , but it is drastically suppressed by a deceleration proportional to the expanding volume, i.e.  $\ddot{a} \propto -1/a^3$ . Such diverging character of the geometrical velocity is required to bring the volume of the Universe from  $V = 0$  in  $a = 0$  to a finite value, in any arbitrarily instant close to  $t = 0$ . In view of further developments, any physical length (for instance an inhomogeneity scale) evolves as the scale factor, i.e.  $L_{\text{phys}} \propto a$ . Since the cosmological horizon is finite and it behaves like  $d_{\text{H}} \propto a^2$ , close enough to the singularity each physical scale is super-horizon sized, i.e.

$$\lim_{a \rightarrow 0} \frac{d_{\text{H}}}{L_{\text{phys}}} = 0. \quad (3.64)$$

On the contrary, when the Universe expands, increasingly large physical scales enter the cosmological horizon, or equivalently the Hubble length. In general, the ratio  $d_{\text{H}}/L_{\text{phys}}$  scales like  $a^{\frac{3\gamma-2}{2}}$ , so the behavior just described takes place if  $\gamma > 2/3$  or, equivalently,  $w > -1/3$ . In general, if in the Universe are present both a radiation and a non-relativistic matter component, with equation-of-state parameters  $w = 1/3$  and  $w = 0$  respectively, at some point the Universe will be matter-dominated, even if it was radiation-dominated at the beginning. In fact, the radiation density  $\rho_{\text{rad}}$  scales like  $a^{-4}$  while the matter density  $\rho_{\text{m}}$  scales like  $a^{-3}$ , so the former decreases faster than the latter during the expansion.

The time  $t_{\text{eq}}$  when  $\rho_{\text{m}} = \rho_{\text{rad}}$  is called the time of “matter-radiation equality” and separates two different regimes in the evolution of the Universe. The corresponding redshift  $z_{\text{eq}} \equiv z(t_{\text{eq}})$  can be expressed in terms of measurable quantities, namely the present densities  $\rho_{\text{m}}^0$  and  $\rho_{\text{rad}}^0$  of matter and radiation. At the time of equivalence, by definition,  $\rho_{\text{m}}(t_{\text{eq}}) = \rho_{\text{rad}}(t_{\text{eq}})$  and, using the scaling of  $\rho_{\text{m}}$  and  $\rho_{\text{rad}}$  with redshift, one gets

$$\rho_{\text{m}}^0 (1 + z_{\text{eq}})^3 = \rho_{\text{rad}}^0 (1 + z_{\text{eq}})^4, \quad (3.65)$$

so that

$$1 + z_{\text{eq}} = \frac{\rho_{\text{m}}^0}{\rho_{\text{rad}}^0} = \frac{\Omega_{\text{m}}^0}{\Omega_{\text{rad}}^0}, \quad (3.66)$$

where  $\Omega_{\text{m}}^0 \equiv \rho_{\text{m}}^0/\rho_{\text{crit}}^0$  and  $\Omega_{\text{rad}}^0 \equiv \rho_{\text{rad}}^0/\rho_{\text{crit}}^0$  denote the present density parameters of matter and radiation, respectively. The observations (see Sec. 4.4) show that  $\Omega_{\text{m}}^0 \simeq 0.25$  and  $\Omega_{\text{rad}}^0 \simeq 8 \times 10^{-5}$  (the latter value includes the contribution from photons and three neutrino families), so that  $z_{\text{eq}} \simeq 3000$ .

### 3.2.3 The de Sitter Solution

A notable cosmological solution of Einstein equations is the de Sitter one, describing an empty Universe with a cosmological constant. Here, we will investigate the geometrical structure of the de Sitter space-time and its cosmological implementation. Let us consider a five-dimensional Minkowski space-time, with line element

$$ds^2 = \eta_{IJ} dx^I dx^J = \eta_{ij} dx^i dx^j - (dx^4)^2, \quad (3.67)$$

where the indices  $I, J$  run from 0 to 4. A sphere of radius  $l$  in such space-time admits the equation

$$\eta_{IJ} x^I x^J = \eta_{ij} x^i x^j - (x^4)^2 = l^2. \quad (3.68)$$

In order to calculate the metric induced on this sphere, we solve Eq. (3.68) to obtain  $x^4$  as

$$x^4 = \pm \sqrt{\eta_{ij} x^i x^j - l^2} \quad (3.69)$$

$$dx^4 = \pm \frac{x_i dx^i}{\sqrt{\eta_{ij} x^i x^j - l^2}}. \quad (3.70)$$

Hence, the line element (3.67) can be restated into that of a Minkowski sphere as

$$ds^2 = \left( \eta_{ij} - \frac{x_i x_j}{\eta_{kl} x^k x^l - l^2} \right) dx^i dx^j. \quad (3.71)$$

To characterize on a cosmological level this line element, let us consider the change of coordinates

$$x^0 = l \sinh\left(\frac{t}{l}\right), \quad x^\alpha = l \cosh\left(\frac{t}{l}\right) \xi^\alpha \quad (3.72)$$

where  $\xi^\alpha$  denotes three-dimensional spherical coordinates; in this new set of variables the line element (3.71) rewrites as

$$ds^2 = dt^2 - l^2 \cosh^2\left(\frac{t}{l}\right) d\Omega_3^2, \quad (3.73)$$

where  $d\Omega_3^2$  denotes  $dl_{\text{RW}}^2$  in the closed case. We are dealing with a non-stationary metric in a synchronous reference and, under the identification  $a(t) = l \cosh(t/l)$ , such line element coincides with the RW one (3.1), having the spatial geometry associated to  $K = 1$ , as in Eq. (3.2). This form of the metric describes the de Sitter space-time associated to an isotropic closed Universe. Indeed, this line element corresponds to a solution of the Friedmann Eq. (3.46), associated to the presence of a cosmological constant term  $\Lambda \equiv 3/l^2$ , i.e.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{l^2} - \frac{1}{a^2}. \quad (3.74)$$

The de Sitter model describes a singularity-free Universe, which collapses towards a minimal volume for  $a = l$  (but this value can be arbitrarily re-scaled) and then re-expands indefinitely. The disappearance of the singularity is a typical effect of the presence of a cosmological constant. In fact, also in the case of a flat RW dynamics we get the peculiar behavior associated to the scheme

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{l^2} \quad (3.75)$$

$$a(t) = a_0 e^{t/l}. \quad (3.76)$$

In this case the volume of the Universe vanishes only in the limit  $t \rightarrow -\infty$  and no real singularity appears. Such flat de Sitter model will be crucial in the study of the inflationary scenario, as we shall see in Chap. 5. An important feature of this model is that it would correspond to an equation of state  $P = -\rho$ , as discussed before in Sec. 3.1.6 and therefore it accelerates as  $\ddot{a} \propto a$ . Furthermore, this model has a constant Hubble length  $L_H = l$ , which significantly differs from the cosmological horizon, which is indeed diverging. Such discrepancy in the behavior of the two fundamental lengths confers an important dynamical and physical role to a de Sitter phase of the Universe evolution. In fact, during such phase, the physical lengths increase with respect to the physical Hubble horizon, even becoming super-horizon sized.

### 3.2.4 Hamiltonian dynamics of the isotropic Universe

We will now restate the dynamics of the FRW Universe in the framework of the Hamiltonian formulation of gravity, developed in Sec. 2.3. This approach is relevant for the canonical quantization of this cosmological model,



as we shall discuss in Sec. 10.1, when dealing with the Wheeler-DeWitt paradigm. The analysis below allows us to outline interesting features of the equations of motion in a generic time gauge.

Even if we could recover the Hamiltonian formulation for the highly symmetric case of the isotropic Universe by simply imposing the appropriate restrictions to the general formulation of Sec. 2.3, nevertheless we will address the problem starting from the Lagrangian approach. Indeed, the homogeneity hypothesis allows to integrate out the spatial dependence of the three-geometry from the action integral. In the case of a closed Universe ( $K = +1$ ), this spatial integral has value  $2\pi^2$  and in the flat and negative curvature cases (that correspond to an open space with infinite volume, unless a non-trivial topology is imposed), we choose for convenience to integrate over a fiducial volume of the same value. This is possible because such a choice does not affect the variational principle.

Let us consider the ADM line element for the homogeneous and isotropic model

$$ds^2 = N(t)^2 dt^2 - a(t)^2 dl_{\text{RW}}^2, \quad (3.77)$$

where the term  $dl_{\text{RW}}^2$  is provided by Eq. (3.2) and we included the lapse function  $N(t)$ , but not the shift vector  $N^\alpha$  because the former ensures the time reparametrization of the dynamics while the latter is forbidden by the isotropy condition (it behaves as a spatial vector and thus would single out a preferred direction). Differently from the generic case discussed in Sec. 2.3, here we derive the Hamiltonian equations starting from the gravitational action for the isotropic case, i.e. we will not use the Gauss-Codazzi relation (2.67). In the presence of an energy density  $\rho = \rho(a)$ , the action takes the form

$$S_{\text{RW}} = \int_{t_1}^{t_2} dt \left[ \frac{6\pi^2}{\kappa N} (\ddot{a}a^2 + a\dot{a}^2 + KN^2a) - 2\pi^2 N \rho a^3 \right]. \quad (3.78)$$

The geometrical part of this action is obtained by direct substitution of the RW metric (3.77) into the Einstein-Hilbert action, with the prescription above for the value of the spatial integral.

The second derivatives with respect to time of the scale factor can be eliminated using the relation  $a^2\ddot{a} = (a^2\dot{a})' - 2a\dot{a}^2$ . Integrating out the total derivative, Eq. (3.78) reads as

$$\begin{aligned} S_{\text{RW}} &= \int_{t_1}^{t_2} dt \mathcal{L}_{\text{RW}}(N, a, \dot{a}) \\ &= \int_{t_1}^{t_2} dt \left[ -\frac{6\pi^2}{\kappa N} a\dot{a}^2 + N \left( \frac{6\pi^2}{\kappa} K a - 2\pi^2 \rho a^3 \right) \right]. \end{aligned} \quad (3.79)$$

Let us perform a Legendre transformation, by defining the momentum  $p_a$  conjugate to the scale factor  $a$  as

$$p_a \equiv \frac{\partial \mathcal{L}_{\text{RW}}}{\partial \dot{a}} = -\frac{12\pi^2}{\kappa N} a \dot{a} \quad \Rightarrow \quad \dot{a} = -\frac{\kappa N}{12\pi^2} \frac{p_a}{a}. \quad (3.80)$$

Hence the Hamiltonian function  $N\mathcal{H}_{\text{RW}} \equiv p_a \dot{a} - \mathcal{L}_{\text{RW}}$  is obtained using Eq. (3.80) and the action (3.79) then rewrites as

$$\begin{aligned} S_{\text{RW}} &= \int_{t_1}^{t_2} dt (p_a \dot{a} - N\mathcal{H}_{\text{RW}}) \\ &= \int_{t_1}^{t_2} dt \left[ p_a \dot{a} - N \left( -\frac{\kappa}{24\pi^2} \frac{p_a^2}{a} - \frac{6\pi^2 K}{\kappa} a + 2\pi^2 \rho a^3 \right) \right]. \end{aligned} \quad (3.81)$$

The Hamilton equations explicitly read as

$$\dot{a} = N \frac{\partial \mathcal{H}_{\text{RW}}}{\partial p_a} = -\frac{\kappa N}{12\pi^2} \frac{p_a}{a}, \quad (3.82a)$$

$$\dot{p}_a = -N \frac{\partial \mathcal{H}_{\text{RW}}}{\partial a} = -\frac{N\kappa}{24\pi^2} \frac{p_a^2}{a^2} + \frac{6\pi^2 N K}{\kappa} - 2\pi^2 N \frac{d(\rho a^3)}{da}, \quad (3.82b)$$

while the variation of Eq. (3.81) with respect to  $N$  provides the following relation

$$\frac{p_a^2}{a^4} + \frac{144\pi^4}{\kappa^2 a^2} K = \frac{48\pi^4}{\kappa} \rho. \quad (3.83)$$

In the particular case  $N = 1$ , Eq. (3.83) coincides with the Friedmann Eq. (3.46) by virtue of Eq. (3.80). Meanwhile, substituting Eq. (3.82a) into Eq. (3.82b), and making use of the continuity equation (3.35) in the form

$$\frac{d(\rho a^3)}{da} = -3a^2 P, \quad (3.84)$$

the system (3.82) is equivalent to the space component of the Einstein Eqs. (3.47).

The variations with respect to the lapse function  $N$  and to the conjugate variables  $(a, p_a)$  provide three independent equations, yielding a complete representation of the dynamics of the isotropic Universe. Equation (3.84), which allows the identification of the pressure function in the equations of motion, has been inferred for the consistence of the Hamilton and Einstein systems, but in the Lagrangian formulation it must be thought of as preliminarily solved to give the relation  $\rho = \rho(a)$ .

In the Hamiltonian formulation, the dynamics of the isotropic Universe resembles that of a one-dimensional point particle, with generalized coordinate  $a$  and momentum  $p_a$ , whose dynamics is governed by a potential term

fixed via the Hamiltonian constraint (3.83) that also states the vanishing nature of the particle energy.<sup>3</sup> The correspondence between the Einstein equations and the equations for a particle motion is a general property of the homogeneous cosmologies, as we shall see in Chap. 8, and offers an interesting scenario for the canonical quantization of this cosmological dynamics.

### 3.3 Dissipative Cosmologies

It is interesting to analyze two relevant examples of dissipative effects that could be able to alter the standard dynamical features discussed in 3.1, i.e. the presence of a *bulk viscosity* in the cosmological fluid and the possibility of *matter creation* during the expansion of the Universe.

These two dissipative effects have different origin, but both provide the same dynamical feature of dealing with a negative pressure term. Such phenomenological issue can have deep implications on the evolution of the early Universe, both with respect to the nature of the singularity, and to the morphology of its causal structure. We will discuss in some details the cosmological pictures emerging from the inclusion of these phenomena, treated via suitable hydrodynamical approaches that make possible their description.

#### 3.3.1 Bulk viscosity

When discussing viscosity effects on the Universe dynamics, we have to distinguish the so-called *shear viscosity* from the bulk viscosity features. In fact, the former emerges as a result of the reciprocal friction that different parts of a system (for example, different layers of a fluid) exert each other. On the other hand, the latter is a macroscopic measurement of the reaction of a fluid to compression or rarefaction and can be related to the difficulty of the system (in our case the Universe) to maintain the thermal equilibrium as it expands.

From this simple description of the two viscosity terms, it appears natural that the shear viscosity coefficient must vanish for a perfectly homogeneous and isotropic cosmological fluid, for which friction among layers is absent by definition (this effect however survives in the presence of inho-

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<sup>3</sup>We stress however that we are dealing with a constrained Hamiltonian framework, i.e.  $\mathcal{H}$  is not straightforwardly related to the energy.

homogeneous perturbations, see Sec. 3.4). On the contrary, a bulk viscosity term is still permitted by the requirements of isotropy and homogeneity. In particular, the bulk viscosity coefficient expectedly receives significant contributions from the primordial phases of the Universe, when the expansion rate is very high and non-equilibrium features were possibly more relevant. Since such region of evolution corresponds to a non-trivial kinetic theory of the cosmological medium (in analogy with what discussed in Sec. 3.1.6), following the Landau School we consider a fluid-dynamical scheme. This approach is based on expressing the bulk viscosity coefficient as a function of thermodynamical parameters, mainly the energy density of the Universe. This way, we provide a phenomenological description of the off-equilibrium behavior of the primordial Universe which allows a simple enough characterization of the viscous cosmologies. Finally, we will not take into account effects due to the finite value of the speed of light, which are indeed relevant when a causal thermodynamics of the Universe is involved. These corrections, ensured by introducing in the problem suitable relaxation times, are important just near the Big Bang, but nevertheless the analysis presented below provides a proper description of the qualitative features emerging in a viscous isotropic cosmology.

The energy-momentum tensor of a fluid characterized by bulk viscosity takes the form

$$T_{ij}^{\text{BV}} = (\rho + P + \Pi_v)u_i u_j - (P + \Pi_v)g_{ij}, \quad (3.85)$$

where  $\Pi_v$  corresponds to a negative pressure-like contribution  $\Pi_v \equiv -\zeta \nabla_i u^i$ . The bulk viscosity coefficient  $\zeta$  can be expressed as proportional to some power  $s$  of the energy density of the fluid, i.e.  $\zeta \equiv \zeta_0 \rho^s$ ,  $\zeta_0$  and  $s$  being constant parameters of the phenomenological model under consideration.

Since in a co-moving frame the relation  $\nabla_i u^i = 3H$  holds, the continuity equation (3.35) for the viscous Universe writes as

$$\dot{\rho} = -3(\rho + P - 3\zeta_0 \rho^s H) H. \quad (3.86)$$

Limiting our attention to the case of a flat model, we can make use of the Friedmann equation (3.46) to replace the Hubble parameter inside the parentheses obtaining

$$\dot{\rho} = -3\left(\rho + P - \sqrt{3\kappa}\zeta_0 \rho^{s+1/2}\right) H. \quad (3.87)$$

By virtue of the equation of state (2.15), Eq. (3.87) eventually rewrites

$$\dot{\rho} = -3\rho\left(\gamma - \sqrt{3\kappa}\zeta_0 \rho^{s-1/2}\right) H. \quad (3.88)$$

From Eq. (3.88), the parameter  $s$  must obey the inequality  $s \leq 1/2$  for large values of the energy density. In fact, for  $s > 1/2$ , in the asymptotic limit  $\rho \rightarrow \infty$  (as expected near the Big Bang), the bulk viscosity term would dominate the continuity equation. Such situation would conflict with the well-grounded idea that viscous effects are a phenomenological outcome of small deviations from thermal equilibrium. To describe strong modifications of the equilibrium, probably occurred in the early Universe, the kinetic treatment would indeed be necessary.

On the other hand, for the case  $s < 1/2$ , the same asymptotic limit would imply a negligible contribution towards the Big Bang singularity. Thus, on a dynamical level, the most interesting situation corresponds to the value  $s = 1/2$ , for which the continuity equation rewrites as

$$\dot{\rho} = -3\rho \left( \gamma - \sqrt{3\kappa}\zeta_0 \right) H \equiv -3\gamma'\rho H. \quad (3.89)$$

In the present model, the main effect of the bulk viscosity on the dynamics of the isotropic Universe consists of re-scaling the index  $\gamma$  as  $\gamma \rightarrow \gamma' \equiv \gamma - \sqrt{3\kappa}\zeta_0$ .

In general, the bulk viscosity is expected to represent only a perturbation to the perfect fluid behavior and in turn  $\zeta_0$  to be small. This implies that  $\gamma' \simeq \gamma = 4/3$  and that no strong deviations from the standard radiation dominated behavior are induced.

### 3.3.2 Matter creation in the expanding Universe

The rapid time variation of the scale factor  $a(t)$  during the early phases of the Universe evolution implies that the cosmic gravitational field, like any other rapidly changing field, is able to create particles, as a result of a quantum effect induced on the microscopic matter fields. On a fundamental level, this rate of particle creation could be calculated by analyzing the states of certain fields, living in the expanding Universe. However, the passage from a microscopic description of the physics of a non-stationary background to a phenomenological macroscopic characterization of the cosmological fluid is highly non-trivial. Thus, like in the case of bulk viscosity, the problem of matter creation, induced by the Universe expansion, has to be addressed on the basis of a phenomenological model.

The proper framework to treat the cosmological particle creation was identified by Prigogine during the '60s, and involves treating the Universe as an open thermodynamical system. We have seen in Sec. 3.1.6 how the continuity equation (3.35) is equivalent to the first principle of thermody-

namics applied to a spatially isothermal and isoentropic co-moving volume. In order to take into account the possibility of particle creation, we have to restate such thermodynamical principle including a non-zero chemical potential  $\mu$

$$dU = \delta Q - PdV + \mu dN, \quad (3.90)$$

where  $N$  denotes the particle number. Expressing  $U = \rho V$  and using the second law of thermodynamics<sup>4</sup>  $\delta Q = TdS = T(\sigma dN + Nd\sigma)$  (where  $S$  denotes the entropy inside the co-moving volume  $V$  and  $\sigma \equiv S/N$  is the relative entropy per particle), the above equation rewrites as

$$d\rho = \frac{TN}{V}d\sigma - (\rho + P)\frac{dV}{V} + (T\sigma + \mu)\frac{dN}{V}. \quad (3.91)$$

Observing that the chemical potential is defined as the Gibbs free-energy  $G \equiv U + PV - TS$  per unit particle, i.e.  $G = \mu N$ , we have that

$$(\rho + P)V = (T\sigma + \mu)N \quad (3.92)$$

so that Eq. (3.91) rewrites as

$$d\rho = \frac{TN}{V}d\sigma - (\rho + P)\left(1 - \frac{d \ln N}{d \ln V}\right)\frac{dV}{V}. \quad (3.93)$$

We are now led to replace the standard request for an isoentropic Universe with the weaker condition that only the entropy per particle be conserved, i.e.  $d\sigma = d(S/N) = 0$ . The entropy and the particle number in a co-moving volume are then linked by a direct proportionality ( $S \propto N$ ). Since in the Universe, treated as an open thermodynamical system, the number of particles changes along the evolution, the total entropy is no longer a conserved quantity and varies accordingly to the processes of matter creation. Dealing with a conserved entropy per particle reduces Eq. (3.93) to the simpler form

$$d\rho = -(\rho + P)\left(1 - \frac{d \ln N}{d \ln V}\right)\frac{dV}{V}. \quad (3.94)$$

In other words, the effect of matter creation is described by an additional negative pressure term  $\Pi_{\text{mc}} \equiv -(\rho + P)d \ln N / d \ln V$ . The analysis of the dynamical implications associated to such negative pressure requires the

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<sup>4</sup>We remind the reader that the Second Law of Thermodynamics states that the equality  $\delta Q = TdS$  holds *for reversible processes only*. However, since  $U$ ,  $V$ ,  $S$  and  $N$  are functions of state, and thus independent of the particular process, Eq. (3.91) and the following ones hold for any transformation. On the other hand, in the presence of irreversible processes like particle creation, entropy is not conserved ( $dS \neq 0$ ) but the expansion is still adiabatic ( $\delta Q = 0$ ).

specification of a phenomenological expression for  $d \ln N / d \ln V$ . In this respect, let us rewrite Eq. (3.94) in the form of a continuity equation as

$$\dot{\rho} = -3(\rho + P) \left( 1 - \frac{1}{3} \frac{d \ln N}{d \ln a} \right) H. \quad (3.95)$$

This equation generalizes Eq. (3.35) in the presence of matter creation and offers the proper tool to get the modified relation between the energy density and the cosmic scale factor  $\rho = \rho(a)$ , necessary to solve the Friedmann equation (which here, as in the bulk viscosity case, retains the usual form (3.46)).

To fix the form of the particle creation rate, we focus our attention on the case of a flat RW model ( $K = 0$ ) and observe that, since the particles are created by the time variation of the cosmological field, a suitable expression for the ansatz we are searching for reads as

$$\frac{1}{3} \frac{d \ln N}{d \ln a} \equiv \left( \frac{H}{\bar{H}} \right)^{2b} = \left( \frac{\rho}{\bar{\rho}} \right)^b, \quad (3.96)$$

where  $b$  is a free parameter of the theory and the two constants  $\bar{H}$  and  $\bar{\rho}$  are related by  $\bar{H}^2 = \kappa \bar{\rho} / 3$ . Substituting this ansatz in Eq. (3.95), and eliminating the synchronous time in favor of the dimensionless variable  $x \equiv 3 \ln a$ , we get the final form of the revised continuity equation

$$\frac{d\rho}{dx} = -(\rho + P) \left[ 1 - \left( \frac{\rho}{\bar{\rho}} \right)^b \right]. \quad (3.97)$$

Considering a generic equation of state of the form (2.15), Eq. (3.97) takes the integrable form

$$\frac{d\rho}{dx} = -\gamma \rho \left[ 1 - \left( \frac{\rho}{\bar{\rho}} \right)^b \right] \quad (3.98)$$

and its solution, restated as  $\rho(a)$ , explicitly reads as

$$\rho(a) = \frac{\bar{\rho}}{[1 + A a^{3b\gamma}]^{\frac{1}{b}}}, \quad (3.99)$$

where  $A$  is a constant. The dynamical implications of such relation can be qualitatively inferred without solving the Friedmann equation. The most relevant modification arises in the finite constant value  $\bar{\rho}$ , taken as the energy density for  $a \rightarrow 0$ , in correspondence to any choice of the index  $\gamma$ . In the present scenario with matter creation, the Universe was born by a singularity-free solution, characterized by a de Sitter phase emerging from  $t \rightarrow -\infty$ , where the scale factor would asymptotically vanish. However,

for a sufficiently large value of  $a$ , i.e. when  $Aa^{3b\gamma} \gg 1$ , the energy density regains the standard form  $\rho \propto a^{-3\gamma}$  and the features of an isentropic Universe with a fixed value of  $N$  are recovered. We see how the present picture has the merit to reconcile a non-singular de Sitter-like Universe in the earliest cosmological phases with a standard picture of the later evolution. Of course, one can expect that in order to completely reproduce the Standard Cosmological Model at later times, a certain fine-tuning of the parameters would be required.

We conclude this section by stressing how the analysis of the dissipative cosmologies developed so far was based on the study of the continuity equation modified to account for an additional negative pressure term. Indeed, in Sec. 3.1.6, we saw how this equation macroscopically accounts for the microscopic structure of the Boltzmann equation and therefore our phenomenological approaches are nothing more than an effective theory of a really complex microphysics, that describes only the general features of the investigated phenomena.

Hence, replacing the old thermostatic pressure with the restated pressure term, the Einstein equations for the isotropic Universe preserve the same structure studied in Sec. 3.2.1. Such similarity of the isotropic dynamics in the dissipative and non-dissipative cases is at the ground of the qualitative dynamical considerations outlined in the two subsections above.

### 3.4 Inhomogeneous Fluctuations in the Universe

#### 3.4.1 *The meaning of cosmological perturbations*

The idea of a perfectly homogeneous and isotropic Universe is mainly a mathematical notion, hardly reconciled with the morphology of the present Universe, unless (initially small) deviations from homogeneity are allowed on different physical scales. Indeed, as we shall see in Chap. 4, at galactic scale the Universe clumpiness appears as a deep modification of the RW geometry. At this level, we speak of non-linear nature of the density fluctuations, because the ratio of the matter fluctuations  $\delta\rho$  to the average background density  $\bar{\rho}$  is much larger than unity, i.e.  $\delta\rho/\bar{\rho} \gg 1$ . In this situation, the dynamics of the Universe cannot be recovered from a linear perturbation theory of the RW metric and non-linear features of the Einstein equations have to be involved. On sufficiently large scales, greater than some hundred of Megaparsecs, the linearity is recovered and the notion



of a homogeneous and isotropic Universe becomes solid. Even if the fluctuations on a given scale are very large, we expect that they were smaller in the past, so that in the early Universe the fluctuations were still in the linear regime. Thus, when  $\delta\rho/\bar{\rho} \ll 1$ , like it is today at the scale of superclusters of galaxies or like it was at the time of recombination (the very small temperature fluctuations of the CMB trace correspondingly small density contrasts) at all scales of cosmological interest, we have a natural approach to describe the evolution of inhomogeneous perturbations, either if they are scalar fluctuations in the energy density, pressure and velocity distributions, as well as if they take the form of vector or tensor disturbances, describing inhomogeneous rotational velocity fields (vortices) or gravitational waves, respectively.

Before facing the proper approaches to perturbation dynamics, we provide a physical insight on why not only inhomogeneities are necessary to explain the observed Universe, but they are also an unavoidable feature of the primordial Universe. In fact, even if we start with a very smooth and isotropic Universe at very early times, a certain degree of fluctuations of the thermodynamical parameters on different Hubble volumes is necessarily implied. The independent evolution of such fine-tuned micro-causal regions has to magnify the relative fluctuations, since the gravitational instability amplifies the density contrast on a given scale, enforcing the clumpiness of the cosmological fluid. At the end, a certain significant perturbation spectrum has to be generated at later stages of evolution even if we start with a very fine-tuned homogeneous Universe.

To shed light on the impossibility to have a perfectly uniform Universe, we stress that it cannot remain in thermal equilibrium arbitrarily close to the initial singularity. Indeed, the mean free path of a given particle species  $l_s$  is provided by the relation

$$l_s \simeq \frac{1}{n_s \sigma_s}, \quad (3.100)$$

$n_s$  and  $\sigma_s$  being the number density of the species and the cross-section of the interactions that are maintaining the equilibrium. Since  $n_s \propto T^3$  and typically  $\sigma_s \propto \langle E_s \rangle^{-2} \sim 1/T^2$  (here  $\langle E_s \rangle$  denotes the mean thermal energy of the particles), then the mean free path scales like  $l_s \propto a$ . Since the Hubble length decreases faster toward the singularity, a primordial instant has to exist when the mean free path is, on average, greater than the microphysical scale and, in practice, it diverges. In such scenario, the Universe cannot be regarded as in a real thermal equilibrium and the request of smoothness does not appear well-grounded. A numerical estimate provides

the constraint  $T > \mathcal{O}(10^{16} \text{ GeV})$  to allow a similar situation. Since this region of the Universe evolution is expected to be in a pre-inflationary phase, the modern idea on the origin of primordial fluctuation origin escapes, in part, this argument, as we shall see in Chap. 5. On the other hand, it well stresses the necessity of a certain degree of inhomogeneity even when the primordial Universe evolution is addressed on a purely classical level (for a discussion of quantum gravity implications in this respect, see Chap. 10). On the basis of the considerations above, we see why the request for a fine-tuned uniform Universe (say immediately after the Planck era) can constitute a physical puzzle, indeed at the ground of the horizon paradox that we will discuss in Chap. 5.

In order to face the problem of cosmological perturbations within the framework of GR, we have to write the metric tensor in the form

$$g_{ik} = \bar{g}_{ik} + \gamma_{ik} \quad (3.101)$$

where  $\bar{g}_{ik}$  denotes the RW background term, while  $\gamma_{ik}$  represents a small perturbation ( $\gamma_{ik} \ll \bar{g}_{ik}$ ) which describes the ripples of the space-time associated to the inhomogeneous features. Starting from such metric tensor, one constructs the first order linearized Einstein tensor  $\delta G_i^k$ , properly simplified by the choice of a suitable gauge, such as the synchronous gauge

$$\gamma_{i0} = 0. \quad (3.102)$$

On the same level, we can perturb the cosmological fluid by a density fluctuation  $\delta\rho$ , a pressure fluctuation  $\delta P$  and a four-velocity disturbance  $\delta u^i$ , never co-moving with the background flow. From these quantities, together with  $\gamma_{ik}$ , we can build up the first-order linearized energy-momentum tensor  $\delta T_i^k$  and hence fixing the perturbation dynamics by the linearized equations  $\delta G_i^k = \kappa \delta T_i^k$ . Of course, this set of equations is coupled to the background dynamics and, in the case the matter behavior is provided by a kinetic theory, we have to couple also the Boltzmann equation, itself separated into background and first-order components.

As far as the scale of perturbations is much smaller than the Hubble horizon and the fluid velocity fields are non-relativistic, we can neglect all GR effects and then we deal with a very simplified system of equations, describing the gravitational instability of the Universe. In such non-relativistic scheme, we can neglect the back-reaction of the matter perturbations on the full tensorial structure of the gravitational field and we limit our attention to the Newton potential  $\Phi$  only (i.e.  $\gamma_{00}$ ). The expansion can be taken into account as a background effect.

The system of non-relativistic equations consists of the continuity equation<sup>5</sup>

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (3.103)$$

( $\vec{v}$  being the velocity field) which ensures the mass conservation of the fluid. The second Newton law of mechanics takes the form of the Euler equation

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} + \frac{1}{\rho} \vec{\nabla} P + \vec{\nabla} \Phi = 0, \quad (3.104)$$

while the relation between the matter distribution and the Newton potential is provided by the Poisson equation

$$\nabla^2 \Phi = \frac{\kappa}{2} \rho, \quad (3.105)$$

where  $\nabla^2$  is the Laplace operator.

In Sec. 3.4.3, we will take into account the Universe expansion in the perturbation dynamics by retaining the same scheme outlined above. This analysis, in spite of its non-relativistic character, has a real cosmological predictivity and it is at the ground of a significant physical insight.

In order to familiarize with the Jeans approach to the gravitational stability of a uniform static fluid, as discussed in the next subsection, we propose a qualitative analysis of the mechanism by which a density contrast evolves. Let us consider an exceeding matter fluctuation  $\delta\rho > 0$  over the background level  $\bar{\rho}$ . The self-gravity of this matter blob tends to induce a collapse toward the formation of a structure, but such a force is contrasted by the pressure gradients, stretching and flattening the denser region. The fate of the matter density, collapse or disappearance, depends on the net resultant of such two forces and, if they are near equilibrium, the fluctuation will expectably oscillate.

We can estimate the condition for a collapse in the case of a spherical inhomogeneity. In fact, the disturbances in the fluid density can lead to the collapse only if their propagation velocity, coinciding with the sound speed  $v_s = (dP/d\rho)^{1/2}$ , is smaller than the limiting value

$$v_s \sim \sqrt{\frac{\kappa M}{8\pi l}}, \quad (3.106)$$

$l$  being the radius of the spherical matter blob, say its linear size. Observing that the mass of the system can be written as  $M \simeq (\bar{\rho} + \delta\rho)l^3 \simeq \bar{\rho}l^3$ , the

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<sup>5</sup>In accordance with the standard literature on the topic, we adopt the vectorial notation. Operation are intended as in the standard Euclidean case.

collapse condition can take place only for matter condensations having a linear size (i.e. a matter content) sufficiently high, namely

$$l \sim v_s \sqrt{\frac{8\pi}{\kappa \bar{\rho}}} . \quad (3.107)$$

This length scale is called the *Jeans length* and represents a natural separation between the perturbations which are obliged to undergo acoustic oscillations and those sufficiently large to be able to collapse because of their self-gravitation and to become the seeds for structure formation. Below we will provide a more rigorous mathematical derivation of this simple scenario, by stressing its relevance in a cosmological implementation.

### 3.4.2 The Jeans length in a static uniform fluid

Let us now study the linear dynamics of a perturbation  $(\delta\rho, \delta\vec{v})$  around a configuration of the fluid characterized by a uniform matter density  $\bar{\rho} = \text{const.}$ , zero velocity  $\bar{v} = 0$  and constant sound velocity  $v_s = \text{const.}$ , so that  $\bar{P} = \text{const.}$  and  $\delta P = v_s^2 \delta\rho$ . Under these hypotheses, the continuity Eq. (3.103) to first order in perturbation terms takes the form

$$\partial_t \delta\rho + \bar{\rho} \vec{\nabla} \cdot \vec{v} = 0 . \quad (3.108)$$

In the same way, the Euler equation (3.104) acquires the linearized form

$$\partial_t \delta\vec{v} + \frac{v_s^2}{\bar{\rho}} \vec{\nabla} \delta\rho + \vec{\nabla} \delta\Phi = 0 , \quad (3.109)$$

where  $\delta\Phi$  denotes the gravitational potential associated to the perturbation and is related to  $\delta\rho$  via the Poisson equation as

$$\nabla^2 \delta\Phi = \frac{\kappa}{2} \delta\rho . \quad (3.110)$$

Taking the divergence of the vector in Eq. (3.109), we get the scalar equation

$$\partial_t \left( \vec{\nabla} \cdot \delta\vec{v} \right) + \frac{v_s^2}{\bar{\rho}} \nabla^2 \delta\rho + \nabla^2 \delta\Phi = 0 . \quad (3.111)$$

Using Eq. (3.108) to express the divergence of  $\delta\vec{v}$  and Eq. (3.110) to remove the Laplacian of  $\delta\Phi$ , we arrive at a linear second order equation in  $\delta\rho$  only

$$\partial_t^2 \delta\rho - v_s^2 \nabla^2 \delta\rho - \frac{\kappa}{2} \bar{\rho} \delta\rho = 0 . \quad (3.112)$$

The solution of this equation can be expanded in a Fourier integral so that we can study the behavior of a generic mode

$$\delta\rho = \delta\rho_{\text{in}} \exp \left\{ i \left[ \frac{2\pi}{\lambda} \vec{n} \cdot \vec{r} - \omega t \right] \right\} , \quad \delta\rho_{\text{in}} = \text{const.} , \quad (3.113)$$

where  $\lambda$  and  $\omega$  denote the wavelength and the frequency of a plane wave, respectively, while  $\vec{n}$  denotes the unit vector of its propagation direction. Substituting Eq. (3.113) into Eq. (3.112), we get the key relation

$$\omega = \pm 2\pi \sqrt{\frac{v_s^2}{\lambda^2} - \frac{\kappa \bar{\rho}}{8\pi^2}}. \quad (3.114)$$

As far as the wavelength of the perturbation is sufficiently small, the frequency is real and the corresponding mode oscillates; on the other hand, when the Jeans condition

$$\lambda > \lambda_J \equiv v_s \sqrt{\frac{8\pi^2}{\kappa \bar{\rho}}} \quad (3.115)$$

holds (this value of  $\lambda_J$  closely resembles the previous estimate (3.107)), the frequency becomes an imaginary number  $\omega = \pm i\alpha$ . In this region of wavelengths, exponentially growing modes appear for  $\omega = i\alpha$  (as well as absorbed ones for  $\omega = -i\alpha$ ) and the corresponding matter perturbation can collapse, according to

$$\delta\rho = \delta\rho_{\text{in}} \exp \left\{ \frac{2\pi i}{\lambda} \vec{n} \cdot \vec{r} + \alpha t \right\}. \quad (3.116)$$

In view of the hypotheses considered here, this analytical behavior is predictive if  $\delta\rho/\bar{\rho} \ll 1$  only. The relevance of this analysis relies on the information that matter perturbations with a linear dimension much larger than the Jeans length are gravitationally unstable and proceed towards a collapse in stable structures. Despite this result has been obtained for a static uniform medium, the Jeans length can be extrapolated to the case of an expanding Universe, preserving its present meaning: this is the main task of the following section.

### 3.4.3 The Jeans length in an expanding Universe

In order to extend the previous Jeans analysis to an expanding background, we have to search for an exact solution of the three equations (3.103), (3.104) and (3.105), under the hypothesis of a matter dominated Universe  $P \ll \rho$ ,  $\gamma \sim 1$ .

For  $\gamma = 1$ , the unperturbed energy density  $\bar{\rho}$  scales as  $1/a^3$  and hence  $\partial_t \bar{\rho} \equiv \dot{\bar{\rho}} = -3H\bar{\rho}$ , and then the continuity Eq. (3.103) provides

$$\vec{\nabla} \cdot \vec{v} = 3H \quad \Rightarrow \quad \vec{v} = H\vec{r}. \quad (3.117)$$

The background velocity of the cosmological fluid corresponds to the geometrical velocity of expansion. On the same level, Eq. (3.105) gives the relation

$$\vec{\partial} \bar{\Phi} = \frac{\kappa}{6} \bar{\rho} \vec{r}. \quad (3.118)$$

The current choice for the background quantities automatically solves Eq. (3.104), as far as Eq. (3.48) is taken into account for the case  $P \simeq 0$ . Finally, the background dynamics is completed by Eq. (3.46), which provides the scale factor evolution.

Linearizing Eq. (3.103) with respect to perturbations, we get

$$\partial_t \delta \rho + 3H \delta \rho + H \vec{r} \cdot \vec{\nabla} \delta \rho + \bar{\rho} \vec{\nabla} \cdot \delta \vec{v} = 0, \quad (3.119)$$

while the Euler Eq. (3.104) up to first order reads as

$$\partial_t \delta \vec{v} + H \delta \vec{v} + H(\vec{r} \cdot \vec{\nabla}) \delta \vec{v} = -\frac{v_s^2}{\bar{\rho}} \vec{\nabla} \delta \rho - \vec{\nabla} \delta \Phi. \quad (3.120)$$

Actually, the sound velocity is no longer a constant as in the previous Subsection. Once assigned a polytropic relation for the perturbation behavior, like  $P \propto \rho^{4/3+\epsilon}$  ( $\epsilon > 0$ ), we get the time evolution of the sound velocity as  $v_s^2 \propto \bar{\rho}^{1/3+\epsilon}$ . Finally, we get the perturbed Poisson equation

$$\nabla^2 \delta \Phi = \frac{\kappa}{2} \delta \rho \quad \Rightarrow \quad \vec{\nabla} \delta \Phi = \left( \frac{\kappa}{6} \delta \rho \right) \vec{r}. \quad (3.121)$$

Taking the curl of Eq. (3.120) and observing that the right-hand side vanishes, the following relation follows

$$\partial_t \left( \vec{\nabla} \wedge \delta \vec{v} \right) + 2H \vec{\nabla} \wedge \delta \vec{v} + H(\vec{r} \cdot \vec{\nabla}) \left( \vec{\nabla} \wedge \delta \vec{v} \right) = 0. \quad (3.122)$$

This equation involves only the rotational component of the velocity perturbation and, as we shall see below, it carries a clear physical information. Analogously, we can take the divergence of equation (3.120) obtaining

$$\partial_t \left( \vec{\nabla} \cdot \delta \vec{v} \right) + 2H \vec{\nabla} \cdot \delta \vec{v} + H(\vec{r} \cdot \vec{\nabla}) \left( \vec{\nabla} \cdot \delta \vec{v} \right) = -\frac{v_s^2}{\bar{\rho}} \nabla^2 \delta \rho - \frac{\kappa}{2} \delta \rho, \quad (3.123)$$

where we also made use of Eq. (3.121). If we now define the fractional density contrast as  $\delta \equiv \delta \rho / \bar{\rho}$ , then Eq. (3.119) takes the simpler form

$$\partial_t \delta + H \vec{r} \cdot \vec{\nabla} \delta + \vec{\nabla} \cdot \delta \vec{v} = 0. \quad (3.124)$$

It is now convenient to expand the spatial dependence of the quantities  $\delta$  and  $\vec{v}$  in plane waves of the expanding Universe. More precisely, let us

analyze the behavior of each single mode of the Fourier transform of the spatial dependence of these objects, i.e.

$$\delta = \delta_k(t) \exp \{ i k_{\text{phys}} \vec{n} \cdot \vec{r} \} \quad (3.125)$$

$$\delta \vec{v} = \vec{v}_k(t) \exp \{ i k_{\text{phys}} \vec{n} \cdot \vec{r} \} . \quad (3.126)$$

Here  $k_{\text{phys}} = 2\pi/\lambda_{\text{phys}}$  denotes the wave number associated to the physical wavelength  $\lambda_{\text{phys}}$  which is proportional to the scale factor since  $d(\lambda_{\text{phys}}^{-1})/dt = -H\lambda_{\text{phys}}^{-1}$ . In view of the expansion above as in (3.125), Eqs. (3.122), (3.123) and (3.124) rewrite respectively as

$$\dot{v}_k^{\parallel} + H v_k^{\parallel} = 0 \quad (3.127a)$$

$$\dot{v}_k^{\perp} + H v_k^{\perp} = \frac{i}{k_{\text{phys}}} \left( -k_{\text{phys}}^2 v_s^2 + \frac{\kappa}{2} \bar{\rho} \right) \delta_k \quad (3.127b)$$

$$\dot{\delta}_k = -i k_{\text{phys}} v_k^{\perp} , \quad (3.127c)$$

where the superscripts  $\parallel$  and  $\perp$  denote the velocity components, parallel ( $v_k^{\parallel} \equiv (\vec{n} \cdot \vec{v}_k) \vec{n}$ ) and transverse ( $v_k^{\perp} \equiv \vec{v}_k - v_k^{\parallel}$ ) to the plane wave direction  $\vec{n}$ , respectively.

Equation (3.127a) states that the rotational modes of the perturbations decay because of Universe expansion, i.e.  $v_k^{\parallel} \propto 1/a$ . The compressional modes, described by the remaining two equations, have a non-trivial dynamics and we can shed light on it by deriving Eq. (3.127c) with respect to  $t$ , getting

$$\ddot{\delta}_k = -i k_{\text{phys}} (\dot{v}_k^{\perp} - H v_k^{\perp}) . \quad (3.128)$$

Using Eq. (3.127b) and taking into account again Eq. (3.127c), we obtain the final fundamental equation for the density contrast

$$\ddot{\delta}_k + 2H\dot{\delta}_k + \left( v_s^2 k_{\text{phys}}^2 - \frac{\kappa}{2} \bar{\rho} \right) \delta_k = 0 , \quad (3.129)$$

which reproduces the Jeans dispersion relation (3.115) for  $a = \text{const}$ .

In order to obtain some information about the perturbation fate, we have to specify Eq. (3.129) in correspondence to a given cosmological model. Since the region of Universe evolution during which the Jeans mass is the relevant scale for the matter dynamics is the one characterized by a negligible spatial curvature, then we can deal with the  $K = 0$  model, without significant loss of generality. For such model with  $P \simeq 0$ , the scale factor behaves as  $a \propto t^{2/3}$  and therefore the energy density takes the time dependence

$$\bar{\rho} = \frac{4}{3\kappa t^2} , \quad (3.130)$$

while the sound velocity  $v_s$  acquires the time evolution  $v_s \sim t^{-1/3-\epsilon}$ . Hence Eq. (3.129) explicitly rewrites as

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k + \left( \frac{C}{t^{2+2\epsilon}} - \frac{2}{3t^2} \right) \delta_k = 0, \quad (3.131)$$

where the constant  $C$  takes account for the amplitude of the first term in parentheses of Eq. (3.129). As time goes by, this term decreases more rapidly than  $\kappa\bar{\rho}/2$  and becomes, sooner or later, negligible. Such situation corresponds just to the time dependent Jeans condition

$$\lambda_{\text{phys}} \gg v_s \sqrt{\frac{8\pi^2}{\kappa\bar{\rho}}}. \quad (3.132)$$

It is immediate to recognize that, under such conditions Eq. (3.131) admits the solution  $\delta_k \sim a(t) \sim t^{2/3}$ , which increases with time. A more careful analysis, based on the behavior of the Bessel functions entering the solution of Eq. (3.131) would refine the inequality (3.132) by a multiplying factor  $\sqrt{3/2}$ .

Let us now briefly discuss the solution of the equation for the density contrast during the radiation era. We are interested in the behavior of the density contrast of matter components (for example baryons), while the main contribution to total density of the Universe is given by the photon component. In this case, Eq. (3.129) takes the form (assuming that the radiation component is unperturbed)

$$\ddot{\delta}_k + \frac{\dot{\delta}_k}{t} + \left( \frac{4\pi^2 v_s^2}{\lambda_{\text{phys}}^2} - \frac{\kappa}{2} \bar{\rho} \right) \delta_k = 0, \quad (3.133)$$

where in this case  $\bar{\rho}$  and  $\delta_k$  denote the background density of matter and the corresponding density fluctuation, respectively.

Let us consider the limit of very large wavelength of the perturbation, i.e. perturbations well above the Jeans length, so that the first term in brackets can be neglected. Moreover, the density of photons  $\rho_\gamma$  during the radiation-dominated era can be calculated from the Friedmann equation and is equal to

$$\rho_\gamma = \frac{3}{4\kappa t^2}. \quad (3.134)$$

The matter density  $\bar{\rho}$  can be written as  $\epsilon\rho_\gamma$  with  $\epsilon \ll 1$ , and then the second term in brackets is of order  $\epsilon\delta_k/t^2$ . This is much smaller than the first derivative term, which is of order  $\delta_k/t^2$ , so that it can also be neglected. The equation can thus be approximated as

$$\ddot{\delta}_k + \frac{\dot{\delta}_k}{t} = 0, \quad (3.135)$$



and admits the solution

$$\delta_k = \delta_k(t_{\text{in}}) \left[ 1 + A \log \left( \frac{t}{t_{\text{in}}} \right) \right]. \quad (3.136)$$

Thus the general solution is the superposition of a constant plus a logarithmic term. This means that the perturbations can grow at most logarithmically, i.e. much slower than they would do in a matter-dominated Universe. This result, known as *Meszaros effect*, is at the basis of the fact that the onset of structure formation has to wait until the Universe becomes matter dominated.<sup>6</sup> The physical reason for this result can be identified with the faster expansion during the radiation-dominated era and then to its more relevant damping effect.

The analysis here addressed shows that the Jeans length retains the same physical meaning and essentially the same structure even in the expanding matter-dominated Universe. Furthermore, we got the important information about the capability of the gravitational instability to magnify density contrasts during the matter-dominated era, according to the law  $\delta \sim a$ , and on the reduced (logarithmic with time) growth of fluctuations during the radiation-dominated era. Such behavior explains how the small perturbations of the early matter-dominated Universe can increase to reach the non-linear regime and therefore they are the seeds from which the structures in the presently observed Universe have been formed.

### 3.5 General Relativistic Perturbation Theory

In this Section we will deal with the full, general relativistic treatment of the evolution of small perturbations over a RW background. As we shall see, this involves writing the metric as the RW metric plus a (small) perturbation. Then we compute the perturbed Einstein tensor  $\delta G_{ij}$ , the perturbed energy-momentum tensor  $\delta T_{ij}$ , and finally obtain the perturbed Einstein equations governing the evolution of the metric, as well as of the matter-energy. In the following we will neglect the spatial curvature, i.e. we will suppose to deal with perturbations at scales much smaller than the curvature radius of the Universe. This is not a strong limitation, since we know that  $|\Omega_K| \lesssim 10^{-2}$ , so that the present curvature radius is at least a

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<sup>6</sup>Of course, modes that enter the horizon much earlier than the time of matter-radiation equality can grow appreciably during the radiation-dominated era, and this has to be taken into account in detailed calculations.

hundred times larger than the Hubble length, and the flat approximation is appropriate even for modes that are well above the horizon (i.e. perturbations with wavelength much bigger than  $L_H$ ). The approximation was even better in the past, when the curvature was less important.

### 3.5.1 Perturbed Einstein equations

We take the unperturbed metric to be the flat RW metric as in Eq. (3.2) where we adopt Cartesian coordinates for the spatial part of the metric, i.e.  $h_{\alpha\beta}^{\text{RW}} = \delta_{\alpha\beta}$ . The only non-vanishing Christoffel symbols are<sup>7</sup>

$$\bar{\Gamma}_{\alpha\beta}^0 = a\dot{a}\delta_{\alpha\beta} \quad (3.137a)$$

$$\bar{\Gamma}_{0\beta}^\alpha = \frac{\dot{a}}{a}\delta_\beta^\alpha \quad (3.137b)$$

while the non-vanishing components of the unperturbed Ricci tensor are

$$\bar{R}_{00} = -3\frac{\ddot{a}}{a}, \quad (3.138a)$$

$$\bar{R}_{\alpha\beta} = (2\dot{a}^2 + a\ddot{a})\delta_{\alpha\beta}. \quad (3.138b)$$

Let us consider a small perturbation  $\gamma_{ij}$  to the RW metric  $\bar{g}_{ij}$  as in Eq. (3.101), and adopt the gauge (3.102) to fix some of the components of  $\gamma_{ij}$ . Equation (3.102) states that the perturbed metric is also synchronous, i.e.  $g_{00} = \bar{g}_{00} = 1$  and  $g_{0\alpha} = \bar{g}_{0\alpha} = 0$  and for this reason such gauge is called *synchronous gauge*. It was the one used in the original paper in 1946 by Lifshitz on cosmological perturbations, and it is still very popular today, mainly because of the numerical stability of the equations written in this gauge. A drawback of this choice is that the condition (3.102) does not fully exhaust the gauge freedom: this gives rise to spurious, unphysical gauge modes among the solutions to the equations that need to be recognized and eliminated.

The perturbation to the inverse of the spatial metric  $\gamma^{\alpha\beta}$  is related to  $\gamma_{\alpha\beta}$  by  $\gamma^{\alpha\beta} = -a^{-4}\gamma_{\alpha\beta}$ , as it can be noted by writing  $g^{\alpha\beta} = \bar{g}^{\alpha\beta} + \gamma^{\alpha\beta}$  and imposing that  $g_{\alpha\mu}g^{\mu\beta} = \delta_\alpha^\beta$ .

As anticipated above, our goal is to write the perturbed Einstein equations that we will analyze in the more convenient form

$$\delta R_{ij} = \kappa \left[ \delta T_{ij} - \frac{1}{2} \delta (g_{ij} T) \right]. \quad (3.139)$$

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<sup>7</sup>Also in this Section, overbar is adopted to denote an unperturbed quantity.

The perturbations to the Christoffel symbols are given, to first order in the small quantities  $\gamma_{\alpha\beta}$ , by

$$\delta\Gamma_{00}^0 = 0 \quad (3.140a)$$

$$\delta\Gamma_{\alpha 0}^0 = 0 \quad (3.140b)$$

$$\delta\Gamma_{00}^\alpha = 0 \quad (3.140c)$$

$$\delta\Gamma_{\alpha\beta}^0 = -\frac{1}{2}\dot{\gamma}_{\alpha\beta} \quad (3.140d)$$

$$\delta\Gamma_{0\beta}^\alpha = -\frac{1}{2a^2} \left( \dot{\gamma}_{\alpha\beta} - 2\frac{\dot{a}}{a}\gamma_{\alpha\beta} \right) \quad (3.140e)$$

$$\delta\Gamma_{\beta\mu}^\alpha = -\frac{1}{2a^2} (\partial_\mu \gamma_{\alpha\beta} + \partial_\beta \gamma_{\mu\alpha} - \partial_\alpha \gamma_{\beta\mu}) \quad (3.140f)$$

The perturbations to the Ricci tensor can be expressed in terms of the  $\delta\Gamma_{jk}^i$  as

$$\begin{aligned} \delta R_{ij} = \delta R^k{}_{ikj} = & \partial_\ell (\delta\Gamma_{ij}^\ell) - \partial_j (\delta\Gamma_{i\ell}^\ell) + (\delta\Gamma_{m\ell}^\ell) \Gamma_{ij}^m \\ & + \Gamma_{m\ell}^\ell (\delta\Gamma_{ij}^m) - (\delta\Gamma_{mj}^\ell) \Gamma_{i\ell}^m - \Gamma_{mj}^\ell (\delta\Gamma_{i\ell}^m). \end{aligned} \quad (3.141)$$

Using Eqs. (3.140) above, we get

$$\delta R_{00} = \frac{1}{2a^2} \left[ \ddot{\gamma}_{\alpha\alpha} - 2\frac{\dot{a}}{a}\dot{\gamma}_{\alpha\alpha} + 2 \left( \frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a} \right) \gamma_{\alpha\alpha} \right] \quad (3.142a)$$

$$\delta R_{0\alpha} = \frac{1}{2} \partial_t \left[ \frac{1}{a^2} (\partial_\alpha \gamma_{\beta\beta} - \partial_\beta \gamma_{\alpha\beta}) \right] \quad (3.142b)$$

$$\begin{aligned} \delta R_{\alpha\beta} = & -\frac{1}{2}\ddot{\gamma}_{\alpha\beta} + \frac{1}{2}\frac{\dot{a}}{a}(\dot{\gamma}_{\alpha\beta} - \dot{\gamma}_{\mu\mu}\delta_{\alpha\beta}) + \frac{\dot{a}^2}{a^2}(-2\gamma_{\alpha\beta} + \gamma_{\mu\mu}\delta_{\alpha\beta}) \\ & + \frac{1}{2a^2}(\partial_\alpha\partial_\beta\gamma_{\mu\mu} + \partial_\mu\partial_\mu\gamma_{\alpha\beta} - \partial_\mu\partial_\alpha\gamma_{\beta\mu} - \partial_\mu\partial_\beta\gamma_{\alpha\mu}) \end{aligned} \quad (3.142c)$$

Here, and for the rest of this Chapter, we adopt the convention that repeated indices are always summed, even if they are both covariant or contravariant. For example, we have that  $\gamma_{\mu\mu} = \gamma_{\mu\nu}\delta_\mu^\nu = \sum_{\mu=1,3}\gamma_{\mu\mu}$ . Of course, this is not the trace  $\gamma_\alpha^\alpha = \gamma_{\alpha\beta}g^{\alpha\beta}$  of  $\gamma_{\alpha\beta}$ , although the two are related by  $\gamma_\alpha^\alpha = -\gamma_{\mu\mu}/a^2$ .

Let us turn our attention to the right-hand side of the Einstein equations. Defining the source tensor  $S_{ij}$

$$S_{ij} \equiv T_{ij} - \frac{1}{2}g_{ij}T, \quad (3.143)$$

the perturbation  $\delta S_{ij}$  to the source tensor is given by

$$\delta S_{ij} = \delta T_{ij} - \frac{1}{2}\gamma_{ij}\bar{T} - \frac{1}{2}\bar{g}_{ij}\delta T. \quad (3.144)$$

Neglecting for the moment dissipative processes like viscosity or heat conduction, we consider for the energy-momentum tensor  $T_{ij}$  the perfect fluid form (2.14), that we rewrite here for convenience

$$T_{ij} = (\rho + P)u_i u_j - P g_{ij}. \quad (3.145)$$

The perturbation to the energy-momentum tensor can be described in terms of the perturbed density  $\rho = \bar{\rho} + \delta\rho$ , pressure  $P = \bar{P} + \delta P$  and four-velocity  $u_i = \bar{u}_i + \delta u_i$  of the cosmological fluid. In particular, one has

$$\delta T_{ij} = (\delta\rho + \delta P)\bar{u}_i \bar{u}_j + (\bar{\rho} + \bar{P})(\bar{u}_i \delta u_j + \bar{u}_j \delta u_i) - \bar{P}\gamma_{ij} - \delta P \bar{g}_{ij}. \quad (3.146)$$

We recall that the zeroth-order four-velocity is given by  $\bar{u}_i = (1, 0, 0, 0)$  and moreover, the identity  $u_i u^i = 1$  (that is true at all orders) can be perturbed to give

$$0 = \delta(u_i u^i) = \delta(g^{ij} u_i u_j) = \delta(g^{00} u_0 u_0) = 2\delta u_0. \quad (3.147)$$

This yields for the perturbations to the energy-momentum tensor

$$\delta T_{00} = \delta\rho, \quad (3.148a)$$

$$\delta T_{0\alpha} = (\bar{\rho} + \bar{P})\delta u_\alpha, \quad (3.148b)$$

$$\delta T_{\alpha\beta} = -\bar{P}\gamma_{\alpha\beta} + a^2 \delta P \delta_{\alpha\beta}. \quad (3.148c)$$

The trace of the energy-momentum tensor is given by  $T = \rho - 3P$  so that the associated perturbation is given by

$$\delta T = \delta\rho - 3\delta P. \quad (3.149)$$

By putting together Eqs. (3.142), (3.148) and (3.149), we finally get the perturbed Einstein equations

$$(00) \quad \ddot{\gamma}_{\alpha\alpha} - 2\frac{\dot{a}}{a}\dot{\gamma}_{\alpha\alpha} + 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)\gamma_{\alpha\alpha} = \kappa a^2(\delta\rho + 3\delta P), \quad (3.150a)$$

$$(0\alpha) \quad \partial_t \left[ \frac{1}{2a^2} (\partial_\alpha \gamma_{\beta\beta} - \partial_\beta \gamma_{\alpha\beta}) \right] = \kappa (\bar{\rho} + \bar{P}) \delta u_\alpha, \quad (3.150b)$$

$$\begin{aligned} (\alpha\beta) \quad & \ddot{\gamma}_{\alpha\beta} - \frac{\dot{a}}{a}(\dot{\gamma}_{\alpha\beta} - \dot{\gamma}_{\mu\mu}\delta_{\alpha\beta}) + 2\frac{\dot{a}^2}{a^2}(2\gamma_{\alpha\beta} - \gamma_{\mu\mu}\delta_{\alpha\beta}) \\ & - \frac{1}{a^2}(\partial_\alpha \partial_\beta \gamma_{\mu\mu} + \partial_\mu \partial_\mu \gamma_{\alpha\beta} - \partial_\alpha \partial_\mu \gamma_{\mu\beta} - \partial_\mu \partial_\beta \gamma_{\alpha\mu}) \\ & = \kappa [(\bar{\rho} - \bar{P})h_{\alpha\beta} - a^2(\delta\rho - \delta P)\delta_{\alpha\beta}]. \end{aligned} \quad (3.150c)$$

These equations can be further simplified by introducing the rescaled metric perturbation  $\tilde{\gamma}_{\alpha\beta} \equiv \gamma_{\alpha\beta}/a^2$ , so that Eqs. (3.150) rewrite as

$$(00) \quad \ddot{\tilde{\gamma}}_{\alpha\alpha} + 2\frac{\dot{a}}{a}\dot{\tilde{\gamma}}_{\alpha\alpha} = \kappa(\delta\rho + 3\delta P), \quad (3.151a)$$

$$(0\alpha) \quad \partial_\alpha \dot{\tilde{\gamma}}_{\beta\beta} - \partial_\beta \dot{\tilde{\gamma}}_{\alpha\beta} = 2\kappa(\bar{\rho} + \bar{P})\delta u_\alpha, \quad (3.151b)$$

$$\begin{aligned} (\alpha\beta) \quad \ddot{\tilde{\gamma}}_{\alpha\beta} + \frac{\dot{a}}{a}(3\dot{\tilde{\gamma}}_{\alpha\beta} + \dot{\tilde{\gamma}}_{\mu\mu}\delta_{\alpha\beta}) - \frac{1}{a^2}(\partial_\alpha\partial_\beta\tilde{\gamma}_{\mu\mu} + \partial_\mu\partial_\mu\tilde{\gamma}_{\alpha\beta} \\ - \partial_\alpha\partial_\mu\tilde{\gamma}_{\mu\beta} - \partial_\mu\partial_\beta\tilde{\gamma}_{\alpha\mu}) = \kappa(\delta P - \delta\rho)\delta_{\alpha\beta}. \end{aligned} \quad (3.151c)$$

In deriving Eq. (3.151c), we have used the two background equations (3.46) and (3.47) to express the unperturbed density  $\bar{\rho}$  and pressure  $\bar{P}$  in terms of the scale factor  $a(t)$  and of its time derivatives.

### 3.5.2 *Scalar-vector-tensor decomposition and Fourier expansion*

It is convenient to decompose the perturbation  $\tilde{\gamma}_{\alpha\beta}$  into its scalar, vector and tensor components. This corresponds to split  $\tilde{\gamma}_{\alpha\beta}$  into parts that behave differently under spatial rotations with the advantage that the scalar, vector and tensor components are decoupled and thus evolve one independently from the other. The spatial metric perturbation (as well as any other symmetric tensor) can be decomposed as

$$\tilde{\gamma}_{\alpha\beta} = \frac{\tilde{\gamma}}{3}\delta_{\alpha\beta} + \tilde{\gamma}_{\alpha\beta}^{\parallel} + \tilde{\gamma}_{\alpha\beta}^{\perp} + \tilde{\gamma}_{\alpha\beta}^{\text{T}}, \quad (3.152)$$

where  $\tilde{\gamma} = \tilde{\gamma}_{\alpha\alpha}$  is the Euclidean trace of  $\tilde{\gamma}_{\alpha\beta}$ . The three components  $\tilde{\gamma}_{\alpha\beta}^{\parallel}$ ,  $\tilde{\gamma}_{\alpha\beta}^{\perp}$  and  $\tilde{\gamma}_{\alpha\beta}^{\text{T}}$  are respectively called the longitudinal, solenoidal and transverse parts of  $\tilde{\gamma}_{\alpha\beta}$ , and are traceless by construction. In addition, they satisfy the conditions

$$\epsilon_{\alpha\beta\mu}\partial_\beta\partial_\nu\tilde{\gamma}_{\nu\mu}^{\parallel} = 0 \quad (3.153a)$$

$$\partial_\alpha\partial_\beta\tilde{\gamma}_{\alpha\beta}^{\perp} = 0 \quad (3.153b)$$

$$\partial_\alpha\tilde{\gamma}_{\alpha\beta}^{\text{T}} = 0. \quad (3.153c)$$

In other words, the first condition states that the divergence of the longitudinal part  $\tilde{\gamma}_{\alpha\beta}^{\parallel}$  is longitudinal itself (i.e. curl-free); the second means that the divergence of the solenoidal part  $\tilde{\gamma}_{\alpha\beta}^{\perp}$  is transverse (i.e. divergence-free); the third that the transverse part  $\tilde{\gamma}_{\alpha\beta}^{\text{T}}$  is, as the name suggests, transverse.<sup>8</sup>

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<sup>8</sup>In fact, another possible nomenclature is to call  $\tilde{\gamma}_{\alpha\beta}^{\parallel}$  doubly longitudinal,  $\tilde{\gamma}_{\alpha\beta}^{\perp}$  singly longitudinal and  $\tilde{\gamma}_{\alpha\beta}^{\text{T}}$  doubly transverse.

From the conditions above, it follows that  $\tilde{\gamma}_{\alpha\beta}^{\parallel}$  can be expressed in terms of a scalar field  $\mu$  as

$$\tilde{\gamma}_{\alpha\beta}^{\parallel} = \left( \partial_{\alpha} \partial_{\beta} - \frac{1}{3} \delta_{\alpha\beta} \partial_{\nu} \partial_{\nu} \right) \mu, \quad (3.154)$$

while  $\tilde{\gamma}_{\alpha\beta}^{\perp}$  can be expressed in terms of a transverse vector field  $V_{\alpha}$

$$\tilde{\gamma}_{\alpha\beta}^{\perp} = \partial_{\alpha} V_{\beta} + \partial_{\beta} V_{\alpha}; \quad \partial_{\alpha} V_{\alpha} = 0. \quad (3.155)$$

The components  $\tilde{\gamma}$  and  $\tilde{\gamma}_{\alpha\beta}^{\parallel}$  (or equivalently  $\mu$ ) represent the *scalar* part of  $\tilde{\gamma}_{\alpha\beta}$ ,  $\tilde{\gamma}_{\alpha\beta}^{\perp}$  (or equivalently  $V_{\alpha}$ ) represents its *vector* part, and finally  $\tilde{\gamma}_{\alpha\beta}^{\text{T}}$ , that cannot be obtained from the gradient of a scalar or a vector, represents its *tensor* part.

A similar decomposition can be done for the velocity perturbation  $\delta u_{\alpha}$  which can be divided into a parallel part  $\delta u_{\alpha}^{\parallel}$  and a transverse, divergenceless part  $\delta u_{\alpha}^{\perp}$ . The parallel can be expressed as the divergence of a scalar field  $\delta u$ , so that we have

$$\delta u_{\alpha} = \partial_{\alpha}(\delta u) + \delta u_{\alpha}^{\perp} \quad (3.156)$$

$$\partial_{\alpha} \delta u_{\alpha}^{\perp} = 0. \quad (3.157)$$

After the metric perturbation and the velocity field have been decomposed in this way, the Einstein equations decompose as ( $\nabla^2 \equiv \partial_{\mu} \partial_{\mu}$ )

- *Scalar modes:*

$$\ddot{\tilde{\gamma}} + 2\frac{\dot{a}}{a}\dot{\tilde{\gamma}} = \kappa(\delta\rho + 3\delta P) \quad (3.158a)$$

$$\partial_{\alpha} (\dot{\tilde{\gamma}} - \nabla^2 \dot{\mu}) = 3\kappa (\bar{\rho} + \bar{P}) \partial_{\alpha} \delta u \quad (3.158b)$$

$$\ddot{\tilde{\gamma}} - \nabla^2 \ddot{\mu} + 3\frac{\dot{a}}{a} (2\dot{\tilde{\gamma}} - \nabla^2 \dot{\mu}) - \frac{1}{a^2} \nabla^2 (\tilde{\gamma} - \nabla^2 \mu) = 3\kappa (\delta P - \delta\rho) \quad (3.158c)$$

$$\partial_{\alpha} \partial_{\beta} \left( \ddot{\mu} + 3\frac{\dot{a}}{a} \dot{\mu} + \frac{1}{3a^2} \nabla^2 \mu - \frac{\tilde{\gamma}}{3a^2} \right) = 0. \quad (3.158d)$$

The scalar modes are compressional modes, involving the perturbations to the density, pressure and to the irrotational part of the velocity field of the fluid. They are thus the most interesting, since they are related to the growth of density fluctuations and then to the linear phase of structure formation. We note that we have four equations for the five unknowns  $\tilde{\gamma}$ ,  $\mu$ ,  $\delta\rho$ ,  $\delta P$  and  $\delta u$ . In order to close the system we have, if possible, to specify the equations of state  $P = P(\rho)$ . For example, for particles with very low thermal

velocities (like cold dark matter) we can simply put  $P = 0$  (and  $\delta u = 0$  as well), while for ultrarelativistic particles like photons or light neutrinos we can use  $P = \rho/3$ . In general, however, a proper kinetic treatment should be performed to close the system, coupling the Einstein equations to the Boltzmann equation describing the evolution of the perturbations to the energy-momentum tensor, as we shall see at the end of this section.

- *Vector modes:*

$$\nabla^2 \dot{V}_\alpha = -2\kappa (\bar{\rho} + \bar{P}) \delta u_\alpha^\perp, \quad (3.159a)$$

$$\partial_\alpha \left( \ddot{V}_\beta + 3\frac{\dot{a}}{a} \dot{V}_\beta \right) = 0. \quad (3.159b)$$

The vector modes represent the vorticity components of the fluid. As we will show below, the conservation of the energy-momentum tensor implies that for a perfect fluid the quantity  $(\bar{\rho} + \bar{P})\delta u_\alpha^\perp$  decays as  $a^{-3}$ . For this reason vector modes are in general not very relevant for the cosmological evolution.

- *Tensor modes:*

$$\ddot{\gamma}_{\alpha\beta}^T + 3\frac{\dot{a}}{a}\dot{\gamma}_{\alpha\beta}^T - \frac{1}{a^2}\nabla^2\gamma_{\alpha\beta}^T = 0. \quad (3.160)$$

It is easy to recognize that this is the wave equation with a damping term proportional to  $H$ . In fact, the tensor modes represent gravitational waves propagating in the expanding Universe.

**Einstein equations in Fourier space.** The next step in simplifying the perturbation equations is to Fourier transform the spatial dependence of all the quantities involved, i.e. the metric and stress-energy perturbations. For example, we consider the Fourier transform  $\mu_k(t)$  of  $\mu(\vec{x}, t)$

$$\mu_k(t) = \int \mu(\vec{x}, t) e^{i\vec{k}\cdot\vec{x}} d^3x \quad (3.161)$$

and analogously for  $\tilde{\gamma}$ ,  $\tilde{\gamma}^\parallel$ ,  $\tilde{\gamma}^T$ ,  $\delta\rho$ ,  $\delta P$ ,  $\delta u$  and  $\delta u_\alpha^\perp$ . With a slight abuse of notation, we drop the subscript  $k$  and keep using the same symbol for a given quantity and its Fourier transform. By doing this, the above equations rewrite as

- *Scalar modes:*

$$\ddot{\gamma} + 2\frac{\dot{a}}{a}\dot{\gamma} = \kappa(\delta\rho + 3\delta P) \quad (3.162a)$$

$$\dot{\gamma} + k^2\dot{\mu} = 3\kappa(\bar{\rho} + \bar{P})\delta u \quad (3.162b)$$

$$\ddot{\gamma} + k^2\ddot{\mu} + 3\frac{\dot{a}}{a}(2\dot{\gamma} + k^2\dot{\mu}) + \frac{k^2}{a^2}(\tilde{\gamma} + k^2\mu) = 3\kappa(\delta P - \delta\rho) \quad (3.162c)$$

$$\ddot{\mu} + 3\frac{\dot{a}}{a}\dot{\mu} - \frac{k^2}{3a^2}\mu - \frac{\tilde{\gamma}}{3a^2} = 0. \quad (3.162d)$$

- *Vector modes:*

$$k^2\dot{V}_\alpha = 2\kappa(\bar{\rho} + \bar{P})\delta u_\alpha^\perp \quad (3.163a)$$

$$\dot{V}_\alpha + 3\frac{\dot{a}}{a}\dot{V}_\alpha = 0. \quad (3.163b)$$

- *Tensor modes:*

$$\ddot{\gamma}_{\alpha\beta}^T + 3\frac{\dot{a}}{a}\dot{\gamma}_{\alpha\beta}^T + \frac{k^2}{a^2}\tilde{\gamma}_{\alpha\beta}^T = 0. \quad (3.164)$$

### 3.5.3 Perturbed conservation equations

Now we will derive the conservation equations satisfied by  $\delta\rho$ ,  $\delta P$  and  $\delta u_\alpha$  for a perfect fluid. The conservation of the energy momentum tensor  $\nabla_i T_j^i = 0$  gives, to first order in perturbations, the expression

$$\delta(\nabla_i T_j^i) = \delta(\partial_i T_j^i) - \bar{\Gamma}_{ji}^\ell \delta T_\ell^i + \bar{\Gamma}_{\ell i}^i \delta T_j^\ell - \delta \Gamma_{ji}^\ell \bar{T}_\ell^i + \delta \Gamma_{\ell i}^i \bar{T}_j^\ell = 0. \quad (3.165)$$

The  $j = 0$  component of this equation gives the equation of energy conservation

$$\partial_t \delta T_0^0 + \partial_\alpha \delta T_0^\alpha + 3\frac{\dot{a}}{a}\delta T_0^0 - \frac{\dot{a}}{a}\delta T_\alpha^\alpha = \frac{\dot{\gamma}}{2}(\bar{\rho} + \bar{P}), \quad (3.166)$$

while the  $j = \alpha$  component gives the equation of momentum conservation

$$\partial_t \delta T_\alpha^0 + \partial_\beta \delta T_\alpha^\beta - a\dot{a}\delta T_0^\alpha + 2\frac{\dot{a}}{a}\delta T_\alpha^0 = 0. \quad (3.167)$$

For a perfect fluid, the conservation equations rewrite as

$$\delta\rho + 3\frac{\dot{a}}{a}(\delta\rho + \delta P) - \frac{\bar{\rho} + \bar{P}}{a^2}\partial_\alpha \delta u_\alpha = \frac{\dot{\gamma}}{2}(\bar{\rho} + \bar{P}), \quad (3.168a)$$

$$\partial_t [(\bar{\rho} + \bar{P})\delta u_\alpha] + 3\frac{\dot{a}}{a}(\bar{\rho} + \bar{P})\delta u_\alpha - \partial_i \delta P = 0. \quad (3.168b)$$



Recalling the decomposition of  $\delta u_\alpha$  into its longitudinal and transverse components, we can split the conservation equations into scalar and vector parts. The equation of energy conservation is already purely scalar and simply rewrites as

$$\dot{\delta\rho} + 3\frac{\dot{a}}{a}(\delta\rho + \delta P) - \frac{\bar{\rho} + \bar{P}}{a^2}\nabla^2\delta u = \frac{\dot{\bar{\gamma}}}{2}(\bar{\rho} + \bar{P}). \quad (3.169)$$

The part of the momentum conservation equation that is proportional to a longitudinal vector (and then to the derivative of a scalar) is

$$\partial_t [(\bar{\rho} + \bar{P})\delta u] + 3\frac{\dot{a}}{a}(\bar{\rho} + \bar{P})\delta u - \delta P = 0. \quad (3.170)$$

Finally, the part of the momentum conservation equation that is proportional to a transverse vector is

$$\partial_t [(\bar{\rho} + \bar{P})\delta u_\alpha^\perp] + 3\frac{\dot{a}}{a}(\bar{\rho} + \bar{P})\delta u_\alpha^\perp = 0. \quad (3.171)$$

In particular, this last equation implies that the quantity  $(\bar{\rho} + \bar{P})\delta u_\alpha^\perp$  scales like  $1/a^3$ .

The conservation equations are not independent of the field equations. However, in the case of non-interacting fluids, they are satisfied separately by each component. For example, if we consider a Universe filled by non-relativistic matter and by radiation not interacting with each other, we can write separate energy and momentum conservation equations for matter and for radiation. In this case, the conservation equations really carry additional information with respect to the field equations. Moreover, when performing a numerical integration of the field equations, it is useful to use the conservation equation to check the validity of the numerical solution.

### 3.5.4 Gauge modes

Here we will show how the synchronous condition  $\gamma_{i0} = 0$  does not exhaust all the gauge freedom in the Einstein equations. Let us consider a coordinate transformation

$$x^i \rightarrow x'^i = x^i + \epsilon^i(x^j) \quad (3.172)$$

where  $\epsilon$  is a small quantity. This induces a transformation in the metric tensor given by

$$g'_{ij}(x') = g_{\ell m}(x) \frac{\partial x^\ell}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} = g_{ij}(x) - g_{i\ell} \frac{\partial \epsilon^\ell}{\partial x^j} - g_{\ell j} \frac{\partial \epsilon^\ell}{\partial x^i}. \quad (3.173)$$

In this equation,  $x$  and  $x'$  that appear on the opposite sides correspond to the same physical point, which has different coordinate labels in the

two reference frames. However, we are interested in the change, after the transformation, of the value  $g_{ij}$  evaluated at the same coordinate value  $x$ , which will correspond in general to two different physical points. The values of the metric tensor in the two points  $x$  and  $x'$  (both referring to the transformed frame) are related by

$$g'_{ij}(x') = g'_{ij}(x) + \frac{\partial g'_{ij}(x)}{\partial x^\ell} \epsilon^\ell(x). \quad (3.174)$$

Then, putting together Eq. (3.173) and Eq. (3.174) we get

$$g'_{ij}(x) = g_{ij}(x) - \bar{g}_{i\ell} \frac{\partial \epsilon^\ell}{\partial x^j} - \bar{g}_{\ell j} \frac{\partial \epsilon^\ell}{\partial x^i} - \frac{\partial \bar{g}_{ij}}{\partial x^\ell} \epsilon^\ell, \quad (3.175)$$

where  $\bar{g}_{ij}$  is the unperturbed metric. In other words, this last formula is taking into account that the metric tensor evaluated at a given coordinate value is changing because the metric tensor at a given physical point is changing according to Eq. (3.173), and also because the physical point associated to the coordinate value has changed according to Eq. (3.174).

Now we can attribute all the change in  $g_{ij}$  to a change in the perturbation  $\gamma_{ij}$ , so to leave the unperturbed metric unchanged

$$\begin{aligned} \Delta\gamma_{ij}(x) &\equiv \gamma'_{ij}(x) - \gamma_{ij}(x) = g'_{ij}(x) - g_{ij}(x) \\ &= -\bar{g}_{i\ell} \frac{\partial \epsilon^\ell}{\partial x^j} - \bar{g}_{\ell j} \frac{\partial \epsilon^\ell}{\partial x^i} - \frac{\partial \bar{g}_{ij}}{\partial x^\ell} \epsilon^\ell. \end{aligned} \quad (3.176)$$

As usual, the spatial part of  $\epsilon_i$  can be decomposed into a parallel and a transverse part as

$$\epsilon_\alpha = \epsilon_\alpha^\parallel + \epsilon_\alpha^\perp = \partial_\alpha E + \epsilon_\alpha^\perp. \quad (3.177)$$

Let us consider a gauge transformation

$$\epsilon_0(x, t) = F(x^\gamma) \quad E(x, t) = -a^2(t)F(x^\gamma) \int \frac{dt}{a^2(t)} \quad (3.178)$$

with  $\epsilon_\alpha^\perp$  arbitrary. Using Eq. (3.176), it is straightforward to check that after this transformation the new metric is still synchronous, i.e.  $\Delta\gamma_{00} = \Delta\gamma_{\alpha 0} = 0$ . However, the spatial part of the metric perturbation changes by the quantity

$$\begin{aligned} \Delta\gamma_{\alpha\beta} &= 2\dot{a}a\epsilon_0\delta_{\alpha\beta} - \partial_\alpha\epsilon_\beta - \partial_\beta\epsilon_\alpha \\ &= 2\dot{a}a\epsilon_0\delta_{\alpha\beta} - 2\partial_\alpha\partial_\beta E - (\partial_\alpha\epsilon_\beta^\perp + \partial_\beta\epsilon_\alpha^\perp) \\ &= \left(2\dot{a}a\epsilon_0 - \frac{2}{3}\nabla^2 E\right)\delta_{\alpha\beta} - 2\left(\partial_\alpha\partial_\beta - \frac{\delta_{\alpha\beta}}{3}\nabla^2\right)E - (\partial_\alpha\epsilon_\beta^\perp + \partial_\beta\epsilon_\alpha^\perp), \end{aligned} \quad (3.179)$$

where the expression on the last line corresponds to the by now familiar decomposition of  $\Delta\gamma_{\alpha\beta}$  into a trace, a longitudinal and a solenoidal part in that order (the transverse part is missing, however). It is now clear that the scalar, vector and tensor perturbations transform according to (we switch to the rescaled perturbation  $\tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}/a^2$ )

$$\Delta\tilde{\gamma} = 6\frac{\dot{a}}{a}\epsilon_0 - \frac{2}{a^2}\nabla^2 E, \quad (3.180a)$$

$$\Delta\mu = -\frac{2E}{a^2}, \quad (3.180b)$$

$$\Delta V_\alpha = -\frac{\epsilon_\alpha^\perp}{a^2}, \quad (3.180c)$$

$$\Delta\tilde{\gamma}_{\alpha\beta}^T = 0. \quad (3.180d)$$

Tensor perturbations are unaffected by gauge transformations, though, on the other hand, the scalar and vector perturbations *do not*, in general, remain unchanged under a gauge transformation. In particular, under the transformation defined by Eq. (3.178) the scalar perturbations transform as

$$\Delta\tilde{\gamma} = 6\frac{\dot{a}}{a}F + 2\nabla^2 F \int \frac{dt}{a^2(t)}, \quad (3.181a)$$

$$\Delta\mu = 2F \int \frac{dt}{a^2(t)}. \quad (3.181b)$$

The components of the perturbed energy-momentum tensor transform in the same way as  $\gamma_{\alpha\beta}$  (see Eq. (3.176)), i.e.

$$\Delta(\delta T_{ij}) = -\bar{T}_{i\ell}\frac{\partial\epsilon^\ell}{\partial x^j} - \bar{T}_{\ell j}\frac{\partial\epsilon^\ell}{\partial x^i} - \frac{\partial\bar{T}_{ij}}{\partial x^\ell}\epsilon^\ell, \quad (3.182)$$

ensuring that the field equations remain unchanged after the gauge transformation.

Equation (3.182) implies that, under the transformation in Eq. (3.178), the scalar energy-momentum perturbations  $\delta\rho$ ,  $\delta P$  and  $\delta u$  transform as

$$\Delta(\delta\rho) = -F\dot{\bar{\rho}}, \quad \Delta(\delta P) = -F\dot{\bar{P}}, \quad \Delta(\delta u) = -F. \quad (3.183)$$

The fact that the field equations are invariant under a gauge transformation, but nevertheless not the metric and energy-momentum tensor perturbations, implies that, if  $\tilde{\gamma}$  or  $\delta\rho$  are solutions, then  $\tilde{\gamma} + \Delta\tilde{\gamma}$  and  $\delta\rho + \Delta(\delta\rho)$  (with  $\Delta\tilde{\gamma}$  and  $\Delta(\delta\rho)$  given by Eq. (3.181a) and by the first of Eq. (3.183), respectively) are also solutions. Moreover, since the field equations are linear,  $\Delta\tilde{\gamma}$  and  $\Delta(\delta\rho)$  are solutions themselves. Of course, they cannot represent any physical disturbance to the metric or to the density field, since they

can be put to zero by a suitable coordinate transformation. However, the gauge ambiguity can be removed if there is a component of the fluid, like cold dark matter, whose particles have very small thermal velocities and are thus essentially at rest in the co-moving frame. In this case, we know that for such a fluid  $P$ ,  $\delta P$  and  $\delta u$  all vanish, so that the right, physical gauge can be found as the one where  $P = \delta P = \delta u = 0$ .

### 3.5.5 Evolution of scalar modes

Here we study the evolution of scalar perturbations, with particular regard to their behavior outside the horizon. The evolution of scalar modes is described by the four Eqs. (3.158) once an equation of state  $P = P(\rho)$  has been specified. However, it is more convenient to trade two of the field equations for the two conservation Eqs. (3.169) and (3.170); in particular, considering Eq. (3.158a) the conservation equations allows to reduce by one the number of unknowns, since the scalar perturbation  $\mu$  does not appear.

Considering adiabatic perturbations (that is appropriate for a single fluid) we have that  $\delta P = v_s^2 \delta \rho$ , where  $v_s$  is the sound speed of the fluid. Equations (3.158a), (3.169) and (3.170) can be rewritten as

$$\ddot{\tilde{\gamma}} + 2\frac{\dot{a}}{a}\dot{\tilde{\gamma}} - 3\frac{\dot{a}^2}{a^2}(1 + 3v_s^2)\delta = 0, \quad (3.184a)$$

$$\dot{\delta} + 3\frac{\dot{a}}{a}(v_s^2 - w)\delta - (1 + w)\left(\frac{\nabla^2}{a^2}\delta u + \frac{\dot{\tilde{\gamma}}}{2}\right) = 0, \quad (3.184b)$$

$$\dot{\delta u} - 3\frac{\dot{a}}{a}w\delta u - \frac{c_s^2}{1 + w}\delta = 0. \quad (3.184c)$$

These three equations form a closed system for the three unknowns  $\tilde{\gamma}$ ,  $\delta$  and  $\delta u$ . The sound speed can be related to the equation of state parameter by noting that

$$v_s^2 = \frac{dP}{d\rho} = w + \rho \frac{dw}{d\rho}. \quad (3.185)$$

In most cases of interest,  $w$  is constant (for example, for non-relativistic matter  $w = 0$ , while for ultrarelativistic matter  $w = 1/3$ ), so that we can safely assume that  $v_s^2 = w$  and the equations reduce to (switching to

$k$ -space, so that  $\nabla^2 \rightarrow -k^2$ )

$$\ddot{\gamma} + 2\frac{\dot{a}}{a}\dot{\gamma} - 3\frac{\dot{a}^2}{a^2}(1+3w)\delta = 0, \quad (3.186a)$$

$$\dot{\delta} + (1+w)\left(\frac{k^2}{a^2}\delta u - \frac{\dot{\gamma}}{2}\right) = 0, \quad (3.186b)$$

$$\delta\dot{u} - 3\frac{\dot{a}}{a}w\delta u - \frac{w}{1+w}\delta = 0. \quad (3.186c)$$

The scalar velocity perturbation  $\delta u$  is more conveniently expressed in terms of the quantity  $\theta$ , defined through

$$\partial_i \delta T_0^i \equiv (\bar{\rho} + \bar{P})\theta, \quad (3.187)$$

so that  $\theta = -\nabla^2 \delta u / a^2$  ( $\theta = k^2 \delta u / a^2$  in  $k$ -space) and the equations can be recast as

$$\ddot{\gamma} + 2\frac{\dot{a}}{a}\dot{\gamma} - 3\frac{\dot{a}^2}{a^2}(1+3w)\delta = 0, \quad (3.188a)$$

$$\dot{\delta} + (1+w)\left(\theta - \frac{\dot{\gamma}}{2}\right) = 0, \quad (3.188b)$$

$$\dot{\theta} + \frac{\dot{a}}{a}(2-3w)\theta - \frac{k^2 w}{a^2(1+w)}\delta = 0. \quad (3.188c)$$

In order to study the behavior of the perturbations outside the horizon (i.e.,  $k_{\text{phys}} \rightarrow 0$ ), we neglect the last term in Eq. (3.188c). Using the fact that  $\dot{a}/a = H \propto 1/t$ , the system of Eqs. (3.188) admits simple power-law solutions of the form

$$\tilde{\gamma}, \delta \propto t^\alpha, \quad (3.189)$$

$$\theta \propto t^{\alpha-1}. \quad (3.190)$$

Since the system is fourth-order, there will be four independent solutions of this kind. In particular, during the matter-dominated (MD) era ( $w = 0$  and  $H = 2/3t$ ), the four solutions correspond to

$$\alpha = \left\{0, -1, -\frac{1}{3}, \frac{2}{3}\right\} \quad (\text{MD}), \quad (3.191)$$

so that a general solution for  $\delta$  is

$$\delta = A + Bt^{-1} + Ct^{-1/3} + Dt^{2/3} \quad (\text{MD}). \quad (3.192)$$

On the other hand, during the radiation-dominated (RD) era ( $w = 1/3$  and  $H = 1/2t$ ) the four solutions are

$$\alpha = \left\{0, -1, \frac{1}{2}, 1\right\} \quad (\text{RD}), \quad (3.193)$$

so that a general solution for  $\delta$  is

$$\delta = A + Bt^{-1} + Ct^{1/2} + Dt \quad (\text{RD}). \quad (3.194)$$

In both matter- and radiation-dominated cases it can be shown that the first two modes (those proportional to  $A$  and  $B$ ) are unphysical, gauge modes that can be put to zero by a suitable coordinate transformation (see Sec. 3.5.4). On the contrary, the modes proportional to  $C$  and  $D$  represent actual density perturbations. Thus, the fastest growing modes evolve like  $t^{2/3}$  and  $t$  during the matter- and radiation-dominated eras, respectively. In terms of the scale factor, since  $a \propto t^{2/3}$  (MD) and  $a \propto t^{1/2}$  (RD), the density contrast is given by

$$\delta \propto \begin{cases} a & (\text{MD}), \\ a^2 & (\text{RD}). \end{cases} \quad (3.195)$$

This remarkable result also holds for imperfect fluids (see Sec. 3.5.7), because dissipative effects cannot operate on scales larger than the Hubble horizon.

### 3.5.6 *Adiabatic and isocurvature perturbations*

In this section we briefly discuss how the density perturbations in the early, radiation-dominated Universe, are characterized. These primordial fluctuations are the initial conditions from which the perturbations are evolved, and can be defined in terms of their power spectrum (see Sec. 4.2.3). The issue of how these primordial fluctuations are generated, and the prediction for their power spectrum, will be dealt with in Sec. 5.6.4 in the framework of the inflationary paradigm. Here we focus on the way in which the initial, super-horizon perturbations can be decomposed according to their physical properties.

There are two different kinds of primordial fluctuations, the so-called *adiabatic* (or isoentropic) and *isocurvature* (or entropic) fluctuations. The distinction between the two is that adiabatic perturbations are perturbations in the total energy density of the system, while in the case of isocurvature perturbations the relative fluctuations between the different components are arranged in order to compensate and leave the total energy density unperturbed. It is clear that the latter can arise only in a system with two or more distinct components. Then adiabatic perturbations represent fluctuations in the intrinsic scalar curvature,<sup>9</sup> that is instead left unperturbed

<sup>9</sup>For this reason, they are also called curvature perturbations.

in the case of isocurvature perturbations (hence the name<sup>10</sup>). A generic fluctuation can be written as a sum of these two kinds of perturbations.

For simplicity, let us consider the simple case where only a radiation ( $w = 1/3$ ) and a matter component ( $w = 0$ ) are present. The adiabatic condition in this case reads as

$$\frac{1}{3}\delta_{\text{m}} = \frac{1}{4}\delta_{\text{rad}}, \quad (3.196)$$

where as above the density contrast  $\delta_i$  of the  $i$ th component is defined as  $\delta_i \equiv \delta\rho_i/\bar{\rho}_i$ . Given that  $\rho_{\text{m}} \propto T^3$  and  $\rho_{\text{rad}} \propto T^4$ , while the number densities  $n_{\text{m}, \text{rad}}$  of both matter and radiation scale like  $T^3$ , so that  $\rho_{\text{m}} \propto n_{\text{m}}$  and  $\rho_{\text{rad}} \propto n_{\text{rad}}^{4/3}$ , the adiabatic condition (3.196) implies

$$\frac{\delta n_{\text{m}}}{\bar{n}_{\text{m}}} = \frac{\delta n_{\text{rad}}}{\bar{n}_{\text{rad}}} = \frac{\delta s}{s}, \quad (3.197)$$

where the last equality follows from the fact that the total entropy density  $s$  is dominated by the radiation component and from Eq. (3.43). Then we have

$$\delta\left(\frac{n_i}{s}\right) = \frac{\delta n_i}{s} - \frac{n_i}{s^2}\delta s = 0, \quad i = \text{m}, \text{rad}. \quad (3.198)$$

This explains the reason why Eq. (3.196) is called adiabatic condition: it implies that the relative fluctuation between the number density of any species and the entropy density vanishes, or, in other words, that the number of particles per comoving volume is left unperturbed. In general, in the presence of  $N$  components of the system  $i_1, i_2, \dots, i_N$ , the adiabatic condition is restated as

$$\frac{\delta_{i_1}}{1 + w_{i_1}} = \frac{\delta_{i_2}}{1 + w_{i_2}} = \dots = \frac{\delta_{i_N}}{1 + w_{i_N}} = \frac{\delta}{1 + w}, \quad (3.199)$$

where  $\delta$  and  $w$  are defined in terms of the total density and pressure of the fluid.

The deviation from the adiabatic condition (3.196) can be expressed by defining the non-adiabatic fluctuation  $\mathcal{S}$  as

$$\mathcal{S} \equiv \delta_{\text{m}} - \frac{3}{4}\delta_{\text{rad}} \quad (3.200)$$

so that in terms of the newly-defined quantity, the condition reads  $\mathcal{S} = 0$ . In order to have isocurvature perturbations, it is thus necessary that  $\mathcal{S} \neq 0$ . For an isocurvature fluctuation,  $\delta\rho = 0$  and then

$$\delta\rho = \delta\rho_{\text{m}} + \delta\rho_{\text{rad}} = \delta_{\text{m}}\bar{\rho}_{\text{m}} + \delta_{\text{rad}}\bar{\rho}_{\text{rad}} = 0 \Rightarrow \delta_{\text{rad}} = -\frac{\bar{\rho}_{\text{m}}}{\bar{\rho}_{\text{rad}}}\delta_{\text{m}}, \quad (3.201)$$

---

<sup>10</sup>In the past, isocurvature fluctuations were mainly referred to as isothermal fluctuations, even if they are not really so, but this name has fallen out of usage.

so that

$$\mathcal{S} = \delta_{\text{m}} \left( 1 + \frac{3}{4} \frac{\bar{\rho}_{\text{m}}}{\bar{\rho}_{\text{rad}}} \right) \simeq \delta_{\text{m}} \neq 0, \quad (3.202)$$

where the last approximate equality holds during the radiation-dominated era, when  $\bar{\rho}_{\text{rad}} \gg \bar{\rho}_{\text{m}}$ . Once  $\delta_{\text{rad}}$  has been fixed, the most general matter fluctuation can be written as a combination of an adiabatic and an isocurvature part

$$\delta_{\text{m}} = \mathcal{A} + \mathcal{S} \quad (3.203)$$

where  $\mathcal{A} = 3\delta_{\text{rad}}/4$ . The two quantities  $\mathcal{A}$  and  $\mathcal{S}$ , or, better, their primordial power spectra  $\mathcal{P}_{\mathcal{A}}(k)$  and  $\mathcal{P}_{\mathcal{S}}(k)$  completely specify the initial conditions in the early Universe from which the perturbations have evolved. This is mathematically equivalent to specify the initial spectra for the matter and radiation components  $\mathcal{P}_{\text{rad}}(k)$  and  $\mathcal{P}_{\text{m}}(k)$ .

For a system with  $N > 2$  components, the entropic perturbation for every pair of components  $(i, j)$  is defined as follows

$$\mathcal{S}_{ij} \equiv \frac{\delta_i}{1 + w_i} - \frac{\delta_j}{1 + w_j} \quad (i, j = 1, \dots, N) \quad (3.204)$$

and adiabatic perturbations are characterized by the vanishing of all the  $\mathcal{S}_{ij}$ 's. In general, there is one adiabatic perturbation mode and  $N - 1$  isocurvature modes, corresponding to the original  $N$  degrees of freedom of the system.

Loosely speaking, purely adiabatic fluctuations are present when the different density perturbations all originate from the same, “fundamental” fluctuation (so that, in some sense, there is only one degree of freedom in the system); this is for example the case in single-field inflationary models, where the density perturbation arise from primordial quantum fluctuations in the scalar field that is responsible for inflation. On the contrary, as noted above, isocurvature fluctuations need the presence of at least one more component. This condition, albeit necessary, is not sufficient by itself for the presence of isocurvature fluctuations: the absence of thermal equilibrium between the extra degree of freedom and radiation is also required. Thus isocurvature fluctuations can be generated in multiple-field models of inflation, or by some dark matter candidates like the axion, that were never in thermal equilibrium with radiation.

### 3.5.7 Imperfect fluids

In the above derivation we assumed that the cosmological fluid can still be described in terms of a perfect fluid. However, this is in general not



true, because dissipative effects like viscosity or heat conduction can in principle be relevant, at least at the perturbation level. These effects can be taken into account by adding a term  $\Pi_{ij}$ , called *anisotropic inertia*, to the energy-momentum tensor, i.e.

$$T_{ij}^{\text{IF}} = T_{ij}^{\text{PF}} + \Pi_{ij} = (\rho + P)u_i u_j - P g_{ij} + \Pi_{ij}. \quad (3.205)$$

This equation must be thought of as a definition for the anisotropic inertia term, encoding the deviations from the perfect fluid behavior. Once the extra term is introduced, an ambiguity arises in the definition of the density, pressure and fluid velocity. This ambiguity is removed firstly by requiring that  $T_{00}^{\text{IF}}$  still gives the energy density of the fluid, i.e.  $T_{00}^{\text{IF}} = \rho$ . Secondly, one requires that  $u^\alpha$  is the velocity of energy transport, so that  $\rho u^\alpha$  is the energy current four-vector. The two conditions imply that only the spatial components of  $\Pi_{ij}$  are different from zero, i.e.  $\Pi_{00} = \Pi_{0\alpha} = 0$ .

Let us define the three-dimensional tensor  $\pi_{\alpha\beta}$  as  $\pi_{\alpha\beta} = -\Pi_{\alpha\beta}/a^2$ ; the term “anisotropic inertia” refers to  $\pi_{\alpha\beta}$  also. The flat three-dimensional metric is used to raise and lower the indices of  $\pi_{\alpha\beta}$ , implying that  $\Pi \equiv \Pi_i^i = \pi_{\alpha\alpha}$ . The anisotropic inertia can be decomposed into a scalar, a vector and a tensor part as

$$\pi_{\alpha\beta} = \frac{\pi_{\mu\mu}}{3} \delta_{\alpha\beta} + \left( \partial_\alpha \partial_\beta - \frac{\delta_{\alpha\beta}}{3} \nabla^2 \right) \pi^S + (\partial_\alpha \pi_\beta^\perp + \partial_\beta \pi_\alpha^\perp) + \pi_{\alpha\beta}^T, \quad (3.206)$$

with

$$\partial_\alpha \pi_\alpha^\perp = 0, \quad \partial_\alpha \pi_{\alpha\beta}^T = 0, \quad \pi_{\alpha\alpha}^T = 0. \quad (3.207)$$

The perturbation to the spatial part of the energy-momentum tensor is then

$$\begin{aligned} \delta T_{\alpha\beta} &= -\bar{P} \gamma_{\alpha\beta} + a^2 \delta P \delta_{\alpha\beta} - a^2 \pi_{\alpha\beta} \\ &= -\bar{P} \gamma_{\alpha\beta} + a^2 \left[ \left( \delta P - \frac{\pi_{\mu\mu}}{3} + \frac{\nabla^2 \pi^S}{3} \right) \delta_{\alpha\beta} - \partial_\alpha \partial_\beta \pi^S \right. \\ &\quad \left. - (\partial_\alpha \pi_\beta^\perp + \partial_\beta \pi_\alpha^\perp) - \pi_{\alpha\beta}^T \right], \end{aligned} \quad (3.208)$$

while the perturbation to the trace is

$$\delta T = \delta \rho - 3\delta P + \pi_{\alpha\alpha}. \quad (3.209)$$

From the above expressions it follows that one of the two scalar degrees of freedom in  $\pi_{\alpha\beta}$  can be eliminated by including the term  $(\nabla^2 \pi^S - \pi_{\alpha\alpha})/3$  into the pressure perturbation  $\delta P$ , i.e. by continuing to define  $a^2 \delta P$  as the coefficient of  $\delta_{\alpha\beta}$  in the sum  $(\delta T_{\alpha\beta} + \bar{P} \gamma_{\alpha\beta})$ . Of course this is not the only

way to eliminate the superfluous degrees of freedom; another popular choice is to take  $\pi_{\alpha\beta}$  to be traceless,  $\pi_{\alpha\alpha} = 0$ . In the following we will make the first choice, and then assume that  $\nabla^2 \pi^S - \pi_{\alpha\alpha} = 0$ . Thus, the reader can verify that under these hypotheses the perturbation equations are modified as follows

- *Scalar modes:*

$$\ddot{\tilde{\gamma}} + 2\frac{\dot{a}}{a}\dot{\tilde{\gamma}} = \kappa(\delta\rho + 3\delta P - \nabla^2 \pi^S), \quad (3.210a)$$

$$\partial_\alpha (\dot{\tilde{\gamma}} - \nabla^2 \dot{\mu}) = 3\kappa (\bar{\rho} + \bar{P}) \partial_\alpha \delta u, \quad (3.210b)$$

$$\begin{aligned} \ddot{\tilde{\gamma}} - \nabla^2 \ddot{\mu} + 3\frac{\dot{a}}{a} (2\dot{\tilde{\gamma}} - \nabla^2 \dot{\mu}) - \frac{1}{a^2} \nabla^2 (\tilde{\gamma} - \nabla^2 \mu) \\ = 3\kappa (\delta P - \delta\rho - \nabla^2 \pi^S), \end{aligned} \quad (3.210c)$$

$$\begin{aligned} \partial_\alpha \partial_\beta \left( \ddot{\mu} + 3\frac{\dot{a}}{a}\dot{\mu} + \frac{1}{3a^2} \nabla^2 \mu - \frac{\tilde{\gamma}}{3a^2} + 6\kappa \frac{\pi^S}{a^2} \right) \\ = 2\kappa \partial_\alpha \partial_\beta \pi^S. \end{aligned} \quad (3.210d)$$

- *Vector modes:*

$$\nabla^2 \dot{V}_\alpha = -2\kappa (\bar{\rho} + \bar{P}) \delta u_\alpha^\perp, \quad (3.211a)$$

$$\partial_\alpha \left( \ddot{V}_\beta + 3\frac{\dot{a}}{a}\dot{V}_\beta \right) = 2\kappa \partial_\alpha \pi_\beta^\perp. \quad (3.211b)$$

- *Tensor modes:*

$$\ddot{\tilde{\gamma}}_{\alpha\beta}^T + 3\frac{\dot{a}}{a}\dot{\tilde{\gamma}}_{\alpha\beta}^T - \frac{1}{a^2} \nabla^2 \tilde{\gamma}_{\alpha\beta}^T = 2\kappa \pi_{\alpha\beta}^T. \quad (3.212)$$

For convenience, we also give the corresponding equations in Fourier space

- *Scalar modes:*

$$\ddot{\tilde{\gamma}} + 2\frac{\dot{a}}{a}\dot{\tilde{\gamma}} = \kappa(\delta\rho + 3\delta P + k^2 \pi^S), \quad (3.213a)$$

$$\dot{\tilde{\gamma}} + k^2 \dot{\mu} = 3\kappa (\bar{\rho} + \bar{P}) \delta u, \quad (3.213b)$$

$$\begin{aligned} \ddot{\tilde{\gamma}} + k^2 \ddot{\mu} + 3\frac{\dot{a}}{a} (2\dot{\tilde{\gamma}} + k^2 \dot{\mu}) + \frac{k^2}{a^2} (\tilde{\gamma} + k^2 \mu) \\ = 3\kappa (\delta P - \delta\rho + k^2 \pi^S) \end{aligned} \quad (3.213c)$$

$$\ddot{\mu} + 3\frac{\dot{a}}{a}\dot{\mu} - \frac{k^2}{3a^2} \mu - \frac{\tilde{\gamma}}{3a^2} = 2\kappa \pi^S \quad (3.213d)$$

- *Vector modes:*

$$k^2 \dot{V}_\alpha = 2\kappa (\bar{\rho} + \bar{P}) \delta u_\alpha^\perp \quad (3.214a)$$

$$\ddot{V}_\alpha + 3\frac{\dot{a}}{a}\dot{V}_\alpha = 2\kappa\pi_\alpha^\perp \quad (3.214b)$$

- *Tensor modes:*

$$\ddot{\tilde{\gamma}}_{\alpha\beta}^T + 3\frac{\dot{a}}{a}\dot{\tilde{\gamma}}_{\alpha\beta}^T + \frac{k^2}{a^2}\tilde{\gamma}_{\alpha\beta}^T = 2\kappa\pi_{\alpha\beta}^T. \quad (3.215)$$

### 3.5.8 Kinetic theory

As we anticipated, the equations for the scalar perturbations contain more unknowns than equations so that an equation of state  $P = P(\rho)$  has to be assigned to close the system. This cannot always be adequate, as in the case of mildly relativistic particles where it is not possible to assign a simple equation of state. When one considers an imperfect fluid, the problem is also more evident, extending also to the vector and tensor modes. One solution is of course to follow in this case a phenomenological approach, parametrizing in some way the dissipative effects encoded in the anisotropic inertia tensor. For example, if one is concerned by viscosity effects, a shear viscosity coefficient can be introduced and expressed in terms of other quantities (i.e.  $\rho$  and  $P$ ). Moreover, sometimes the interactions between different components of the fluid need to be taken into account, like in the case of the cosmological baryon-photon fluid. As long as the interactions between baryons and photons are very frequent, the two components are tightly coupled and can be treated as a single fluid. However, when the time-scale for the collisions is of order of the Hubble length, the single-fluid approximation breaks down and dissipative effects have to be taken into account.

The proper way to deal with this problem is to turn to a microscopical description of the energy-momentum tensor. The fundamental quantity that describes, from a statistical point of view, the state of the fluid is the distribution function in phase space  $f(x^\alpha, p_\beta, t)$ . The phase space is described by three positions  $x^\alpha$  and their conjugate momenta  $p_\beta$ . The distribution function is the density in phase space, i.e. given a six-dimensional infinitesimal volume element  $d\mathcal{V} \equiv dx^1 dx^2 dx^3 dp_1 dp_2 dp_3$  around the point  $(x^\alpha, p_\beta)$  at time  $t$ , containing  $dN$  particles, then

$$f(x^\alpha, p_\beta, t)d\mathcal{V} = dN. \quad (3.216)$$

The time evolution of the distribution function is described by the Boltzmann equation (Sec. 3.1.6)

$$\frac{df}{ds} = \frac{dx^\alpha}{ds} \frac{\partial f}{\partial x^\alpha} + \frac{dp_\beta}{ds} \frac{\partial f}{\partial p_\beta} = \hat{C}_s[f]. \quad (3.217)$$

The right-hand side represents the change in the distribution function due to the effects of collisions in a unit of proper time (hence the subscript  $s$ ). For our purposes, it is more convenient to change the momentum variable from the conjugate momentum  $p_\alpha$  to the proper momentum  $p'_\alpha$  measured by a co-moving observer. The two are related by

$$p_\alpha = \sqrt{-g_{\alpha\beta}} p'^\beta = a \left( \delta_{\alpha\beta} - \frac{\gamma_{\alpha\beta}}{2a^2} \right) p'^\beta = \left( \delta_{\alpha\beta} - \frac{\gamma_{\alpha\beta}}{2a^2} \right) q^\beta, \quad (3.218)$$

where we have also defined the rescaled momentum  $q^\alpha \equiv a p'^\alpha$ . We note that the indices of the proper and rescaled momenta are raised and lowered using the flat metric, so that  $p'^\alpha = p'_\alpha$  and  $q^\alpha = q_\alpha$ . We can write  $q_\alpha$  as  $q_\alpha = q n_\alpha n^\alpha$ , where  $q \equiv \sqrt{q^\alpha q_\alpha}$ , and the  $n_\alpha$  are unit vectors, i.e.  $n_\alpha n^\alpha = 1$ . We can change the momentum variables from  $p^i$  to  $(q, q_\alpha)$  and replace<sup>11</sup>  $f(x^\alpha, p_\beta, t)$  by  $f(x^\alpha, q, n_\beta, t)$ , so that the left-hand side of the Boltzmann equation, i.e. the Liouville operator, rewrites as

$$\frac{d}{ds} f(x^\alpha, q, n_\beta, t) = \frac{dt}{ds} \frac{\partial f}{\partial t} + \frac{dx^\alpha}{ds} \frac{\partial f}{\partial x^\alpha} + \frac{dq}{ds} \frac{\partial f}{\partial q} + \frac{dn_\alpha}{ds} \frac{\partial f}{\partial n_\alpha}. \quad (3.219)$$

The term  $dq/ds$  can be computed from the geodesic equation as follows. First of all, we note that

$$\frac{dq}{ds} = \frac{d}{ds} \sqrt{q_\alpha q^\alpha} = n^\alpha \frac{dq_\alpha}{ds}. \quad (3.220)$$

Multiplying both sides of Eq. (3.218) by  $(\delta_{\alpha\mu} + \gamma_{\alpha\mu}/2a^2) q_\alpha$  can be expressed in terms of  $p_\alpha$  and  $p^\alpha$  as

$$q_\alpha = \left( \delta_{\alpha\beta} + \frac{\gamma_{\alpha\beta}}{2a^2} \right) p_\beta = \left( -a^2 \delta_{\alpha\beta} + \frac{\gamma_{\alpha\beta}}{2} \right) p^\beta. \quad (3.221)$$

Then it results that

$$\begin{aligned} \frac{dq_\alpha}{ds} &= \frac{d}{ds} \left[ \left( -a^2 \delta_{\alpha\beta} + \frac{\gamma_{\alpha\beta}}{2} \right) p^\beta \right] \\ &= \frac{d}{ds} \left[ \left( -a^2 \delta_{\alpha\beta} + \frac{\gamma_{\alpha\beta}}{2} \right) \right] p^\beta + \left( -a^2 \delta_{\alpha\beta} + \frac{\gamma_{\alpha\beta}}{2} \right) \left( \frac{dp^\beta}{ds} \right) \\ &= \left( -2a\dot{a}p^0 \delta_{\alpha\beta} + \frac{p^0}{2} \partial_t \gamma_{\alpha\beta} + \frac{p^\mu}{2} \partial_\mu \gamma_{\alpha\beta} \right) p^\beta + \left( -a^2 \delta_{\alpha\beta} + \frac{\gamma_{\alpha\beta}}{2} \right) \left( \frac{dp^\beta}{ds} \right). \end{aligned} \quad (3.222)$$

<sup>11</sup>Since  $x^\alpha$  and  $q_\alpha$  are *not* conjugate variables,  $d^3x d^3q$  is not the phase-space volume element and  $f d^3x d^3q$  is not the particle number.

The geodesic equation gives

$$\begin{aligned} \frac{dp^\alpha}{ds} = & -\Gamma_{ij}^\alpha p^i p^j = -2\frac{\dot{a}}{a}p^0 p^\alpha + \frac{1}{a^2} \left( \partial_t \gamma_{\alpha\beta} - 2\frac{\dot{a}}{a} \gamma_{\alpha\beta} \right) p^0 p^\beta \\ & + \frac{1}{2a^2} (2\partial_\nu \gamma_{\alpha\mu} - \partial_\alpha \gamma_{\mu\nu}) p^\mu p^\nu, \end{aligned} \quad (3.223)$$

and then

$$\begin{aligned} \frac{dq_\alpha}{ds} = & \left( -2a\dot{a}p^0 \delta_{\alpha\beta} + \frac{p^0}{2} \partial_t \gamma_{\alpha\beta} + \frac{p^\mu}{2} \partial_\mu \gamma_{\alpha\beta} \right) p^\beta \\ & + 2a\dot{a}p^0 p^\alpha - \left( \partial_t \gamma_{\alpha\beta} - 2\frac{\dot{a}}{a} \gamma_{\alpha\beta} \right) p^0 p^\beta - \frac{1}{2} (2\partial_\nu \gamma_{\alpha\mu} - \partial_\alpha \gamma_{\mu\nu}) p^\mu p^\nu - \frac{\dot{a}}{a} \gamma_{\alpha\beta} p^0 p^\beta \\ = & -\frac{1}{2} p^0 p^\beta \partial_t \gamma_{\alpha\beta} + \frac{\dot{a}}{a} p^0 p^\beta \gamma_{\alpha\beta} - \frac{1}{2} (\partial_\nu \gamma_{\alpha\mu} - \partial_\alpha \gamma_{\mu\nu}) p^\mu p^\nu. \end{aligned} \quad (3.224)$$

Putting everything together, we finally get

$$\frac{dq}{ds} = q n^\alpha n^\beta p^0 \left( \frac{1}{2a^2} \dot{\gamma}_{\alpha\beta} - \frac{\dot{a}}{a} \frac{\gamma_{\alpha\beta}}{a^2} \right) = \frac{1}{2} q n^\alpha n^\beta p^0 \dot{\tilde{\gamma}}_{\alpha\beta}, \quad (3.225)$$

where we remember that  $\tilde{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta}/a^2$ . For what concerns  $dn_\alpha/ds$ , it results that

$$\frac{dn_\alpha}{ds} = \frac{d}{ds} \left( \frac{q_\alpha}{q} \right) = \frac{1}{q} \frac{dq_\alpha}{ds} - \frac{q_\alpha}{q^2} \frac{dq}{ds}. \quad (3.226)$$

Since both  $dq/ds$  and  $dq_\alpha/ds$  are  $\mathcal{O}(\gamma_{\alpha\beta})$ , also  $dn_\alpha/ds$  is of order  $\mathcal{O}(\gamma_{\alpha\beta})$  at least. As we shall see, this is enough for our purposes.

Let us write the distribution function  $f(x^\alpha, q, n_\beta, t)$  as the sum of an unperturbed part  $\bar{f}$  plus a small perturbation  $\delta f \equiv \bar{f}\Psi$ . The background homogeneity implies that  $\bar{f}$  cannot depend on the spatial position  $x^\alpha$ , while the background isotropy implies that it can depend on the momentum only through its magnitude  $q$ . Finally, recalling again that  $dq/ds = \mathcal{O}(\gamma_{\alpha\beta})$ , the zeroth-order Boltzmann equation for  $\bar{f}$  rewrites as

$$\frac{d\bar{f}}{ds} = \frac{\partial \bar{f}}{\partial s} = p^0 \frac{\partial \bar{f}}{\partial t} = 0, \quad (3.227)$$

so that  $\bar{f}$  does not depend on time either ( $\bar{f} = \bar{f}(q)$ ) and

$$\begin{aligned} f(x^\alpha, q, n_\beta, t) = & \bar{f}(q) + \delta f(x^\alpha, q, n_\beta, t) \\ = & \bar{f}(q) [1 + \Psi(x^\alpha, q, n_\beta, t)]. \end{aligned} \quad (3.228)$$

We can assume the zeroth-order distribution function as given, in the co-moving frame, by an equilibrium Bose-Einstein or Fermi-Dirac distribution with temperature  $T$  and zero chemical potential

$$\bar{f} = \frac{g^{\text{dof}}}{2\pi^3} \frac{1}{e^{\frac{E}{k_B T}} \pm 1}. \quad (3.229)$$

This form of the distribution is appropriate for species that are at kinetic equilibrium and with vanishing chemical potential. However, most of the formulas in the following are independent of the particular choice of  $\bar{f}$ .

The energy  $E$  that appears in Eq. (3.229) is the energy measured by a co-moving observer so that it is related to the proper momentum  $p'_\alpha$  and to  $q^\alpha$  by  $E^2 = p'_\alpha p'^\alpha + m^2 = (q/a)^2 + m^2$ . Let us introduce the rescaled energy variable  $\epsilon$  defined as the proper energy  $E$  times the scale factor  $a$ , so that  $\epsilon = \sqrt{q^2 + a^2 m^2}$ . We note that Eq. (3.227) implies that, for a Fermi-Dirac or Bose-Einstein distribution

$$\frac{d}{dt} \left( \frac{E}{T} \right) = 0 \quad \Rightarrow \quad \frac{E}{T} = \text{const.} \quad (3.230)$$

Finally, we can rewrite the first-order Liouville operator (3.219) as

$$\frac{df}{ds} = p^0 \bar{f} \left[ \frac{\partial \Psi}{\partial t} - \frac{q^\alpha}{a\epsilon} \frac{\partial \Psi}{\partial x^\alpha} + \frac{1}{2} n^\alpha n^\beta \dot{\gamma}_{\alpha\beta} \frac{d \ln \bar{f}}{d \ln q} \right] \quad (3.231)$$

considering that, to zeroth order,  $p_0 = \epsilon/a$  and  $p^\alpha = -q^\alpha/a^2$ . The term  $(dn_\alpha/ds)(\partial f/\partial n_\alpha)$  in Eq. (3.219) does not appear because both  $(dn_\alpha/ds)$  and  $(\partial f/\partial n_\alpha)$  are  $\mathcal{O}(\gamma_{\alpha\beta})$  and then their product is  $\mathcal{O}((\gamma_{\alpha\beta})^2)$ . Using  $p^0 ds = dt$ , the Boltzmann equation reads as

$$\frac{\partial \Psi}{\partial t} - \frac{q^\alpha}{a\epsilon} \frac{\partial \Psi}{\partial x^\alpha} + \frac{1}{2} n^\alpha n^\beta \dot{\gamma}_{\alpha\beta} \frac{d \ln \bar{f}}{d \ln q} = \frac{1}{\bar{f}} \hat{C}_t[f] \quad (3.232)$$

where, as usual, the subscript  $t$  in the collision term on the right-hand side represents the fact that  $\hat{C}_t[f]dt$  gives the variation of the distribution function due to the collisions in a small interval of time  $dt$ . The form above makes explicit that the Boltzmann equation is coupled to the Einstein equations by the presence of the metric perturbation  $\tilde{\gamma}_{\alpha\beta}$ . In Fourier space, the Boltzmann equation rewrites as

$$\dot{\Psi} + i \frac{q}{a\epsilon} (k_\alpha n^\alpha) \Psi + \frac{1}{2} n^\alpha n^\beta \dot{\gamma}_{\alpha\beta} \frac{d \ln \bar{f}}{d \ln q} = \frac{1}{\bar{f}} \hat{C}_t[f] \quad (3.233)$$

where we continue to use  $\Psi$  to denote the Fourier transform of the perturbation to the distribution function.

The next step to close the Einstein-Boltzmann system is to express the components of the energy momentum tensor in terms of integrals of the distribution function. In general, the energy-momentum tensor is related to the distribution function by

$$T_j^i = \frac{1}{\sqrt{-g}} \int \frac{p^i p_j}{p^0} f(x^\alpha, p_\beta, t) d^3 p \quad (3.234)$$

where  $g = -a^6(1 - \tilde{\gamma})$  is the determinant of the metric. For consistency, we need to express the integrand in terms of  $q$  and  $n_\alpha$ . From Eq. (3.218), the Jacobian matrix of the transformation from the  $p_\alpha$  to the  $q^\beta$ , is equal to  $dp_\alpha/dq^\beta = (\delta_{\alpha\beta} - \gamma_{\alpha\beta}/2a^2)$ , so that the Jacobian is, to first order in  $\gamma$ , equal to  $(1 - \tilde{\gamma}/2)$ , and

$$d^3p = \left(1 - \frac{\tilde{\gamma}}{2}\right) d^3q = \left(1 - \frac{\tilde{\gamma}}{2}\right) q^2 dq d\Omega, \quad (3.235)$$

where  $d^3p/\sqrt{-g} = a^{-3}d^3q$ . Here  $d\Omega$  is the infinitesimal element of solid angle around the direction  $n_\alpha$ . It results that

$$T_0^0 = \frac{1}{a^4} \int \epsilon \bar{f}(q)(1 + \Psi) q^2 dq d\Omega, \quad (3.236a)$$

$$T_\alpha^0 = \frac{1}{a^3} \left( \delta_{\alpha\beta} - \frac{\tilde{\gamma}_{\alpha\beta}}{2} \right) \int q n_\beta \bar{f}(1 + \Psi) q^2 dq d\Omega, \quad (3.236b)$$

$$T_\beta^\alpha = -\frac{1}{a^4} \int \frac{q^2 n^\alpha n_\beta}{\epsilon} \bar{f}(1 + \Psi) q^2 dq d\Omega, \quad (3.236c)$$

where  $p^0 = p_0 = \epsilon/a$ . In the second of these equations, the angular integral of the unperturbed part is just  $\int_\Omega n_\beta d\Omega$  that vanishes identically. We then have

$$T_0^0 = \frac{1}{a^4} \int \epsilon \bar{f}(q)(1 + \Psi) q^2 dq d\Omega \quad (3.237a)$$

$$T_\alpha^0 = \frac{1}{a^3} \int q n_\alpha \bar{f} \Psi q^2 dq d\Omega \quad (3.237b)$$

$$T_\beta^\alpha = -\frac{1}{a^4} \int \frac{q^2 n^\alpha n_\beta}{\epsilon} \bar{f}(1 + \Psi) q^2 dq d\Omega \quad (3.237c)$$

which makes explicit that the convenience of replacing the conjugate momentum  $p_\alpha$  with the proper momentum  $q_\alpha$  is that the metric perturbations disappear from the expressions of the components of  $T_j^i$ .

The mixed components of the energy-momentum tensor are related to the energy density, pressure, velocity and anisotropic inertia of the fluid by

$$T_0^0 = \rho = \bar{\rho} + \delta\rho \quad (3.238a)$$

$$T_\alpha^0 = (\bar{\rho} + \bar{P})\delta u_\alpha \quad (3.238b)$$

$$T_\beta^\alpha = -P\delta_\beta^\alpha + \Pi_\beta^\alpha = -(\bar{P} + \delta P)\delta_\beta^\alpha + \Pi_\beta^\alpha. \quad (3.238c)$$

Comparing with Eqs. (3.237), the background quantities  $\bar{\rho}$  and  $\bar{P}$  stand as

$$\begin{aligned} \bar{\rho} &= \frac{4\pi}{a^4} \int q^2 \sqrt{q^2 + a^2 m^2} \bar{f}(q) dq \\ \bar{P} &= \frac{4\pi}{3a^4} \int \frac{q^4}{\sqrt{q^2 + a^2 m^2}} \bar{f}(q) dq \end{aligned} \quad (3.239)$$

where  $\int d\Omega = 4\pi$  and  $\int n_\alpha n_\beta = 4\pi\delta_{\alpha\beta}/3$ . For the perturbations, it results that

$$\delta\rho = \frac{1}{a^4} \int q^2 \sqrt{q^2 + a^2 m^2} \bar{f}(q) \Psi dq d\Omega \quad (3.240a)$$

$$(\bar{\rho} + \bar{P})\delta u_\alpha = \frac{1}{a^3} \int q n_\alpha \bar{f} \Psi q^2 dq d\Omega \quad (3.240b)$$

$$\delta P \delta_\beta^\alpha - \Pi_\beta^\alpha = \frac{1}{a^4} \int \frac{q^4 n^\alpha n_\beta}{\sqrt{q^2 + a^2 m^2}} \bar{f} \Psi dq d\Omega. \quad (3.240c)$$

Once the collision term on the right-hand side of the Boltzmann equation has been specified, the Einstein equations (3.213), (3.214) and (3.215) and the Boltzmann equation<sup>12</sup> (3.233), together with the relations (3.240) form a closed system for the coupled evolution of  $\tilde{\gamma}_{\alpha\beta}$  and  $\Psi$  that can be solved without any further assumption once initial conditions are given. In the form presented here, this system takes the form of an integro-differential system, since integrals of the distribution function  $f$  appear as sources of the differential equations for  $\tilde{\gamma}_{\alpha\beta}$ .

### 3.6 The Lemaître-Tolmann-Bondi Spherical Solution

The Lemaître-Tolmann-Bondi (LTB) spherical solution can be thought of as a generalization of the RW line element in which the requirement of homogeneity is dropped, while that of isotropy is kept. It is worth noting that the LTB model can be isotropic but not homogeneous (see Sec. 3.1.1) because a preferred point is singled out, i.e. the space is isotropic only when viewed from this particular point, but not from any other point. For this reason, it is more correctly described as a spherically symmetric, inhomogeneous solution of the Einstein equations. In particular, it describes the evolution of a zero-pressure spherical overdensity in the mass distribution, and thus the resulting solution is different from the Schwarzschild one. In the synchronous reference system (see Sec. 2.4), the spherically symmetric line element can be written as

$$ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.241)$$

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<sup>12</sup>In general, for a fluid made of many uncoupled (or, more precisely, not perfectly coupled) components, one should write a Boltzmann equation for every component. These Boltzmann equations can be coupled, other than by the presence of the metric perturbation, representing the effect of the gravitational field, by the collision terms, representing the effects of impulsive interactions between the various components. For example, the Boltzmann equations for baryons and photons are coupled by the collision term for Thomson scattering. However, this does not change the sense of the discussion.



where  $\alpha = \alpha(r, t)$  and  $\beta = \beta(r, t)$ , while the identity  $\sqrt{-g} = \sin \theta e^{\alpha+2\beta}$  follows.

Originally, this kind of solution was discussed under the assumption that the perfect fluid energy-momentum tensor (2.14) is dominated by pressureless dust ( $P = 0$ ) and by a cosmological constant term  $\Lambda$ . In this scheme, the Einstein field equations rewrite as

$$\kappa T_1^0 = 0 = -2\dot{\beta}' - 2\dot{\beta}\beta' + 2\dot{\alpha}\beta' = G_1^0 \quad (3.242a)$$

$$\kappa T_1^1 = \Lambda = 2\ddot{\beta} + 3\dot{\beta}^2 + e^{-2\beta} - (\beta')^2 e^{-2\alpha} = G_1^1 \quad (3.242b)$$

$$\begin{aligned} \kappa T_0^0 &= \kappa\rho + \Lambda \\ &= \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta} + e^{-2\beta} - e^{-2\alpha}[2\beta'' + 3(\beta')^2 - 2\alpha'\beta'] = G_0^0, \end{aligned} \quad (3.242c)$$

where the  $\dot{(\ )}$  and the  $(\ )'$  denote derivatives with respect to time and to the radial coordinate  $r$ , respectively. Let us note that these are the only independent equations because the following relation stands

$$G_2^2 = G_1^1 + [G_1^1]' / 2\beta'. \quad (3.243)$$

Since  $T_1^0$  vanishes, Eq. (3.242a) rewrites as

$$\dot{\beta}' / \beta' = \partial_t \ln \beta' = \dot{\alpha} - \dot{\beta}, \quad (3.244)$$

which admits the solution  $\beta' = f(r) e^{\alpha-\beta}$  giving  $e^\beta \beta' = \partial_r e^\beta = f(r) e^\alpha$ . Let us now define the scale factor  $a(r, t)$  by using the parametrization

$$e^\beta = r a(r, t), \quad f(r) = [1 - r^2 K^2]^{1/2}, \quad (3.245)$$

where  $K = K(r)$ . Using the expressions above, the LTB line element (3.241) rewrites as follows

$$ds^2 = dt^2 - \frac{[(ar)']^2}{1 - r^2 K^2} dr^2 - (ar)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.246)$$

If  $a$  and  $K$  are independent of the radial coordinate  $r$ , Eq. (3.246) corresponds to the RW line element (3.1) and it can be constructed if and only if  $T_1^0$  vanishes. The function  $K^2$  has been written as a square, to conform with the standard notation for the isotropic models, but of course  $K^2$  can be negative, as in the open isotropic model.

The field equations (3.243) and (3.242b) rewrite now as

$$(\kappa\rho + \Lambda)[(ar)^3]' = 3[\dot{a}^2 ar^3 + ar^3 K^2]', \quad (3.247a)$$

$$\Lambda = \frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K^2}{a^2}, \quad (3.247b)$$

respectively. The last expression, if multiplied by  $a^2\dot{a}$ , is a total time derivative, which can be integrated getting

$$\dot{a}^2 a + aK^2 - \frac{\Lambda}{3}a^3 = F(r) . \quad (3.248)$$

Let us now suppose that the cosmological constant term is small enough for the function  $F(r)$  to be positive. We can also change the radial coordinates according to

$$\bar{r} = r \left[ \frac{F(r)}{A} \right]^{1/3} , \quad \bar{a} = a \left[ \frac{A}{F(r)} \right]^{1/3} , \quad \bar{K} = K \left[ \frac{A}{F(r)} \right]^{1/3} , \quad (3.249)$$

( $A$  being a constant) leaving the form of the line element (3.246) unchanged. Furthermore, the right-hand side of Eq. (3.248), in the new coordinate system, is constant. In this scheme, Eqs. (3.247a) and (3.248) rewrite respectively as

$$\kappa\rho(r, t) = \frac{3A}{(ar)'\dot{a}^2} , \quad (3.250)$$

$$\dot{a}^2 a + aK^2 - \frac{\Lambda}{3}a^3 = A , \quad (3.251)$$

where we have dropped the bars for the sake of simplicity.

Let us consider the Lagrangian formulation of LTB space-time. The Lagrangian density can be obtained from the expression of the Ricci scalar and by avoiding total derivatives it reads as

$$\mathcal{L}_{\text{LTB}} = \frac{2\pi}{\kappa} \int (-r^3 W a \dot{a}^2 + r^2 V' a \dot{a} - V) dt dr , \quad (3.252)$$

where

$$W = \left( \frac{1}{1 - r^2 K^2} \right)' , \quad V = \frac{[(ar)']^2}{1 - r^2 K^2} . \quad (3.253)$$

The momentum conjugated to  $a$  is given by

$$p_a = \frac{2\pi}{\kappa} (-2W r^3 a \dot{a} + r^2 a V') , \quad (3.254)$$

such that the Hamiltonian turns out to be

$$\mathcal{H}_{\text{LTB}} = -\frac{\kappa}{8\pi} \frac{p_a^2}{ar^3 W} + \frac{V'}{2rW} p_a - \frac{1}{2\kappa} \frac{(V')^2 r a}{W} - \frac{2\pi}{\kappa} V . \quad (3.255a)$$

### 3.7 Guidelines to the Literature

There are many textbooks on GR that deal with the kinematics and dynamics of the isotropic Universe, studied in Secs. 3.1 and 3.2, starting from the classic one by Landau & Lifshitz [301] and by Weinberg [462]. The more recent books by Kolb & Turner [290] and Peebles [378] are more directly devoted to physical cosmology, with the first one putting quite an emphasis on topics at the interface between cosmology and particle physics. Another book that covers a wide range of topics is that by Peacock [374]. The recent ones by Dodelson [155] and by Weinberg [464] include many recent developments in the field.

A textbook on kinetic theory and the Boltzmann equation, introduced in Sec. 3.1.6 is the book by Lifshitz & Pitaevskij [317]. The Boltzmann equation and the topic of the kinetic theory and thermodynamics of the expanding Universe are covered in the book by Bernstein [82], as well as in the above mentioned by Kolb & Turner, Dodelson, and Weinberg.

For what concerns the dissipative cosmologies studied in Sec. 3.3, we refer the reader to Landau & Lifshitz *Fluid Mechanics* [300], and in particular to Chap. XV. The issue of the effects of bulk viscosity on the cosmological evolution has been studied in [41, 42, 62, 67, 111, 331, 371, 461]. The phenomenological description of the process of matter creation in an isotropic expanding Universe was firstly formulated by Prigogine [351]. The influence of the matter creation term on the dynamics of the Universe has been studied in [148, 351].

There is a vast literature on the theory of cosmological perturbations, discussed in Sec. 3.4. The Jeans mechanism, both in a static and in an expanding Universe, is described in many textbooks, like Weinberg [462] and Kolb & Turner [290]. The general relativistic treatment of small fluctuations over a RW background was first formulated, in the synchronous gauge, by Lifshitz [311]. This work was later reviewed by Lifshitz & Khalatnikov [312]. The conformal Newtonian gauge was introduced by Mukhanov and collaborators in [358]. The gauge-dependent treatment (either in the synchronous or Newtonian gauge, or in both) is summarized, among others, in the books by Weinberg [462, 464], Landau & Lifshitz [301], and Dodelson [155], as well as in many reviews, like that of Bertschinger [84] (see also [85]). The gauge invariant formulation, that we did not address here, was formulated by Bardeen [38] and later reviewed by Kodama & Sasaki [288]. The systematic treatment of the coupled Boltzmann and Einstein equations, including all species of cosmological interest, has been

given by Ma & Bertschinger [329]. A discussion of the Lemaître-Tolmann-Bondi solution can be found, among others, in the book by Peebles [378]. A comprehensive treatment of the evolution of cosmological perturbations, including also evolution in the non-linear regime, is given by Padmanabhan [370], which is devoted to cosmological structure formation.

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## Chapter 4

# Features of the Observed Universe

In this Chapter we review the present observational knowledge of the Universe, introducing the main cosmological observables and discussing how these can be used to extract information on the main parameters describing the Universe.

We start by briefly reviewing the so-called  $\Lambda$ CDM (or “concordance”) cosmological model, that is able to account for all observations, although yet not explaining the precise nature of the dark matter and dark energy components that provide most of the energy content of the Universe.

The relevance of the distribution of large-scale structures in the Universe is treated arguing how their existence is not at variance with the isotropy and homogeneity requirements of the cosmological principle. We briefly recall the main features of the mechanism of gravitational instability, and show how the existence of galaxies already provides an indirect evidence for the presence of a non-baryonic matter component. We introduce the quantitative tools necessary to study the distribution of matter in the Universe, such as the two-point correlation function and its Fourier transform, or the matter power spectrum.

In the following section, we will show how the Hubble diagram can be used to infer the matter content of the Universe. The observations of supernova Ia indicate that the Universe is accelerating and we will briefly discuss the theoretical implications.

Finally, the last section is devoted to the study of the Cosmic Microwave Background (CMB). We first discuss its black body frequency spectrum that demonstrates that the early Universe was in thermal equilibrium, and then its extreme isotropy and how the small anisotropies carry important cosmological information. We briefly describe the mechanism by which these anisotropies are produced and introduce the power spectrum of the

anisotropies, discussing how the acoustic oscillations present in the primeval plasma left a distinct pattern in the spectrum, made of alternating peaks and dips. We conclude by discussing the effect of the cosmological parameters on the spectrum, and giving their values obtained by the most recent CMB observations.

#### **4.1 Current Status: The Concordance Model**

For many years, cosmology has been a data-starved science and, until a few decades ago, the observational basis for the standard cosmological model, although robust, consisted of just a handful of observations, basically given by:

- (i) the spectrum of distant galaxies is shifted towards the red;
- (ii) the existence of an isotropic background of thermal radiation in the microwave range;
- (iii) the distribution of galaxies;
- (iv) the measured abundances of the light elements.

In the last couple of decades, however, the observational data has grown in quality and quantity. Observational cosmologists have been able to test the above-mentioned pieces of evidence even further. The expansion history of the Universe has been probed up to redshifts of the order of 1.8. Satellites like BOOMERanG and WMAP have measured the tiny angular anisotropies in the temperature of the cosmic microwave background radiation, disclosing a wealth of information about the Universe as it was nearly 400,000 years after the Big Bang. Galaxy surveys, like the 2dF and SDSS, have increased in volume, allowing to collect a sample of  $\sim 1$  million objects with measured spectra, thus mapping the distribution of matter in the Universe with high precision. Finally, the abundances of light elements have been measured with increasing precision.

In the last years, new ways have been designed for the ongoing study of the Universe: for example the measure of abundance of neutral hydrogen by radio telescope arrays looking at the characteristic 21 cm line emission, to probe the “dark ages” in the history of the Universe; the polarization of the cosmic microwave background, providing new informations to detect gravitational waves produced in the early Universe; such background of gravitational waves is also a possible target for detection by interferometers like LISA, and for the so-called Pulsar Timing Arrays.

The observational knowledge of the Universe has very much increased in the last twenty years. Has our basic understanding of the cosmo increased as well? The answer is twofold. On one hand, the observations point to a very simple picture. Our Universe is very well described, at least at large scales, by a flat Robertson-Walker geometry. Nowadays, its energy content is given by some form of “dark” matter, making up roughly 20% of the total, and by an equally unknown (and even more exotic) form of “dark” energy, making up 75% of the total. Normal matter composes just the remaining 5%. The present Universe is very cold (2.7 K) but, since it is expanding, it was much hotter in the past. The light elements (hydrogen, helium and, to a lesser amount, lithium) present today were produced in this very dense and hot phase, as hypothesized by Gamow, when the temperature was around 10 MeV. The microwave radiation observed is the red-shifted relic of this early phase, released when the free protons and electrons recombined to form neutral hydrogen atom, thus allowing the photons to propagate freely. The structures observed today - galaxies, clusters, superclusters - have been grown by small “seeds” through the Jeans mechanism of gravitational instability. According to the inflationary scenario, these seeds have been produced from quantum fluctuations in the early Universe. As we have noted above, this “concordance model”, as it is currently called, can safely explain all the pieces of evidence briefly discussed here. However, the model is unsatisfactory in many ways. First of all, it is not yet known what dark matter really is, although there is no shortage of well-motivated particle physics candidates, starting from the supersymmetric neutralino. There is hope that in the next decade or so the dark matter particle will be detected either directly (by producing it in accelerators, or revealing it in specifically-designed detection experiments) or indirectly (through the observation of its decay/annihilation products in an astrophysical or cosmological setting), thus shedding light on its nature. The situation is worse for what concerns dark energy. In this case, it is fair to say that there is not a strongly motivated candidate, at least from the theoretical point of view, although many proposals have been made. From the observational point of view, the simpler explanation of the currently available data in the framework of an FRW model is still given by a cosmological constant-like fluid. This problem seems to point out that either we miss something very fundamental from the point of view of particle physics, or that the standard cosmological model is incomplete, or both. Many scientists have tried alternative ways to explain the observations without invoking any dark en-



ergy component.<sup>1</sup> Since the main evidence for the presence of dark energy comes from the dynamics of the Universe on the largest scales (in particular from acceleration), a natural approach is to assume that General Relativity is modified on cosmological scales, as proposed by  $f(R)$  theories.

Another possibility is that the observed acceleration is an artifact, due either to our location in an underdense region, or to the breaking of the homogeneity assumption underlying the FRW model at small scales. Although there is not yet a shared consensus if these approaches can provide a satisfactory explanation to the acceleration, without at the same time spoiling other observations, at the present time they still represent a possible alternative to the dark energy models.

## 4.2 The Large-Scale Structure

The Standard Cosmology is based on the cosmological principle, namely on the assumption that the Universe is homogeneous and isotropic. The best evidence for this is the great degree of isotropy of the cosmic microwave background radiation: the fractional difference in temperature between two directions in the sky is smaller than  $10^{-4}$ . This is also a proof of homogeneity, because the temperature variations in the CMB track the density fluctuations at the time of photon decoupling (see Sec. 4.4 below). This is in fact an evidence that the Universe was very homogeneous and isotropic at the time of last scattering, roughly 400,000 years after the big bang (corresponding to a redshift  $z \sim 1100$ ).

What about the present-day Universe? On a first look, it would seem very far from homogeneity. If we look at the sky, we see stars that are located into bound systems (the galaxies) separated by large, empty regions. The density inside a galaxy is roughly  $10^5$  times larger than the average density of the Universe, so that a galaxy cannot certainly be considered a small fluctuation of the background density. From this point of view the galaxies in some sense constitute the “elementary particles” of cosmology, since they can be taken as free falling in the cosmological gravitational field. Galaxies themselves tend to form groups called galaxy clusters (with an average density  $10^2$ – $10^3$  times the background), which in turn can form larger (not yet virialized) structures called superclusters. The density in the superclusters is estimated to be of the order of the background density, so that at those scales the density perturbations are presumably only in the

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<sup>1</sup>Sometimes even without dark matter, but this is a much harder task.

mildly nonlinear regime.

However, even if the distribution of luminous matter in the Universe is inhomogeneous, nevertheless these inhomogeneities become smaller as we look at the Universe at the largest scales. More precisely, if we adopt a coarse-grained description of the Universe, considering the density of luminous matter averaged over a given fiducial volume, the fluctuations in the coarse-grained density become smaller as the fiducial volume increases. The existence of such a “homogeneity” scale has been somewhat debated until recently, but now the scientific consensus is that the present Universe is homogeneous on scales larger than  $\sim 100$  Mpc.

The good approximation of the Universe homogeneity and isotropy does not mean that the existence of structures is irrelevant for cosmology. The origin and evolution of galaxies and larger structures is an issue of paramount importance in modern cosmology, as it provides key information on the evolution of the Universe. Here we will give some ideas about galaxies formation so that the reader can understand the overall picture.

#### 4.2.1 *Deviations from homogeneity*

If the Universe were perfectly homogeneous at all scales, it would not be possible to form any kind of structure. Indeed, small deviations from homogeneity are needed as starting “seeds” from which structures are formed. For the moment we will leave the existence of such primordial seeds as an assumption, coming back later to how they formed. The mechanism of growth for the initial inhomogeneities is the Jeans mechanism of gravitational instability, that we have described in detail in Sec. 3.4. The idea is that, if an overdense region is present in an otherwise homogeneous fluid, it will correspond to a potential well of the gravitational field, attracting other particles inside the well. This will increase the overdensity and further deepen the well, and so on. However, this is just a part of the story; the gravitational collapse is countered by the pressure forces inside the fluid, that increase with the overdensity. The final fate of the initial small inhomogeneity is decided by the (un)balance between gravity and pressure: if gravity dominates, the inhomogeneity will grow, become non-linear and eventually a structure will be formed; on the contrary, the amplitude of the inhomogeneity will just oscillate (and eventually decay once “real-life” dissipative effects are taken into account). These two behaviors are separated by a critical length called the *Jeans length*  $\lambda_J$  [given in Eq. (3.132)], that is a function of the density and of the speed of sound of the fluid

(see Sec. 3.4). Perturbations with a linear size larger than  $\lambda_J$  will collapse, while those smaller than  $\lambda_J$  will oscillate. Although the detailed behavior is in general more complex, this simple picture captures the essence of the mechanism of structure formation.

The full theory of cosmological perturbations has been developed in Sec. 3.5, where we have written the equations describing the coupled linear evolution of the perturbations in the metric and in the energy-momentum tensors. Once the initial inhomogeneities are given, the only missing piece of information to fully compute the linear evolution is the composition of the Universe.

#### 4.2.2 Dark matter

The evidence of galaxies existence is a strong hint to the fact that baryonic matter is not the only kind of matter present in the Universe. In fact, baryons were tightly coupled to photons via Thomson scattering until the time of decoupling, when the CMB radiation was emitted (see Sec. 4.4). Before that time, baryons and photons were behaving as a single fluid, with a very large pressure given by the photon component. Such a large radiation pressure was extremely effective in contrasting the gravitational instability, making the Jeans length roughly equal to the size of the cosmological horizon.<sup>2</sup> This means that perturbations in the baryonic component could not grow until decoupling, which occurred when the scale factor  $a$  of the Universe was  $\simeq 10^{-3}$  of its present value. However, the baryon density contrast at the time of decoupling has to be of the same order of magnitude as the temperature fluctuations observed in the CMB (the factor of 4 comes from the proportionality to  $T^4$  of the energy density):

$$\left. \frac{\delta\rho_b}{\rho_b} \right|_{\text{dec}} \simeq 4 \frac{\delta T}{T} \lesssim 10^{-4}. \quad (4.1)$$

Combining this with the fact that, as seen in Sec. 3.4.3, during the matter-dominated era the perturbations grow linearly with  $a$ , we obtain that the present density contrast should be  $\sim 10^{-1}$ . This implies that the non-linear evolution should not have started yet, and no structures would have formed at all - actually, if it were so, they will not form before some other ten billion years!

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<sup>2</sup>More quantitatively, this can be understood by noting that Eq. (3.132) with  $v_s = \sqrt{3}$  (the speed of sound in an ultrarelativistic fluid, with equation of state  $P = \rho/3$ ) and  $\rho_B = \rho_c = 3H^2/\kappa$  gives  $\lambda_J \simeq H^{-1}$ .

The solution to this apparent paradox is that some other kind of matter exists that does not couple to photons (and hence it is “dark”).<sup>3</sup> The density fluctuations of such dark matter component are not hindered by the pressure of photons and can start growing well before the time of decoupling, creating the potential wells where baryons will fall later, eventually leading to the formation of galaxies.

From the point of view of theoretical particle physics, there is no shortage of candidates for the role of dark matter. For the purpose of the present section, it will suffice to say that nearly all candidates belong to one of two broad classes: *hot* and *cold* dark matter. The distinction is based on the damping length of the particles, namely on the characteristic length below which dissipative effects become important and perturbations are erased. In the case of collisionless dark matter, this damping effect is provided by Landau damping, or *free streaming*. In other words, free streaming is due to the fact that, in a collisionless fluid, the particles can stream from overdense to underdense regions, in the process smoothing out the inhomogeneities. Since fast (hot) particles can cover larger distances, collisionless damping is more important for dark matter candidates with a large velocity dispersion. In more detail, one defines as hot dark matter (HDM) those candidates with a damping length  $\lambda_D$  of the order of the size of the horizon at the time of matter-radiation equality  $\lambda_{EQ}$ . This is the case for ultrarelativistic relics like neutrinos. On the other hand, cold dark matter (CDM) candidates have  $\lambda_D \ll \lambda_{EQ}$ . This is the case for non-relativistic relics like the supersymmetric neutralino. The importance of the time of matter-radiation equality is due to the fact that structure formation cannot start earlier, since during the radiation dominated regime the fast cosmological expansion nearly freezes the growth of *all* fluctuations - including those in the dark matter component. This is the Meszaros effect described at the end of Sec. 3.4.3.

The difference in the damping scale leads to different scenarios of structure formation between hot and cold dark matter-dominated Universes. In the case of hot dark matter, all perturbations below the (very large) damping length are erased, so that only the perturbations on the very largest scales survive. This implies that the largest structures in the Universe (like superclusters) are formed first, and smaller structures are formed later via a fragmentation process. In particular, the first structures to form have a

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<sup>3</sup>The fact that it is not possible to form the present cosmological structures of course is not the only reason to introduce dark matter but we think it nicely illustrates that the large-scale structure encodes fundamental information about the Universe.

mass of roughly  $10^{15}M_{\odot}$ , much larger than the typical mass of a galaxy ( $\sim 10^{11} - 10^{12}M_{\odot}$ ). This is called a *top-down* process of structure formation. On the other hand, for cold dark matter the damping length is effectively zero so there is no damping of small scale perturbations. Thus small structures (on subgalactic scales,  $\sim 10^6M_{\odot}$ ) form first, and eventually merge to form larger structures. This is called a *bottom-up*, or *hierarchical*, process of structure formation. The modern observations rule out the HDM scenario for at least two reasons. The first one is the prediction of more structures on large scales than actually seen. The second is that small structures seem actually to be *older* than large structures. The currently favored scenario is then with structures formed via a bottom-up process driven by CDM. This does not however exclude that a small HDM fraction could be present as a subdominant dark matter component.

#### 4.2.3 The power spectrum of density fluctuations

The quantity describing the distribution of matter in the Universe is the density contrast  $\delta(\vec{x})$ :

$$\delta(\vec{x}) \equiv \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}} \quad (4.2)$$

where  $\rho(\vec{x})$  is the density in  $\vec{x}$ , and  $\bar{\rho}$  is the average density of the Universe (in general, we will use bars to denote background quantities). It is useful to consider the Fourier transform  $\delta_k$  of  $\delta(\vec{x})$

$$\delta_k = \int \delta(\vec{x}) e^{i\vec{k} \cdot \vec{x}} d^3\vec{x}. \quad (4.3)$$

In the previous paragraph, we have given a very qualitative description of how the quantity  $\delta$  evolves with time. In Sec. 3.4 we have given the exact equations describing the full evolution of  $\delta$  in the linear regime  $\delta \ll 1$ .

The main question to answer is how to compare theory with observations: of course, a qualitative approach would be to compare the distribution of galaxies in the real, observed sky with a map produced by a numerical simulation. However, one does not compare the observed and theoretical matter distributions *per se*, but rather their statistical properties. The basic quantity is then the two-point correlation function  $\xi(\vec{r})$

$$\xi(\vec{r}) \equiv \langle \delta(\vec{x}) \delta(\vec{x} + \vec{r}) \rangle = \frac{1}{V} \int_V \delta(\vec{x}) \delta(\vec{x} + \vec{r}) d^3\vec{x}, \quad (4.4)$$

namely the autocorrelation function of the density field. Here the brackets denote an average over some fiducial volume  $\mathcal{V}$ . The two-point correlation function evaluates how the density fluctuations in pairs of points separated by  $\vec{r}$  are correlated. We stress that  $\xi$  does not depend on the absolute position  $\vec{x}$ , but on the points separation  $\vec{r}$ , since by construction it is a volume average.<sup>4</sup> Moreover, the isotropy of the universe implies that  $\xi$  depends only on the modulus  $r$  of  $\vec{r}$ , i.e. on the distance between the points.

Taking the Fourier transform of the two-point correlation function, one obtains the power spectrum  $P(k)$  as

$$P(k) = \int \xi(\vec{r}) e^{i\vec{k} \cdot \vec{r}} d^3\vec{r}. \quad (4.5)$$

It can be shown that the relation between the power spectrum and the Fourier transform  $\delta_k$  of the density contrast

$$\delta_k \delta_{k'}^\dagger = (2\pi)^3 P(k) \delta_D^3(\vec{k} - \vec{k}'), \quad (4.6)$$

holds, where  $\delta_D^3$  is the three-dimensional Dirac  $\delta$  function. It is then clear that  $P(k)$  is related to the variance of the density field in  $k$ -space:

$$P(k) = \frac{|\delta_k|^2}{(2\pi)^3}. \quad (4.7)$$

The power spectrum at a given wave number  $k$  is a measure of the “clumpiness” at a scale  $\lambda \sim 2\pi/k$ . In particular, let us compute the average density inside a sphere of radius  $\lambda$  centered around a given point in space. This will smooth out any inhomogeneities at scales much smaller than  $\lambda$ . Then, let us move the center of the sphere and repeat the procedure for every point in space. If the sample of values obtained in this way has a large variance, then  $P(k \sim \lambda/2\pi)$  will be large as well, and vice versa. Repeating the procedure for different sizes of the sphere will give the complete power spectrum  $P(k)$ .

The power spectrum is the key quantity when comparing the theory with observations of the large scale distribution of galaxies. A fundamental requirement of any successful cosmological model is then to predict a matter power spectrum in good agreement with the observations. The fluctuations

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<sup>4</sup>Strictly speaking, the brackets in Eq. (4.4) should denote an ensemble average. However, the homogeneity of the Universe makes it reasonable to assume that an ensemble average corresponds to a volume average.

at a given scale are often expressed in terms of the more convenient dimensionless quantity  $\Delta_k^2 \equiv k^3 P(k)/2\pi^2$ , whose relevance is related to having

$$\langle \delta(\vec{x})^2 \rangle = \frac{1}{(2\pi)^3} \int P(k) d^3 \vec{k} = \int_0^\infty \frac{k^3 P(k)}{2\pi^2} d \ln k, \quad (4.8)$$

so that  $\Delta^2(k)$  can be regarded as the contribution to the real space variance from a given logarithmic interval in  $k$ .

We remark that the two-point correlation function (or equivalently the power spectrum) encodes all the information on the statistical properties of the density field only if the fluctuations in the density field are Gaussian. A non-Gaussian field is also defined by its higher order moments, starting from the three-point correlation function (and so on), while for a Gaussian field the higher-order moments are either vanishing (if odd) or can be expressed in terms of the two-point correlation function (if even).

As a consequence of the central limit theorem, a field is Gaussian if the phases of the different Fourier modes  $\delta_k$  are uncorrelated and random. Since the linear evolution does not change the phases, this amounts to the requirement that the *initial* perturbations are Gaussian, as in the case of fluctuations produced during inflation. However, in general, the Gaussianity of the fluctuations is an assumption that has to be tested. This has been done, finding that the initial fluctuations were indeed highly Gaussian.

Another property to consider when comparing theory with observations is that the power spectrum  $P_m$  defined above refers to the *whole matter distribution*, i.e. including both baryons and dark matter. However, since we cannot directly observe dark matter, what is measured is just the power spectrum of the *luminous matter*, i.e. the galaxy power spectrum  $P_{\text{gal}} \neq P_m$ . The minimal assumption to obtain  $P_m$  from a measurement of  $P_{\text{gal}}$  is

$$P_{\text{gal}} = b^2 P_m \quad (4.9)$$

where  $b$  is a constant (i.e. scale-independent) *bias* parameter. Thus, the matter and galaxy spectra coincide, apart from their overall normalization. This assumption can be restated by saying that *light faithfully traces mass*. If this is true, the only effect of the mismatch between the matter and galaxy spectra is the introduction of an additional, effective parameter that has to be taken into account when analyzing the experimental data. However, in the past few years it has become increasingly clear that the assumption of a scale-independent bias is unsatisfactory, and a dependence of the bias parameter on  $k$  has to be introduced to better model the relationship between the dark and luminous matter distributions.

### 4.3 The Acceleration of the Universe

The most puzzling fact about our Universe is probably its presently accelerating expansion, whose main evidence comes from the observations of distant type Ia supernovae (SNe Ia). SNe Ia are standard candles, i.e. they are objects whose intrinsic luminosity is known, so that a measurement of their flux allows a determination of their luminosity distance  $d_L$ . By measuring also the redshift of the SN, the distance-redshift relationship can be reconstructed. Since it depends on the past expansion history, its determination allows to measure the energy budget of the Universe. In the small redshift limit, the distance-redshift relationship is given by the Hubble law  $z = H_0 d_L$ , as seen in Sec. 3.1.4. Going to larger redshift, some deviations from the linear behavior can be observed. Another advantage of using SNe Ia, apart from being standard candles, is their brightness (the typical luminosity is of the order of that of a galaxy) so that they can be observed up to high redshifts (currently, up to  $z \sim 1.8$ ).

In the 1990s, two research groups, the Supernova Cosmology Project and the High- $z$  Supernova Search, independently reported evidence that the Universe is accelerating. Both groups found that distant supernovae are dimmer than they should if the Universe were decelerating. In the framework of a FRW cosmology, their results pointed to  $\Omega_\Lambda > 0$ . We know from Sec. 3.2.1, and in particular from Eq. (3.49b) for the deceleration parameter, that, in the framework of FRW cosmology, the acceleration of the Universe cannot be produced by a normal matter or radiation component, i.e. a component with  $w > 0$ . In fact, acceleration requires  $w < -1/3$  as it can be seen from Eq. (3.48) or Eq. (3.49b).

Let us explain in more detail how the distance-redshift relationship can constrain the matter-energy content of the Universe. For simplicity, we restrict the discussion to the case of a flat Universe ( $K = 0$ ). We have seen in Sec. 3.1.4 that the luminosity distance  $d_L$  of a source is related to the coordinate distance  $r$  by Eq. (3.17), i.e.  $d_L = a_0 r(1+z) = d(1+z)$ , where  $d$  is the proper distance to the source today.

The proper distance  $d = a_0 r$  as a function of redshift can be calculated using the fact that for a photon  $ds^2 = dt^2 - a(t)^2 dr^2 = 0$ , so that

$$d = a_0 \int_0^r dr' = a_0 \int_{t_1}^{t_0} \frac{dt}{a} = a_0 \int_{a_1}^{a_0} \frac{da}{a^2 H(a)} = \int_0^z \frac{dz'}{H(z')}, \quad (4.10)$$

where  $t_1$  and  $a_1$  are the time and scale factor at the time the photon was emitted,  $dt = da/\dot{a} = da/aH$  and  $1+z = a_0/a$ . For a flat Universe composed by non-relativistic matter, radiation and by an exotic component



(“dark energy”) with a constant equation-of-state parameter  $w_{\text{de}}$  (so that its energy density  $\rho_{\text{de}}$  scales like  $(1+z)^{3(1+w_{\text{de}})}$ ), with respective density parameters  $\Omega_{\text{m}}$ ,  $\Omega_{\text{rad}}$  and  $\Omega_{\text{de}}$ , the Friedmann equation Eq. (3.46) can be put in the form

$$H(z) = H_0 \sqrt{\Omega_{\text{m}}(1+z)^3 + \Omega_{\text{rad}}(1+z)^4 + \Omega_{\text{de}}(1+z)^{3(1+w_{\text{de}})}}, \quad (4.11)$$

so that

$$d(z) = H_0^{-1} \int_0^z \frac{dz'}{\sqrt{\Omega_{\text{m}}(1+z')^3 + \Omega_{\text{rad}}(1+z')^4 + \Omega_{\text{de}}(1+z')^{3(1+w_{\text{de}})}}}. \quad (4.12)$$

Using the fact that the radiation density is negligible at  $z \ll z_{\text{eq}} \sim 10^4$ , and the flatness condition (implying  $\Omega_{\text{de}} = 1 - \Omega_{\text{m}}$ ), we finally get for the luminosity distance as a function of redshift

$$d_L(z) = H_0^{-1}(1+z) \int_0^z \frac{dz'}{\sqrt{\Omega_{\text{m}}(1+z')^3 + (1 - \Omega_{\text{m}})(1+z')^{3(1+w_{\text{de}})}}}, \quad (4.13)$$

which depends on the matter content of the Universe and on the dark energy equation-of-state parameter  $w_{\text{de}}$ . The cosmological constant case can be recovered by putting  $w_{\text{de}} = -1$  in Eq. (4.13). In Fig. 4.1, we show the luminosity distance (multiplied by  $H_0$ ) as a function of redshift for different values of the matter content of the Universe and of the dark energy equation-of-state parameter.

The relation between the luminosity distance and redshift is often more conveniently restated as a relation between the distance modulus and redshift. The distance modulus  $\mu$  is the difference between the apparent magnitude  $m$  (defined as the logarithm of the ratio of the flux to a reference flux) and the absolute magnitude  $M$  (defined as the logarithm of the ratio of the luminosity to a reference luminosity), i.e.

$$\mu(z) \equiv m - M = 5 \log_{10}(d_L/10\text{pc}). \quad (4.14)$$

A plot of  $\mu(z)$  vs.  $z$  is called a *Hubble diagram*.

#### 4.4 The Cosmic Microwave Background

The CMB is a unique cosmological observable providing a “snapshot” of the Universe as it was nearly 400,000 years after the Big Bang, corresponding to a redshift  $z \sim 1100$ , when the photons of the CMB last interacted with matter; after that time, they have been freely streaming until the present time. Thus, the CMB radiation carries a wealth of information about the

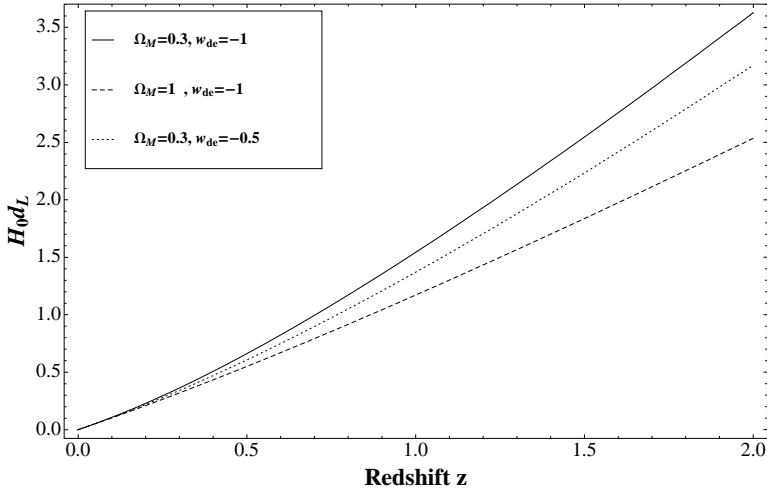


Figure 4.1 The effects of the matter content  $\Omega_M$  and of the dark energy equation-of-state parameter  $w_{de}$  on the luminosity distance  $d_L$  as a function of redshift  $z$ .

physical conditions in the early Universe and a great deal of effort has gone into measuring its properties since its serendipitous discovery by Arno Penzias and Robert Wilson in 1965. In fact, as we have seen in Chap. 1, the very existence of the CMB was enough to make the Hot Big Bang model prevail over the Steady State Universe model. The first, fundamental information that can be inferred from the observation of the CMB is that the photons in the early Universe were in thermal equilibrium. This arises from the blackbody spectrum of the CMB with a temperature of 2.725 K all across the sky, providing the more perfect blackbody spectrum observed in nature. Figure 4.2 reports the spectrum of the cosmic microwave radiation measured by the Far InfraRed Absolute Spectrometer (FIRAS) instrument on board the COBE satellite.

The CMB blackbody spectrum is a consequence of the fact that the frequent Thomson scattering of photons over electrons maintained the thermal equilibrium of the plasma. The scattering was very effective until the electrons recombined with the free protons in the plasma to form neutral hydrogen atoms, when the temperature of the Universe was  $T \simeq 3500 \text{ K} \simeq 0.3 \text{ eV}$ , corresponding to the value  $z \simeq 1100$ ; at that time, the photons scattered over electrons for the last time.<sup>5</sup> For this reason, the spatial (hyper-) sur-

<sup>5</sup>More precisely, since the Universe will later get reionized by the UV light of the first stars, the photons have been able to scatter again before the present time.

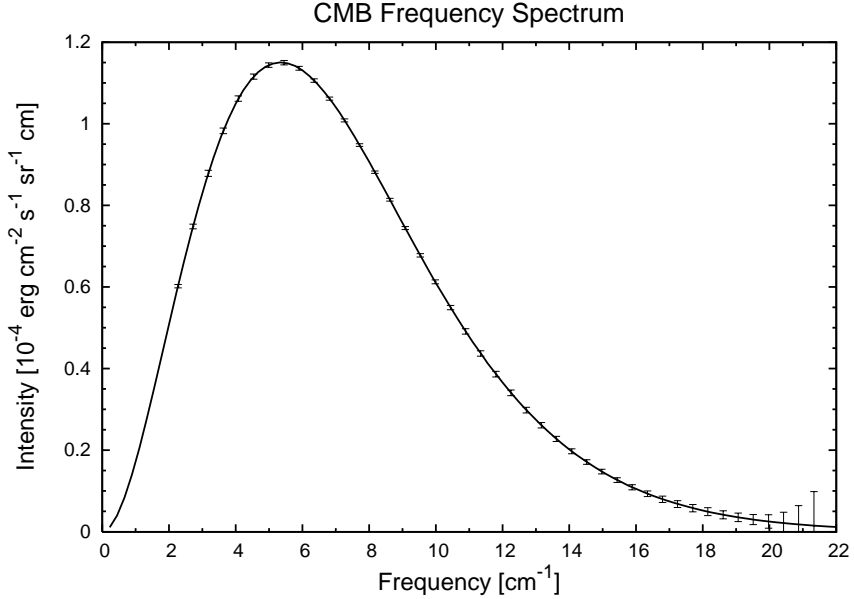


Figure 4.2 Intensity of the CMB radiation as a function of frequency, measured by the FIRAS instrument, shown with  $100\sigma$  errorbars. The solid curve is the theoretical expectations for a blackbody at  $T = 2.725$  K.

face at the time  $t(z = 1100)$  is called the *last scattering surface*. The reason why the temperature at the time of last scattering is so much smaller than the hydrogen ionization threshold  $E_H = 13.6$  eV is the tiny value of the baryon-to-photon ratio  $\eta \equiv n_b/n_\gamma \sim 10^{-10}$ . Thus, even when the average photon energy is well below 13.6 eV, there is still a sufficiently large number of photons with energy  $E > E_H$  able to photoionize the hydrogen atoms and prevent recombination. After recombination, the photons can travel almost freely through the Universe, so that the present-day microwave sky gives a faithful image of the last scattering surface, apart from the redshift of the photon energy due to the cosmological expansion.

The second remarkable property of the CMB is its extreme isotropy. The deviation  $\Delta T(\hat{n}) = T(\hat{n}) - \bar{T}$  from the average temperature at a given direction  $\hat{n}$  in the sky is everywhere  $\lesssim 200 \mu\text{K}$ , corresponding to a fractional deviation  $\Delta T(\hat{n})/T < 10^{-4}$ . This high degree of isotropy can be traced back to the equally high homogeneity of the cosmological plasma at the time of recombination. However, the CMB is not *perfectly* isotropic, implying that

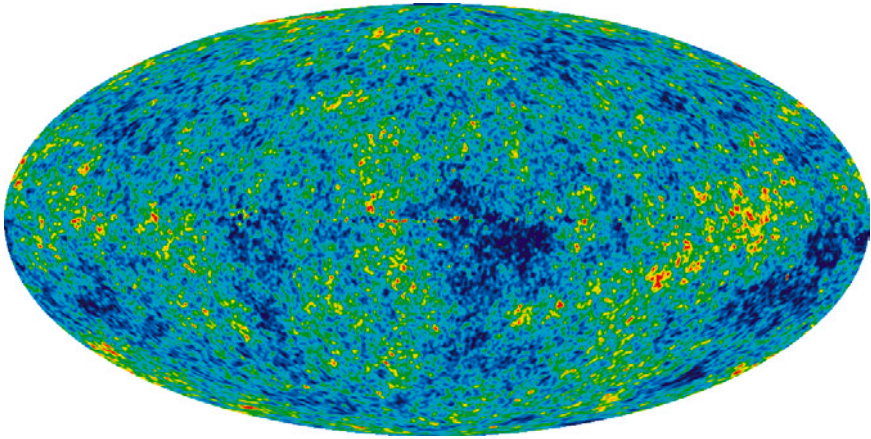


Figure 4.3 The microwave sky as seen by the WMAP experiment after seven years of observations. The image shows a temperature range of  $\pm 200 \mu\text{Kelvin}$ . (Source: <http://map.gsfc.nasa.gov/>).

there were indeed small perturbations in the plasma. In some sense, one should expect this, since, as explained in the previous section, galaxies are thought to be formed through the growth of small primordial density fluctuations. The anisotropies of the CMB are in fact related to the density perturbations at the last scattering surface. Similarly to what happens for the distribution of matter, a successful cosmological model should be able to explain the angular distribution of CMB anisotropies. A further advantage is given by the fact that the perturbations were still linear at the time of recombination (as testified by  $\Delta T/T \ll 1$ ), so that complications related to the non-linear stages of evolution are absent. For all these reasons, a great deal of observational effort has been invested to measure the anisotropy pattern.

#### 4.4.1 Sources of anisotropy

Let us briefly look at the mechanisms through which density and velocity perturbations give rise to the temperature anisotropies. It is customary to distinguish between *primary* anisotropies, already present at the time of last scattering, and *secondary* anisotropies, created along the photon's path from the last scattering surface to the observer. There are three sources of primary anisotropies. The first ones are the density fluctuations themselves: where the plasma is denser, it is also hotter. The second source is

given by the velocity perturbations. A patch that is moving towards us will appear hotter due to the Doppler shift, and vice versa a patch moving in the opposite direction will seem colder. The third source is given by the perturbations to the gravitational potential. Photons coming from potential wells will appear colder, since they lose more energy to climb out of the well. This is known as Sachs-Wolfe (SW) effect. For what concerns the secondary anisotropies, the most important is the *integrated* Sachs-Wolfe (ISW) effect. The physical mechanism is exactly the same at the basis of the SW effect, i.e. a difference in the gravitational potential, but this time the difference arises from the time variation of the potential as the photons travel towards the observer (hence the term “integrated”). Since the gravitational potentials are constant in a matter dominated Universe, this effect is relevant either at early times (just after recombination), when the radiation contribution to the total density is still important, or at late times (close to the present day) when the contribution of dark energy becomes relevant. The two effects are referred to as “early” and “late” ISW.

Another source of secondary anisotropies is reionization. Once the first stars were formed, around redshift 10, the UV radiation emitted by stars reionized the neutral hydrogen and helium in the Universe. We know from the observations of the absorption spectra of distant quasars that the Universe was completely reionized at least from redshift  $z \sim 6$ . When the Universe is even partially reionized, free electrons are present and the CMB photons are scattered again. Roughly speaking, this new “last” scattering will mix up photons coming from different points of the last scattering surface at  $z = 1100$  and thus will tend to smear out anisotropies on scales below the horizon at the time of reionization (i.e. an angular scale  $\theta \sim 5^\circ$  for a reionization occurring at  $z \sim 10$ ). The temperature fluctuations below this angular scale are suppressed by a factor  $e^{-\tau}$  ( $\tau$  is the optical depth to the last scattering surface) which is the fraction of unscattered photons. Finally, the Sunyaev-Zel’dovich effect generates secondary anisotropies due to the scattering of the CMB photons over the free electrons present in the hot intracluster medium.

#### 4.4.2 *The power spectrum of CMB anisotropies*

As it was the case for the spatial fluctuations of the cosmological density field, it is not possible to directly compare the exact pattern of temperature fluctuations that we observe with the predictions of a certain theory. This is because what we see in the sky is just *one* particular realization of the ran-

dom process from which the temperature fluctuations originated.<sup>6</sup> Instead, successful theories are required to predict the right statistical properties of the temperature field. As usual, if the fluctuations are Gaussian, all the statistical properties of the CMB are encoded in the power spectrum, related to the two-point correlation function. A difference with respect to the power spectrum of galaxies is that the observed temperature field is two-dimensional, depending only on the direction of observation  $\hat{n}$ .

The first step is to expand the temperature field  $\Theta(\hat{n}) \equiv \Delta T(\hat{n})/\bar{T}$  in spherical harmonics as

$$\Theta(\hat{n}) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\hat{n}), \quad (4.15)$$

which is the analogue on the two-sphere of the Fourier transform in three-dimensional space. The functions  $Y_{lm}$  form a complete basis over the two-sphere and the set of  $a_{lm}$  encodes all the information present in the original function. The spherical harmonics satisfy the orthonormality relation

$$\int Y_{lm}(\hat{n}) Y_{l'm'}^{\dagger}(\hat{n}) d\Omega = \delta_{ll'} \delta_{mm'}, \quad (4.16)$$

where  $d\Omega$  is the infinitesimal element of solid angle spanned by  $\hat{n}$ . This can be used to invert Eq. (4.15) and write the  $a_{lm}$  in terms of  $\Theta$

$$a_{lm} = \int \Theta(\hat{n}) Y_{lm}^{\dagger}(\hat{n}) d\Omega. \quad (4.17)$$

As noted above, we cannot make predictions directly for the  $a_{lm}$  [or equivalently for  $\Theta(\hat{n})$ ] but only for their statistical properties. The mean value of the  $a_{lm}$ 's vanishes because the average of fluctuations is zero by definition. The temperature angular power spectrum  $C_l$  is given by the variance of the  $a_{lm}$ 's as

$$\langle a_{lm} a_{l'm'}^{\dagger} \rangle \equiv \delta_{ll'} \delta_{mm'} C_l. \quad (4.18)$$

If the temperature fluctuations (and thus the  $a_{lm}$ 's) follow a Gaussian distribution, the  $C_l$ 's completely define the temperature field, at least from a statistical point of view. Let us stress one subtlety in the definition of the  $C_l$ 's, with important observational consequences. Since we have just *one* Universe, we can observe just *one* temperature field and the corresponding  $a_{lm}$ 's. However, the definition of the  $C_l$ 's involves an average operation — in particular, the brackets in Eq. (4.18) denote an ensemble average, i.e.

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<sup>6</sup>This random process is related to the presence of quantum fluctuations in the early Universe, as we shall see in more detail in Chap. 5.

an average over many independent realizations of the underlying random process. If we were in a laboratory, we could repeat this process many times under the same conditions observing the results at every realization. For the CMB this is clearly impossible — so how can we perform the average? As indicated by Eq. (4.18), the  $C_l$ 's do not depend on  $m$ , i.e. for a given  $l$  all the  $a_{lm}$ 's have the same variance. This (as well as the constraints  $l = l'$  and  $m = m'$ ) follows from the assumption of statistical isotropy, since the  $C_l$ 's cannot depend on the orientation of the coordinate system. For a given  $l$ , the parameter  $m$  can assume  $2l + 1$  possible values, providing as many samples drawn from the same distribution. We can construct the unbiased estimator of  $C_l$

$$\hat{C}_l = \frac{1}{2l + 1} \sum_{m=-l}^l a_{lm}^\dagger a_{lm}. \quad (4.19)$$

The possibility to sample the distribution of the  $a_{lm}$ 's only with a limited number of values (equal to  $2l + 1$  for a given  $l$ ), implies an intrinsic limitation to the measurement accuracy for a given  $C_l$ . This effect, known as *cosmic variance*, is more important at low  $l$ 's where the number of  $a_{lm}$ 's samples is smaller. In particular, the minimum variance of a measured  $C_l$  is given by  $2C_l^2/(2l + 1)$ , so that the relative uncertainty related to the cosmic variance is

$$\left( \frac{\Delta C_l}{C_l} \right)_{CV} = \sqrt{\frac{2}{2l + 1}}, \quad (4.20)$$

where CV stands for “cosmic variance”. Another effect that introduces an uncertainty in the measurement of the  $C_l$ 's is the *sample variance* due to the fact that an experiment does not observe the full sky, but covers a solid angle  $A < 4\pi$ . Quantitatively, this increases the cosmic variance by a factor of  $4\pi/A$ . This is also relevant for full-sky experiments, since usually some parts of the sky that are very contaminated by foreground emission, like the Galactic plane, are removed when analyzing CMB fluctuations. Both cosmic and sample variance are present independently of the resolution and sensitivity of the instrument. In particular, the cosmic variance represents an intrinsic limitation in the measure of the  $C_l$ 's. For this reason, when performing forecasts for the accuracy with which future CMB observations will be able to constrain a given parameter, it is often customary to consider as the most optimistic case that of an ideal, “cosmic variance-limited” experiment, i.e. one where the only source of error is that in Eq. (4.20).

#### 4.4.3 *Acoustic oscillations*

We will now give a qualitative description of how the distinctive sequence of oscillating peaks in the CMB angular spectrum is generated. This structure is due to standing pressure waves in the plasma prior to recombination. At that time, the photons were tightly coupled to baryons through Thomson scattering and the pressure of the plasma was mainly given by that of the photon component, i.e.  $P = \rho_\gamma/3$ . This corresponds to a sound speed  $v_s$  of the order of the speed of light, i.e.  $v_s = \sqrt{\partial P / \partial \rho} = 1/\sqrt{3}$ , whose large value prevented the growth of baryon density perturbations. In terms of the Jeans mechanism (see Sec. 3.4), the Jeans length of the baryon-photon fluid is comparable in size to the cosmological horizon. Then all the perturbations inside the horizon at the time of recombination are oscillating, while those outside are “frozen” to their initial values. If perturbations at all scales have the same initial conditions as  $t \rightarrow 0$ , i.e. if all modes oscillate with the same phase  $\phi$ , some of them will be caught at a maximum or minimum of the oscillation at recombination and will correspond to peaks in the angular power spectrum (since the spectrum is proportional to the variance of the temperature, both maxima and minima give rise to a peak). On the other hand, modes that are caught at the zero of the oscillation correspond to dips in the spectrum. Since all waves have the same phase, the distance between peaks and dips follows a harmonic pattern.

Let us give a more quantitative description of the physics of acoustic oscillations. The equations for the evolution of temperature fluctuations in the tight coupling limit were put in the form of an oscillator equation in a classic work by Hu and Sugiyama in 1996. In a slightly simplified form, neglecting for the moment the dynamical effects of baryons, the equation for the temperature fluctuation  $\Theta_k$  in Fourier space takes the forced oscillator form

$$\ddot{\Theta}_k + v_s^2 k^2 \Theta_k = \mathcal{F}, \quad (4.21)$$

where dots denote derivatives with respect to the conformal time  $\eta$ ,  $k$  is the wave number of the perturbation and the forcing term  $\mathcal{F}$  takes into account the effects of gravity. Neglecting the forcing term, the more general solution to the homogeneous equation is simply  $\Theta_k(\eta) = \bar{\Theta}_k \cos(v_s k \eta + \phi_k)$ , where the integration constants  $\bar{\Theta}_k$  and  $\phi_k$  depend on the initial conditions. For the moment, let us assume that  $\phi_k$  does not depend on  $k$  and that  $\bar{\Theta}_k$  is a simple featureless power law in  $k$ . As we shall see in Chap. 5, these are the initial conditions predicted by inflation. We also take  $\phi_k = 0$ , i.e.  $\dot{\Theta}_k(\eta = 0) = 0$ , thus neglecting the initial velocity perturbations. At a fixed



instant in time, and in particular at the time  $\eta_*$  of recombination (asterisks denote quantities evaluated at recombination) the temperature distribution as a function of  $k$  is given by

$$\Theta_k(\eta_*) = \bar{\Theta}_k \cos(ks_*) , \quad (4.22)$$

where  $s(\eta) \simeq v_s \eta \simeq \eta/\sqrt{3}$  is the distance a sound wave can travel in a time interval  $\eta$ , usually called the sound horizon (it is the acoustic equivalent of the causal horizon discussed in Sec. 3.1.5).

The form of  $\Theta_k(\eta_*)$  shows that the acoustic oscillations generate a cosine-like structure, superimposed on the featureless initial conditions  $\bar{\Theta}_k$ . The variance  $\Theta_k^2$  will exhibit a series of alternating peaks, starting with the first peak at  $k = \pi/s_*$  (corresponding to the mode with wavelength equal to twice the sound horizon) and with subsequent peaks at integer multiples of the first. At wavelengths much larger than the sound horizon,  $ks_* \ll 1$ , the perturbations will still be tracing their primordial values  $\bar{\Theta}_k$ , as illustrated in Fig. 4.4.

In the limit of constant gravitational potentials, the forcing term  $\mathcal{F}$  is equal to  $-k^2\Psi_k/3$ , where the curvature perturbation  $\Psi_k$  coincides with the Newtonian gravitational potential at scales well below the horizon. In this case, the oscillator equation can be put again in the homogeneous form by defining an effective temperature  $\Theta'_k = \Theta_k + \Psi_k$ . All the results obtained until now still hold as long as they are stated in terms of the effective temperature  $\Theta'_k$ , which is also the actual observed quantity. The reason is that after recombination photons have to climb out their potential wells to reach the observer, so that they lose energy proportionally to the value of the gravitational potential. This is the Sachs-Wolfe effect at the last scattering surface, briefly discussed above. In the case of time-varying gravitational potentials (the case when the Universe is not perfectly matter-dominated), the forcing term also includes terms proportional to the time derivative of the potential which give rise to the integrated Sachs-Wolfe effect also discussed above.

The effect of including baryons is twofold. First of all, they reduce the sound speed to  $v_s = 1/\sqrt{3(1+R)}$ , where  $R \simeq 3\rho_b/4\rho_\gamma$  is the baryon-to-photon momentum density ratio. This shifts all the peaks to larger  $k$ , i.e. they are now at  $k = n\pi\sqrt{3(1+R)}/\eta_*$  instead of  $k = n\pi\sqrt{3}/\eta_*$ . Secondly, the baryons shift the zero of the acoustic oscillations to  $\Theta_k = -(1+R)\Psi_k$ ,

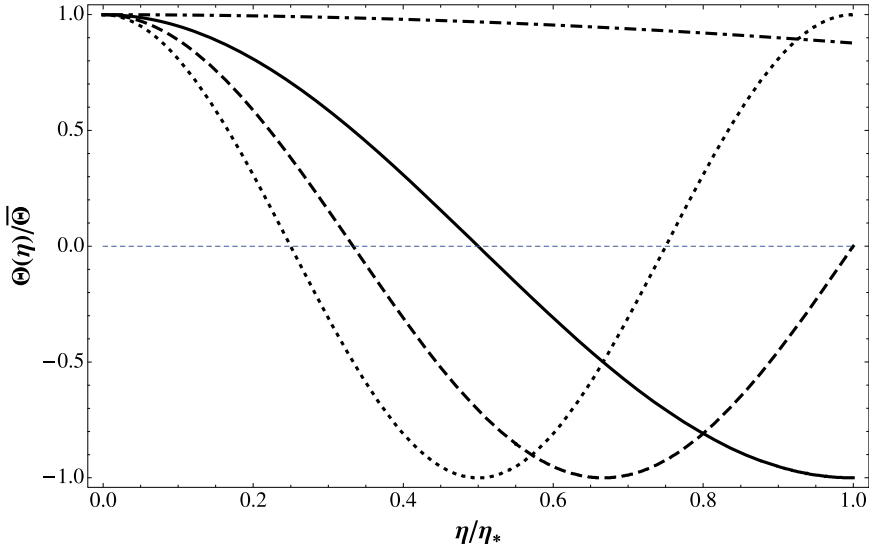


Figure 4.4 Behavior of different  $k$ -modes of temperature fluctuations as a function of the conformal time  $\eta$ . We show a mode that at recombination is caught at a minimum of the oscillation ( $k = \pi/s_*$ , solid line), one that is caught in phase with the background ( $k = 3\pi/2s_*$ , dashed line) and one that is caught at a maximum of oscillation ( $k = 2\pi/s_*$ , dotted line). These modes correspond to the first peak, first dip and second peak in the anisotropy spectrum. We also show a mode with  $ks_* \ll 1$  (dot-dashed thin line), i.e. with a wavelength much larger than the sound horizon, that had no time to evolve and is still tracing the initial condition  $\bar{\Theta}$ .

since they increase the inertia of the plasma. In the limit of constant  $R$ , the effective temperature field at recombination is

$$\Theta'_k(\eta_*) = \bar{\Theta}_k \cos(ks_*) - R\Psi_k, \quad (4.23)$$

with  $s_* \simeq \eta_*/\sqrt{3(1+R)}$ . This form breaks the symmetry between odd peaks (corresponding to the maximum compression of the plasma) and even peaks (corresponding to maximum rarefaction). In particular, odd peaks are enhanced while even peaks are suppressed.

Another effect that should be taken into account is radiation damping, better known as *Silk damping*, due to the fact that the baryon-photon fluid is not a perfect fluid. In particular, shear viscosity and heat conduction effects become important at scales below the mean free path of photons  $\lambda_\gamma$ , in particular close to recombination when the tight coupling approximation breaks down. Simply speaking, Silk damping is due to the fact that not-so-tightly coupled photons can diffuse out of overdense regions and into underdense regions and then cancel small-scale fluctuations in the radiation

density. The mean free path of photons is  $\lambda_\gamma = 1/(n_e\sigma_T)$  where  $n_e$  is the number density of electrons and  $\sigma_T$  is the Thomson cross section. In a Hubble time  $H^{-1}$ , a photon will scatter on average  $n_e\sigma_T/H$  times, so that the mean total distance traveled in that interval will be

$$\lambda_d = \lambda_\gamma \sqrt{\frac{n_e\sigma_T}{H}}. \quad (4.24)$$

We expect that perturbations below the damping scale  $\lambda_d$  are canceled. A careful numerical integration of the Boltzmann equation is required in order to follow the evolution of  $\lambda_d$  as the photons decouple from baryons and  $\lambda_\gamma \rightarrow \infty$ , however the calculations show that inhomogeneities are damped by a factor  $\exp(-k^2/k_d^2)$ , with the critical wave number  $k_d$  of the order of  $10/s_*$ .

Finally, after recombination, photons are no longer coupled to the baryons and can travel almost freely. This is when the secondary anisotropies like the ISW effect and reionization, briefly discussed above, come into play.

Let us briefly discuss how the 3-D field  $\Theta'_k$  translates into the anisotropy spectrum. In a flat Universe, a temperature *spatial* fluctuation with wavelength  $\lambda = 2\pi/k$  will roughly correspond to *angular* fluctuations at the scale  $\theta \sim \lambda/(\eta_0 - \eta_*) \simeq \lambda/\eta_0$ , where  $\eta_0$  is the conformal time today and thus  $\eta_0 - \eta_*$  is the comoving distance between us and the last scattering surface. When considering the multipole expansion of the temperature field on the sphere [see Eq. (4.15)], a given angular scale corresponds (roughly) to a multipole  $l \sim 1/\theta$ . Summarizing, inhomogeneities on a scale  $k$  are mapped onto anisotropies at the multipole  $l \simeq k\eta_0$ . In particular, the peaks will be located at multipoles  $l_n \simeq n\pi\eta_0/s_*$ . Putting the numbers,<sup>7</sup> one gets that the first peak should be at  $l \simeq 180$ , in agreement with the observed position of the first peak. In a non-flat Universe, a given angular scale  $k$  would not be projected onto an angle  $\theta \simeq (k\eta_0)^{-1}$ , but on a larger or smaller angle in the case of a closed or open Universe, respectively. The position of the first peak is, in fact, a powerful way to measure the curvature of the Universe. The whole argument basically relies on the knowledge of the distance to the last scattering surface and of the size of the acoustic horizon at decoupling, so that the angle under which it gets projected provides information on the spatial curvature. In jargon, the sound horizon at decoupling is a “standard ruler”.

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<sup>7</sup>This can be done by taking  $s_* = \eta_*/\sqrt{3}$  and noting that during the matter dominated era  $\eta$  grows like  $(1+z)^{-1/2}$ , so that  $\eta_0/\eta_* \simeq \sqrt{z_*} \simeq 30$ .

Of course, the general calculation of the exact temperature pattern  $\Theta_k$  will require to follow numerically the evolution of the perturbations, including all the effects neglected so far (like varying gravitational potentials, the dependence of the gravitational potentials themselves on the photon density and thus on  $\Theta$ , etc.). Several codes have been developed to this aim. The first was CMBFAST, written by U. Seljak and M. Zaldarriaga (partially based on E. Bertschinger's COSMICS package). To date, the most widely used is the CAMB code by A. Lewis and A. Challinor, itself partially based on CMBFAST.

Let us discuss the issue of the initial conditions, and in particular the assumption of coherent oscillations, i.e. that the phase  $\phi_k$  is the same for all  $k$ -modes. The initial conditions have to be given by the theory that explains how the primordial fluctuations have been generated, such as inflation which in fact predicts that  $\phi_k$  is the same for all  $k$ 's. Although we will discuss inflation in more detail in the next chapter, we can anticipate that the reason is that all the perturbations are generated at the same time independently of  $k$ . In models when this does not happen, for example in topological defects models that were extensively studied during the '90s as a possible alternative to inflation, the phases  $\phi_k$  are uncorrelated. The resulting incoherent oscillations lead to a washing out of the acoustic peaks. In fact the observations of the peak structure of the CMB anisotropy spectrum, first made by the BOOMERanG experiment in the late '90s, ruled out topological defects as the main mechanism of generation of the primordial fluctuations, in favor of the inflationary paradigm.

#### 4.4.4 *Effect of the cosmological parameters*

The main features of the CMB anisotropy spectrum can be qualitatively predicted through the arguments presented above, however the precise values of the  $C_l$ 's depend on the cosmological parameters. This is the reason why the CMB angular spectrum is such a powerful tool to measure the cosmological parameters. We will briefly discuss their effect.

The first step is to choose *which* parameters should be used to describe our Universe. The simplest model able to explain the WMAP data is a 6-parameter, flat  $\Lambda$ CDM model with adiabatic initial conditions. This minimal model is often dubbed “vanilla”  $\Lambda$ CDM. The parameters of the vanilla model are the physical<sup>8</sup> baryon density  $\omega_b \equiv \Omega_b h^2$  (where  $h$  is the Hubble

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<sup>8</sup>The term *physical density* (denoted with  $\omega$ ) is a jargon to indicate the density parameter  $\Omega = \rho/\rho_c$  multiplied by  $h^2$ , in order to eliminate the uncertainty related to the fact

constant  $H_0$  in units of  $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$ ), the physical cold dark matter density  $\omega_c \equiv \Omega_c h^2$ , the cosmological constant density  $\Omega_\Lambda$ , the amplitude of the primordial curvature perturbations  $\Delta_{\mathcal{R}}^2$  at the scale  $k_0 = 0.002 \text{ Mpc}^{-1}$ , the slope of the primordial spectrum  $n_s$ , and the optical depth  $\tau$  to the last scattering surface. Of these six parameters, three ( $\omega_b$ ,  $\omega_c$  and  $\Omega_\Lambda$ ) describe the matter-energy content of the Universe, two ( $\Delta_{\mathcal{R}}^2$  and  $n_s$ ) describe the initial conditions from which the perturbation evolution started, and the remaining one ( $\tau$ ) is related to the reionization of the Universe at  $z \sim 10$ . The value of  $h$  is fixed requiring that the Universe should be flat, i.e.  $\Omega_b + \Omega_c + \Omega_\Lambda = (\omega_b + \omega_c)/h^2 + \Omega_\Lambda = 1$ . A different combination of the parameters could have been chosen, for example selecting  $\Omega_b$ ,  $\Omega_c$  and  $H_0$  in place of  $\omega_b$ ,  $\omega_c$  and  $\Omega_\Lambda$ . In this case,  $\Omega_\Lambda$  would be the derived parameter fixed by requiring flatness, i.e.  $\Omega_\Lambda = 1 - \Omega_b - \Omega_c$ . In general, there is no “correct” choice for the parameter set. However, one tries to choose the parameters such that each of them has a unique, peculiar effect on the anisotropy spectrum (even if that is not always possible). Thus, one can disentangle the effect of each parameter, useful for pedagogical purposes, other than when performing “real analyses” of the data. The one presented here is a fairly common choice of the minimal parameter set. A good parameter choice for one particular observable, for example the CMB, could not be as good for another one, for example the matter power spectrum. For these reasons, the choice of the parameters used to describe the cosmological model is often a matter of compromise. The vanilla model is just the simplest choice, so that it can be expanded by considering additional parameters beyond the minimal set. Some examples include, but are not limited to, the neutrino mass,<sup>9</sup> the effective number of relativistic species, the running of the spectral index, tensor modes, the fraction of isocurvature perturbations, the equation-of-state parameter for dark energy.

After this necessary *caveats*, let us examine the effect of the parameters on the spectrum.

*Dark matter density.* The lower  $\omega_c$ , the longer the radiation-dominated era lasts,<sup>10</sup> so that matter-radiation equality occurs closer to recombination. Since the gravitational potentials decay during the radiation-dominated

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that  $\rho_c \propto h^2$ . Of course, there is nothing really *unphysical* about the  $\Omega$ ’s.

<sup>9</sup>The vanilla  $\Lambda$ CDM indeed includes neutrino, but considers them massless. Even though we know, from oscillation experiments, that neutrinos do have a mass, this is a reasonable approximation for the minimal cosmological model since the effects of a finite mass are small.

<sup>10</sup>Recall that  $1 + z_{\text{eq}} = \omega_m/\omega_{\text{rad}}$ .

era, this will lead to a smaller gravitational potential at recombination and thus to a *larger* temperature fluctuation.<sup>11</sup> Another effect is the early integrated SW that occurs immediately after recombination due to the residual radiation. Both effects sum up to increase the amplitude of the spectrum for smaller  $\omega_c$ .

*Baryon density.* Changing  $\omega_b$  also produces the effects described above for  $\omega_c$ , since they only depend on the time of matter-radiation equality and thus on  $\omega_m = \omega_b + \omega_c$ . However,  $\omega_b$  also has a very peculiar effect on the spectrum since the presence of baryons is responsible for the alternating structure of the peaks (odd peaks are higher than even peaks<sup>12</sup>) and a larger value of  $\omega_b$  makes this asymmetry stronger, enhancing odd peaks and suppressing even peaks. This peculiar character makes this effect very easy to be isolated, and in fact  $\omega_b$  is one of the parameters that are better measured from the CMB. Another effect of an increased baryon density is a reduced diffusion damping (the photon mean free path is smaller), so that the small scale (high  $l$ 's) spectrum is larger.

*Cosmological constant.* The effect of a cosmological constant mainly comes from the late integrated SW effect, because the potential is not constant at late times, when the Universe is not perfectly matter dominated. A large value of  $\Omega_\Lambda$  thus enhances the large-scale anisotropies. However, since the spectrum is usually normalized at the large scales, the net effect of increasing  $\Omega_\Lambda$  is to suppress the small-scale anisotropies.

However, all the parameters considered until now have a small effect in the location of the peaks. In the case of  $\omega_c$  and  $\omega_b$  this is due to their effect on the sound horizon at recombination  $s_*$  and on the conformal time today  $\eta_0$ , while the cosmological constant density  $\Omega_\Lambda$  has no effect on  $s_*$  but changes  $\eta_0$ .

*Amplitude of the primordial curvature perturbations.* The effect of changing the amplitude of the primordial spectrum of fluctuations is to modify the normalization of the CMB spectrum.

*Spectral index.* Changing the spectral index  $n$  affects the relative height between the small and large scales. The Harrison-Zel'dovich spectrum  $n = 1$  generates a large-scale CMB spectrum that scales as  $[l(l+1)]^{-1}$ . This is in fact the reason why the spectrum is usually plotted in terms of the quantity

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<sup>11</sup>This somewhat counterintuitive result is due to the fact that, even if the smaller potential corresponds to a less dense (i.e. colder) region, the photons have to climb a less deep potential well and this will more than compensate for the smaller temperature. In other words,  $\Theta_k$  becomes smaller but the observed temperature  $\Theta_k + \Psi_k$  becomes larger.

<sup>12</sup>Before diffusion damping is taken into account.

Table 4.1 Cosmological parameters from WMAP7. Mean values of the parameters of the minimal (“vanilla”)  $\Lambda$ CDM model from the analysis of the 7-year WMAP data. The errors show the 68% confidence region. The primordial curvature fluctuation  $\Delta_{\mathcal{R}}^2$  is normalized at the pivot point  $k_0 = 0.002\text{Mpc}^{-1}$ . Adapted from Ref. [304].

Parameter	Mean WMAP
$100\Omega_b h^2$	$2.258^{+0.057}_{-0.056}$
$\Omega_c h^2$	$0.1109 \pm 0.0056$
$\Omega_\Lambda$	$0.734 \pm 0.029$
$n_s$	$0.963 \pm 0.014$
$\tau$	$0.088 \pm 0.015$
$10^9 \Delta_{\mathcal{R}}^2(k_0)$	$2.43 \pm 0.11$

$\mathcal{C}_l = l(l+1)C_l$ , so that at small  $l$ ’s it will reach a plateau. In terms of  $\mathcal{C}_l$  all multipoles receive the same contribution from an initial Harrison-Zel’dovich spectrum. If the spectrum is tilted ( $n \neq 1$ ), the contribution to  $\mathcal{C}_l$  will scale like  $l^{n-1}$ . If the initial spectrum is blue ( $n > 1$ ), the small scales will have more (primordial) power with respect to the large scales; vice versa if the spectrum is red ( $n < 1$ ). Considering again that the spectrum is normalized at small  $l$ ’s, we have that  $n > 1$  ( $< 1$ ) will increase (decrease) the overall power at large  $l$ ’s. Since it represents the slope of the spectrum,  $n$  can be measured more and more precisely as smaller scales become accessible to observations.

*Optical Depth.* A non-zero value of the optical depth  $\tau$  to the last scattering surface represents the integrated effect of the scattering of photons over free electrons after the Universe gets reionized at  $z \sim 10$ . As explained above, this tends to cancel the anisotropies at scales below the horizon at the time of reionization (roughly  $l \gtrsim 10$ ), with a suppression factor given by  $e^{-\tau}$ .

The measurement of the CMB anisotropy spectrum is a powerful tool to constrain the values of the cosmological parameters. In Tab. 4.1 we show the values of the six parameters of the minimal  $\Lambda$ CDM determined from the analysis of the 7-year WMAP data. In Fig. 4.5 we also show the best-fit anisotropy spectrum along with the WMAP data. The picture emerging from the CMB is that of a Universe with only 5% of baryons, 22% of cold dark matter and 73% made of a cosmological constant-like component, consistently with that coming from other observations.

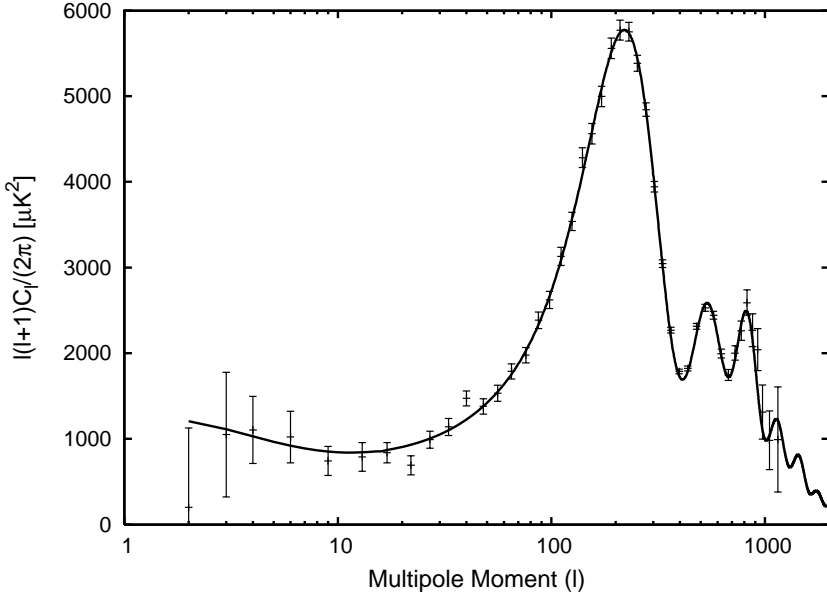


Figure 4.5 CMB anisotropy spectrum corresponding to the best-fit model of the 7-year WMAP analysis. The points show the WMAP7 data with the relative errorbars.

#### 4.5 Guidelines to the Literature

Many of the textbooks recommended in the previous Chapter also deal, to some extent, with the phenomenology of the observed Universe. Since this topic is strictly connected to the advances in the observational field, recent books provide a picture of the current observational status. In general, we refer the interested reader to Dodelson's book [155], that puts a focus on the quantitative comparison between theory and observations, with some bias towards the CMB and the matter distribution. It also provides an excellent introduction to the basic techniques that are used in the analysis of cosmological data.

The topic of large-scale structure discussed in Sec. 4.2 is the subject of many textbooks, like the classic one by Peebles [377] or the more recent one by Padmanabhan [370]. An introduction can also be found in the already mentioned books by Kolb & Turner [290] and Peebles [378]. The review [159] also covers in detail the topic of galaxy formation. The effect of collision-less damping on the evolution of perturbations in a hot dark matter component (massive neutrinos) was studied in [99,100]. We refer the reader



interested in the topic of dark matter to the review [83]. The galaxy power spectrum has been measured in the last decade by the *Two Degree Field* (2dF) galaxy survey (<http://msowww.anu.edu.au/2dF/index{2dF}GRS/>) and by the *Sloan Digital Sky Survey* (SDSS) ([www.sdss.org/](http://www.sdss.org/)). The cosmological implications of the observations of the 2dF and SDSS surveys are discussed in [123, 158, 167, 233, 375, 382, 383, 409, 448] and [412, 413, 433, 434], respectively. The most recent determination of the homogeneity scale, using the luminous red galaxy sample of the SDSS, can be found in [239].

The acceleration of the Universe, discussed in Sec. 4.3, including its observational evidence and the possible interpretations, is the subject of [178]. The papers reporting the first evidences for the acceleration are [392] and [386]. See also the websites [www.supernova.lbl.gov/](http://www.supernova.lbl.gov/) and [www.cfa.harvard.edu/supernova/HighZ.html](http://www.cfa.harvard.edu/supernova/HighZ.html). More recent SNIa data and their cosmological interpretation can be found in [296].

An introduction to the physics of the CMB discussed in Sec. 4.4 can be found in [246, 249] and [295]. Here we did not address the topic of the polarization of the CMB, for which we refer the interested reader to [107, 265, 294]. The description of the recombination process was first made by Peebles in [376] and the original paper by Silk on diffusion damping is of the same year [418]. The evolution of the perturbations in the baryon-photon fluid was first computed in [380]. Dark matter was introduced only later in the computations, see e.g. [453]. The physics of acoustic oscillations was investigated in detail in a series of papers by Hu & Sugiyama [247, 248]. The computational framework for the integration of the coupled Einstein-Boltzmann equation has been established by Ma & Bertschinger [329]. The calculation of the CMB anisotropies has been sped up after the introduction of the line-of-sight integration approach by Seljak & Zaldarriaga [414] and of their computer program *CMBFast* ([lambda.gsfc.nasa.gov/toolbox/tb\\_cmbfast\\_ov.cfm](http://lambda.gsfc.nasa.gov/toolbox/tb_cmbfast_ov.cfm)). The method has been refined by Lewis, Challinor & Lasenby [309] and implemented in the code *CAMB* ([camb.info/](http://camb.info/)). The measurements of the frequency spectrum of the CMB, shown in Fig. 4.2, were made by the FIRAS experiment on board the COBE satellite [173, 337, 338]. The most recent measurements of the CMB anisotropy spectrum shown in Fig. 4.5, along with their cosmological interpretation, made by the *WMAP* satellite ([map.gsfc.nasa.gov/](http://map.gsfc.nasa.gov/)) after seven years of observations, can be found in [72, 198, 264, 291, 304, 460].

## Chapter 5

# The Theory of Inflation

In this Chapter we will discuss the inflationary scenario, focusing on the most general features of this paradigm for the evolution of the early Universe. The idea of an early phase of inflationary expansion was developed between the end of the '70s and the beginning of the '80s, in order to overcome some critical shortcomings of the Standard Cosmological Model. Despite its standing success in solving basic paradoxes of the standard hot Big Bang model, the inflationary scenario still has an *ad hoc* taste. This is due to the need for a certain amount of fine-tuning of the model parameters (especially concerning the flatness of the scalar field potential) and, on the other hand, to the many alternative proposals for the detailed evolution of the self-interacting scalar field at the ground of the whole idea.

However, inflation can be regarded as a cosmological *theory* because its basic framework is well-motivated at the level of fundamental physics and its predictions, other than solving conceptual questions, are in agreement with the present observational knowledge of the Universe.

In the following, we will concentrate on the most general features of the inflationary scenario, which have been largely unaffected by the later developments of the theory and are the most relevant for the primordial history of the Universe, which is the main subject of this Book.

We start by discussing the basic shortcomings of the Standard Cosmology which require the introduction of a new paradigm. Then we will provide a brief description of the ideas characterizing the theory of elementary particles which offer the physical motivation and the dynamical tools to implement the key role of a phase transition during the evolution of the Universe, making available a dominant *vacuum energy*. The real inflationary evolution is implemented by describing the different dynamical regimes in the evolution of the self-interacting scalar field. We show how the infla-

tion solves the main puzzles of the standard cosmology and the resulting predictions. In particular, the mechanism by which this paradigm provides a perturbation spectrum for the isotropic Universe is treated in some detail. This Chapter ends with a brief discussion of the late acceleration of the Universe, a timely question related to the inflationary paradigm, which could be explained by the presence of an exotic component called *dark energy*, or by modifications to GR.

## 5.1 The Shortcomings of the Standard Cosmology

The SCM provides a successful representation of the Universe in terms of the Robertson-Walker (RW) geometry underlying the large-scale evolution of a homogeneous and isotropic thermal bath. As the Universe expands, the temperature of the bath decreases and a series of departures from equilibrium and phase transitions happen, like for example baryogenesis, nucleosynthesis and hydrogen recombination. In the latter phase of the evolution of the Universe, corresponding to the matter-dominated regime, the Jeans mechanism (see Sec. 3.4.3) explains the magnification of the primordial density perturbations, eventually resulting in the formation of cosmological structures once the perturbations reach the non-linear regime. The SCM is expected to fail close to the Planck era, when the quantum gravity effects have to be taken into account. Moreover, it has been argued that in the present Universe the small-scale inhomogeneities could induce significant deviations from the RW background even on very large scales, implying that the homogeneity hypothesis at the basis of the SCM is not completely correct. However, these are not failures of the SCM *per se*, but instead just limits for its domain of applicability. On the other hand, the SCM leads to paradoxical results when it is applied to the very early Universe just after the Planck time. All such paradoxes, described in more detail in the following, are somewhat related to the very particular initial conditions needed to obtain the present day Universe.

In the following Chapters we will describe a rather different scenario for the very early Universe with respect to the homogeneous and isotropic framework. Inflation plays a crucial role in reconciling these very general dynamical perspectives with the SCM phenomenology. However, we stress that even if the main aim of this Book is to investigate a very general (anisotropic and inhomogeneous) nature of the Big Bang, nevertheless the request for an inflationary phase of the Universe arises from internal incon-

sistencies of the SCM, emerging as soon as even qualitative observational evidences are critically analyzed.

This section is then devoted to the analysis and discussion of four fundamental shortcomings of the SCM, the so-called *horizon* and *flatness paradoxes* and the *entropy* and *unwanted relics problems*.

### 5.1.1 The horizon and flatness paradoxes

**The horizon paradox** The evidence of a conceptual problem in the understanding of the Friedmann-Robertson-Walker (FRW) Universe (essentially characterized only by the radiation- and matter-dominated eras) arose immediately after the discovery of the CMB and of its high degree of isotropy (the temperature angular fluctuations are less than one part in  $10^4$ , see Sec. 4.4). The observation of such isotropic thermal radiation provided a compelling evidence in favor of the hot Big Bang theory, but it was soon realized that the spatial uniformity of such black body was indeed problematic.

To understand the paradox, one needs to relate the CMB isotropy to the notion of causality. As explained in Sec. 3.1.5, in a Friedmann Universe the size of the causal horizon coincides, apart from factors of order unity, with the Hubble length  $L_H$ . Since  $H \propto 1/t$  and during the matter-dominated era  $a \propto t^{2/3}$ , the Hubble length at the time of hydrogen recombination was  $L_H(t_{\text{re}}) = H_{\text{re}}^{-1} = H_0^{-1}(1+z_{\text{re}})^{-3/2} = L_H^0(1+z_{\text{re}})^{-3/2}$ , where  $z_{\text{re}} \simeq 1100$ . The corresponding physical distance today is  $d = (1+z_{\text{re}})L_H(t_{\text{re}}) = (1+z_{\text{re}})^{-1/2}L_H^0$ .

To estimate how many independent causal regions are contained in the CMB sphere, we observe that the latter has a surface of the order  $4\pi^2(cH_0^{-1})^2$ , thus the number of the observed independent causal regions is

$$\text{n.c.r.} \sim \left( \frac{L_H^0}{L_H(t_{\text{re}})(1+z_{\text{re}})} \right)^2 \sim (1+z_{\text{re}}) \sim 10^3. \quad (5.1)$$

In other words, when we look at the extremely uniform microwave sky, we are actually looking at  $\sim 1000$  independent causal regions at the time of recombination, which never had the chance to be in thermal contact; in spite of this, they all have the same temperature within one part in  $10^{-4}$ . The question at the basis of the paradox is: *why have these regions such a fine tuned temperature if they had never been in thermal contact among themselves at the time when the CMB was emitted?*

A possible answer is that the homogeneity was part of the initial conditions. In fact, one could ask: *Why is the CMB isotropy so strange if we considered a RW geometry?*

The point is that if we assign, at a given instant, the initial conditions for the cosmological fluid, we unavoidably deal with uncertainties on the fundamental matter fields. Despite their smallness, such uncertainties independently evolve on disconnected causal regions. After a certain interval of time, the matter fields eventually acquire a degree of inhomogeneity due to the specific thermodynamical evolution of each horizon. Thus, to interpret the CMB isotropy as a consequence of the initial conditions, we need an estimate of the density contrast in a primordial stage, say at the Planck era.

The fractional temperature fluctuation of the CMB is  $\delta T/T \lesssim 10^{-4}$  and traces the density fluctuations in the cosmological fluid at the time of recombination  $t_{\text{re}}$  (see Sec. 4.4 for more details). Since the fluid was in thermal equilibrium, one has  $\rho \propto T^4$  and  $\delta \equiv \delta\rho/\rho = 4\delta T/T$ . Thus, a reliable estimate of the degree of inhomogeneity of the Universe at the recombination, for  $z \simeq 10^3$ , is given by the value  $\delta_{\text{re}} \simeq 10^{-4}$ . As discussed in Sec. 3.5.5, during the matter-dominated era the density perturbations outside the horizon grow like  $a \propto t^{2/3}$  and during the radiation-dominated era as  $a^2 \propto t$ , one can compute the density contrast  $\delta_{\text{P}}$  at the Planck era. At the time of matter-radiation equality it was  $z \simeq 10^4$  [ $t_{\text{eq}} \sim \mathcal{O}(10^{12})$  s], and therefore  $\delta_{\text{eq}} \sim \mathcal{O}(10^{-5})$ . Hence, at the Planck era [ $t_{\text{P}} \sim \mathcal{O}(10^{-44})$  s], we get  $\delta_{\text{P}} \sim \mathcal{O}(10^{-5}) \times t_{\text{P}}/t_{\text{eq}} \sim \mathcal{O}(10^{-61})$ ! Such a Planckian value of the density contrast is too small to be physically acceptable as an initial condition, especially in view of the quantum fluctuations characterizing those primordial phases. The horizon paradox is therefore a real and deep conceptual inconsistency of the SCM, unless an extreme fine tuning of the initial conditions is accepted as an *a priori* prescription of the Nature.

**The flatness paradox** The present value of the spatial curvature of the Universe is very small; as discussed in Chap. 4, the CMB observations indicate the present value of the critical parameter  $\Omega = 1.01 \pm 0.02$ , or, in other words,  $|\Omega - 1| \lesssim 10^{-2}$ , thus the Universe is very close to being flat, even if the sign of the curvature is still unknown. The flatness paradox emerges from analyzing the structure of the relation (3.52), which is restated here for convenience

$$\Omega - 1 = \frac{K}{H^2 a^2}. \quad (5.2)$$

This formula, recalling that the matter and radiation energy densities behave as  $\rho_m \sim 1/a^3$  and  $\rho_{\text{rad}} \sim 1/a^4$  respectively, allows us to get the behavior of the quantity  $(\Omega - 1)$  as a function of redshift. Today, the curvature term in the Friedmann equation is negligible with respect to the matter source and this was even more true in the past, since the curvature term scales like  $a^{-2}$  while the matter and radiation terms scale as  $a^{-3}$  and  $a^{-4}$  respectively. The Friedmann Eq. (3.46) states the proportionality between  $H^2$  and  $\rho$ , so that we can write

$$\text{Matter dominated Universe: } (\Omega - 1) \propto (1 + z)^{-1} \quad (5.3)$$

$$\text{Radiation dominated Universe: } (\Omega - 1) \propto (1 + z)^{-2}. \quad (5.4)$$

If today we have  $|\Omega_0 - 1| \lesssim 10^{-2}$ , at the time of equivalence  $z_{\text{eq}} \simeq 10^4$  we get  $|\Omega_{\text{eq}} - 1| \lesssim 10^{-6}$  and finally we gain the surprising Planck value ( $z_P \sim 10^{32}$ )

$$\Omega_P - 1 \lesssim \mathcal{O}(10^{-62}). \quad (5.5)$$

As for the density contrast in the previous subsection, we find again that the initial condition compatible with the present flatness of the Universe requires an extreme fine tuning of its initial value at the Planck time. From a physical point of view, this result is equivalent to require that the Universe is appropriately described by the flat RW geometry, i.e. by  $K = 0$ . Such situation is paradoxical since tiny, local density fluctuations (naturally expected after a quantum regime of the Universe), would have affected drastically the present structure of the cosmological space, but this is not the case. Thus, as for the horizon paradox, we deal with a subtle conceptual puzzle, which calls attention for its solution in a new cosmological framework.

### 5.1.2 The entropy problem and the unwanted relics paradox

**The entropy problem** The entropy problem can be stated noting that the entropy of the observable Universe is enormous. We know from Eq. (3.44) that the entropy density  $s \sim T^3$ . The size of the observable Universe, in a Friedmann Universe is roughly given by the Hubble length  $L_H = H^{-1}$  and then the total entropy  $S$  inside the presently observable Universe is

$$S = (T_\gamma^0 L_H)^3 \sim 10^{87} \quad (5.6)$$

where we have used the present-day photon temperature  $T_\gamma^0 \simeq 2.73 \text{ K} \sim 10^{-13} \text{ GeV}$  and  $L_H \sim 10^{28} \text{ cm} \sim 10^{42} \text{ GeV}^{-1}$ .

Such a large value of the entropy is especially puzzling if the expansion is taken to conserve entropy. This would imply that the Universe has started with an enormous entropy  $S \sim 10^{87}$ , and this appears like a very particular initial condition.

**The unwanted relics paradox** The unwanted relics paradox is related to the fact that if we allow the early Universe to have an arbitrarily high temperature at that time (or at least a temperature as high as the Planck energy), we expect very heavy degrees of freedom (for example new particle species predicted by unified gauge theories or by supersymmetry). Such relics could survive until today with an actual abundance that would be in gross contradiction with observations.

Let us briefly describe how to determine the present cosmological abundance of a species that was in thermal equilibrium in the early Universe. Considering in the early Universe a very heavy (for example,  $m_X > 100 \text{ GeV}$ ) particle  $X$ , the  $X$ 's were kept at equilibrium with the other particles in the cosmological plasma (call them in general  $Y$ ) by rapid annihilations of the type  $X\bar{X} \leftrightarrow Y\bar{Y}$ . This happens as long as the annihilation rate  $\Gamma_{\text{ann}}$  is fast enough with respect to the expansion rate, given by the Hubble parameter  $H$ , thus  $\Gamma_{\text{ann}} \gg H$ .

The annihilation rate  $\Gamma_{\text{ann}}$  is given by  $n_X \langle \sigma_{\text{ann}} v \rangle$ , where  $\langle \sigma_{\text{ann}} v \rangle$  is the thermally averaged cross section multiplied by velocity. When the annihilation rate drops below the expansion rate, i.e.  $\Gamma_{\text{ann}} \lesssim H$ , the annihilations are no longer effective in coupling the  $X$ 's to the other species in the plasma, and the number of  $X$ 's per comoving volume is “frozen” at the value it had at the time  $t_{\text{f.o.}}$  such that  $\Gamma_{\text{ann}}(t_{\text{f.o.}}) \simeq H(t_{\text{f.o.}})$ . This process is called *freeze-out* and after it, the number density  $n_X$  is diluted by the Universe expansion, i.e.  $n_X(t > t_{\text{f.o.}}) = n_X(t_{\text{f.o.}})[a(t_{\text{f.o.}})/a(t)]^3$ .

In more detail, we can distinguish three phases:

- Initially,<sup>1</sup>  $\Gamma_{\text{ann}} \gg H$  and  $T \gg m_X$ . The first condition ensures that the annihilations are very effective in maintaining the  $X$ 's at the equilibrium with the plasma. The second condition states that the  $X$ 's are ultrarelativistic, so that their number density  $n_X \simeq T^3$ . In particular, this implies that  $n_X \sim n_\gamma$  (this is basically a consequence of the equipartition theorem). Assuming for definitiveness that  $\langle \sigma_{\text{ann}} v \rangle = \text{const.} = \sigma_0^2$ , we have that  $\Gamma_{\text{ann}} \propto T^3$ . On the other hand, during the radiation-dominated era, from the Fried-

<sup>1</sup>In this Chapter, for convenience we put  $K_B = 1$ .

<sup>2</sup>This is the case for  $s$ -wave annihilations.

mann equation one has that  $H = \sqrt{\kappa \rho_{\text{rad}}^4/3} \propto T^2$  and the annihilation rate decreases faster than the expansion rate.

- As the Universe expands and cools down, it enters a second phase, corresponding to the regime when the annihilations are still effective,  $\Gamma_{\text{ann}} \gg H$ , but the  $X$ 's are non-relativistic,<sup>3</sup>  $T \lesssim m_X$ . The  $X$ 's still have an equilibrium distribution, but the equilibrium number density in the non-relativistic regime is  $n_X \simeq (m_X T)^{3/2} \exp(-m_X/T)$ , so that the number density exponentially decreases with temperature. This is also a consequence of the fact that when  $T < m_X$  the average energy of the other particles is not large enough to produce the  $X$ 's, so that the annihilations are not compensated by the inverse creation process, i.e. one has  $X\bar{X} \rightarrow Y\bar{Y}$  but not  $Y\bar{Y} \rightarrow X\bar{X}$ .
- Finally, when the temperature decreases enough,  $\Gamma_{\text{ann}} \lesssim H$  and the annihilations become ineffective, the  $X$ 's are not destroyed and their number per comoving volume is conserved.

From this picture one expects that the larger the annihilation cross section the smaller the freeze-out abundance, because annihilations will “switch off” later and the abundance will be more reduced with respect to its high-temperature value. One expects that for fixed  $\langle \sigma_{\text{ann}} v \rangle$ , a smaller mass will produce a larger relic abundance because the exponential decay of the density will start at a correspondingly lower temperature  $T \simeq m_X$ , and thus closer to freeze-out, leaving less time for the annihilations to operate. In fact, when the freeze-out process is worked out, the integral of the relevant Boltzmann equation, leads to<sup>4</sup>:

$$N_X \propto \frac{1}{m_X \langle \sigma_{\text{ann}} v \rangle_{\text{f.o}}} = \frac{1}{m_X \sigma_0}, \quad (5.7)$$

where the last equality holds for constant  $\langle \sigma_{\text{ann}} v \rangle$ . The present number density  $n_X^0$  is also proportional to  $1/(m_X \sigma_0)$ , so that the present energy density  $\rho_X^0 = m_X n_X^0$  only depends on  $\sigma_0$ .

The “unwanted relics” paradox comes from the very small annihilation cross section of very heavy particles,  $\sigma_0 \propto 1/m_X$ , so that in the end the  $X$ 's

<sup>3</sup>We are implicitly assuming that the freeze-out happens when the particles are non-relativistic, so that  $T \simeq m_X$  before  $\Gamma_{\text{ann}} \simeq H$ . This makes the  $X$  particle a cold relic. The opposite case, when  $T \simeq m_X$  is later than  $\Gamma_{\text{ann}} \simeq H$  (i.e. a freeze-out that occurs when the particles are ultrarelativistic), is called that of a hot relic. In practice, a hot relic skips the second phase so that its number density is frozen to the high temperature value  $n_X \sim n_\gamma$ .

<sup>4</sup>There is actually also an additional, logarithmic dependence on  $m_X$  that we neglect for our discussion.



density parameter is  $\Omega_X \equiv \rho_X^0 / \rho_c \propto m_X$ . Very heavy, stable particles tend to overclose the Universe, i.e.  $\Omega_X \gg 1$ , at variance with the observations which indicate that  $\Omega_{\text{tot}} \simeq 1$ .

## 5.2 The Inflationary Paradigm

This Section is devoted to define the general framework of the inflationary model. We start from the description of the *spontaneous symmetry breaking* process at the ground of the inflationary phase transition and then we develop the details of the scalar field dynamics.

### 5.2.1 *Spontaneous symmetry breaking and the Higgs phenomenon*

At low energies, the electromagnetic and weak interactions appear as separate physical phenomena, although the Standard Model of elementary particles predicts a unified electroweak interaction which accounts for all the observation in a common theoretical picture. However, the electroweak model relies on a fundamental symmetry which is not directly observed in Nature. The process which allows this transition from a more general symmetry to a restricted one is known as spontaneous symmetry breaking (SSB) and it corresponds to the fact that a quantum theory can be invariant under a certain symmetry, while at the same time the associated vacuum state is not invariant under such symmetry. The low energy limit, near that vacuum state, is unable to reveal the global symmetry of the model, that is then said to be *spontaneously* broken, because no additional terms, violating this symmetry, are present in the Lagrangian of the model.

Furthermore, the observed gauge bosons  $Z_0$  and  $W_{\pm}$  carrying the weak interaction are massive particles, but they emerge from the massless bosons of the electroweak model (only photon remain massless after the SSB process is implemented on a linear combination of the fundamental four gauge fields). Since a massive vector boson has three independent degrees of freedom, instead of two like the corresponding massless field (for which longitudinal states are forbidden), we have to answer the question: *where do the three degrees of freedom come from in the SSB of the electroweak model?*

An appropriate answer is provided by the so-called Higgs phenomenon, according to which the scalar field responsible for the SSB transition pro-

vides also some degrees of freedom to give mass to the vector bosons; in particular, we will see below how the single degree of freedom of a Goldstone boson becomes available to induce longitudinal states of a vector field.

The SSB process and the Higgs phenomenon represent the fundamental physics motivation for the inflationary paradigm, deriving a more detailed discussion, despite not essential for the overall cosmological dynamics. For the sake of simplicity, we will treat these two concepts in the simplified case of an Abelian  $U(1)$  symmetry, instead of the real non-Abelian scheme of the electroweak model. Furthermore, we limit our presentation to a semiclassical framework, avoiding additional quantum features. Despite these simplifications, the SSB process and the Higgs phenomenon are traced in the following analysis in all their formal elegance and power.

Let us consider the Lagrangian density of a complex scalar field  $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$

$$\mathcal{L}_\phi = \eta^{ij} \partial_i \phi^\dagger \partial_j \phi - V_H(|\phi|), \quad (5.8)$$

with the Higgs potential term

$$V_H = \frac{\alpha}{2} |\phi|^2 + \frac{\lambda}{4} |\phi|^4, \quad (5.9)$$

$\alpha$  being a negative number ( $\alpha = -\mu^2$ ) and  $\lambda$  a positive one. The Lagrangian density (5.8) is invariant under a rotation in the  $\{\phi_1, \phi_2\}$ -plane, i.e. for

$$\phi_1' = \phi_1 \cos \psi + \phi_2 \sin \psi \quad (5.10)$$

$$\phi_2' = -\phi_1 \sin \psi + \phi_2 \cos \psi, \quad (5.11)$$

where  $\psi$  is a constant rotation angle ( $\psi = \text{const.}$ ).

The vacuum state of the field  $\phi$  has to correspond to the state of minimal energy for the system. Considering that the energy density has the form

$$\rho_\phi = |\partial_t \phi|^2 + |\nabla \phi|^2 + U(|\phi|), \quad (5.12)$$

the vacuum is obtained for a constant value of  $\phi$  which gives the lowest local minimum of the potential. On a quantum level, the field fluctuates around such a constant value, and in the semiclassical treatment one deals with expectation values of the field around the vacuum, denoted as  $\langle \phi_0 \rangle$ .

In correspondence to the quartic potential (5.9), the case  $\alpha > 0$  yields as vacuum state  $\langle \phi_0 \rangle = 0$ , which is invariant under the symmetry (5.11) and no SSB takes place in the field dynamics. Instead, when  $\alpha = -\mu^2$ , we get an infinite array of degenerate vacuum states, namely those associated to the circumference  $\langle \phi_1 \rangle^2 + \langle \phi_2 \rangle^2 = \langle r_0 \rangle^2 \equiv \mu^2/\lambda$ .

In such situation, a SSB arises because the rotation (5.11) maps the different vacuum states one onto another. Without loss of generality, if we choose the vacuum state as  $\langle \phi_1 \rangle = \langle r_0 \rangle$ ,  $\langle \phi_2 \rangle = \langle 0 \rangle$ , then it is manifestly non-invariant under the fundamental symmetry of the model, while the Lagrangian density is invariant under the same symmetry. In order to outline two relevant implications of this SSB scenario, let us introduce a new representation of the complex scalar field, having the form

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} = r \exp(i\theta). \quad (5.13)$$

Using this parametrization of the field  $\phi$ , the Lagrangian density (5.8) rewrites as

$$\mathcal{L}_\phi = \eta^{ij} (\partial_i r \partial_j r + r^2 \partial_i \theta \partial_j \theta) + \frac{\mu^2}{2} r^2 - \frac{\lambda}{4} r^4. \quad (5.14)$$

In such representation, the symmetry (5.11) reduces to the invariance of the Lagrangian density (5.14) under the transformation  $\theta \rightarrow \theta' = \theta + \psi$ . In order to analyze the features of the Lagrangian near the vacuum state chosen above (that in the new variables reads as  $\langle r \rangle = \langle r_0 \rangle$  and  $\langle \theta \rangle = \langle 0 \rangle$ ), let us define the fields  $\bar{r} = r - \langle r_0 \rangle$  and  $\bar{\theta} = \theta$ , such that they have vanishing vacuum expectation values. In terms of these fields, adapted to the chosen vacuum, the Lagrangian density (5.14) takes the form

$$\begin{aligned} \mathcal{L}_\phi = & \eta^{ij} \left[ \partial_i \bar{r} \partial_j \bar{r} + (\bar{r} + \langle r_0 \rangle)^2 \partial_i \bar{\theta} \partial_j \bar{\theta} \right] \\ & + \frac{\mu^2}{2} (\bar{r} + \langle r_0 \rangle)^2 - \frac{\lambda}{4} (\bar{r} + \langle r_0 \rangle)^4. \end{aligned} \quad (5.15)$$

From a careful analysis of the Lagrangian density, the two fundamental statements follow:

- *The model is no longer invariant under the symmetry  $r \rightarrow -r$ , because in the Lagrangian density (5.15) linear and cubic terms appear.* Despite the full model is still invariant under this (discrete) reflection symmetry, an observer living near the chosen vacuum does not realize it: this is the real physical content of the SSB phenomenon. Such discrete symmetry is equivalent, in the present context, to the transformation  $\theta \rightarrow \theta + \pi$ , but we are interested in its existence and in the spontaneous breaking as referred to field  $r$  alone. In fact, we will show that the degree of freedom associated to  $\theta$  is absorbed by the longitudinal mode of a vector boson. Therefore, the scalar field responsible for the inflationary scenario has a potential invariant under the parity symmetry, see Fig. 5.1.

However it is spontaneously broken near one of the two degenerate minima, as outlined above.

- The scalar field  $\theta$  corresponds to a massless boson (coupled to the field  $r$ ): this feature induced by the SSB is known as the emergence of a Goldstone boson.

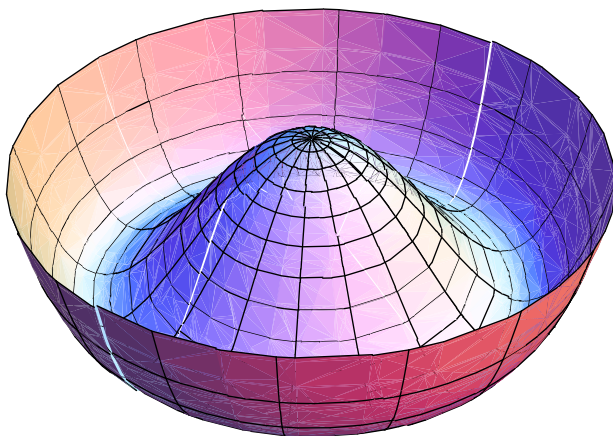


Figure 5.1 The Higgs potential  $V_H(\phi)$  of the Lagrangian (5.15), also known as *mexican hat*. The rotational invariance of  $V_H(\phi)$  can be noticed, as well as the presence of infinite equivalent minima.

Finally, to illustrate the Higgs mechanism, we promote the transformation on  $\theta$  to a gauge symmetry, by requiring the angle  $\psi$  to be a space-time function, i.e. we deal with the change  $\theta' = \theta + \psi(x^l)$ .

According to Sec. 2.2.4, the lost invariance of the Lagrangian density under such local symmetry is restored by introducing an Abelian gauge boson  $A_i$ , which correspondingly transforms as  $A_i' = A_i - \partial_i \psi$  and allows to set the covariant gauge derivative  $\mathcal{D}_i \theta \equiv \partial_i \theta + A_i$ . A Lagrangian density

invariant under the above gauge symmetry explicitly reads as

$$\begin{aligned} \mathcal{L}_\phi = & \eta^{ij} \left[ \partial_i \bar{r} \partial_j \bar{r} + (\bar{r} + \langle r_0 \rangle)^2 (\partial_i \theta + A_i) (\partial_j \theta + A_j) \right] \\ & + \frac{\mu^2}{2} (\bar{r} + \langle r_0 \rangle)^2 - \frac{\lambda}{4} (\bar{r} + \langle r_0 \rangle)^4 - \frac{1}{4g^2} F_{ij} F^{ij}, \end{aligned} \quad (5.16)$$

where  $F_{ij} = \partial_i A_j - \partial_j A_i$  is the gauge tensor associated to  $A_i$ , and  $g$  is the associated coupling constant. Looking at the form of Eq. (5.16), we are naturally led to define the new gauge boson  $B_i \equiv A_i + \partial_i \theta$ , which allows to rewrite the electromagnetic tensor as  $F_{ij} = \partial_i B_j - \partial_j B_i \equiv F_{ij}(B_l)$  (indeed the definition of  $B_i$  corresponds to a gauge transformation for  $A_i$ ), and hence the Lagrangian density can be rewritten as

$$\begin{aligned} \mathcal{L}_\phi = & \eta^{ij} \left( \partial_i \bar{r} \partial_j \bar{r} + (\bar{r} + \langle r_0 \rangle)^2 B_i B_j \right) \\ & + \frac{\mu^2}{2} (\bar{r} + \langle r_0 \rangle)^2 - \frac{\lambda}{4} (\bar{r} + \langle r_0 \rangle)^4 - \frac{1}{4g^2} F_{ij} F^{ij}(B_l). \end{aligned} \quad (5.17)$$

The  $\theta$  boson disappears from the theory, but its degree of freedom is incorporated within the massive boson  $B_i$ , which has an additional (longitudinal) degree of freedom with respect to the original massless gauge boson  $A_i$ . Such massive gauge boson is a typical feature of the Higgs phenomenon associated to a SSB process. Summarizing, we started with a theory invariant under a given internal symmetry which is spontaneously broken by the quartic Higgs potential and, near a vacuum state, such symmetry was lost. However, if we upgrade that symmetry on a gauge level, the SSB process is able to provide a mass for the corresponding boson, eliminating the Goldstone boson from the theory. It is exactly by the non-Abelian version of this Higgs mechanism that the  $Z_0$  and  $W_\pm$  electroweak bosons acquire a non-zero mass.

In the following subsection we will see how the SSB process provides a physical framework to the inflationary paradigm, especially in view of the transition from a single minimum of the Higgs potential to a configuration with two degenerate minima, that happens in correspondence to the change of the parameter  $\alpha$  from a positive to a negative value. We will implement the phase transition associated to the new dynamical regime through the coupling of the Higgs field, i.e. the relic field  $\bar{r}$ , with the thermal bath of the primordial Universe.

### 5.3 Presence of a Self-interacting Scalar Field

The general idea at the ground of the inflationary paradigm is the presence in the early Universe of a real self-interacting scalar field, whose potential has a direct dependence on the temperature of the thermal bath (accounting, in a phenomenological way, for the relative interaction between these two components), i.e. we deal with a Lagrangian density of the form

$$\mathcal{L}_\phi = \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi - V(\phi, T). \quad (5.18)$$

At sufficiently high temperatures (this concept will be more precise in the next subsection), the dynamics of the scalar field is dominated by its kinetic term, while the potential term is expected to be negligible. According to Sec. 2.2.2, the energy density and pressure of a scalar field  $\phi = \phi(t)$  living in an expanding isotropic Universe are given by the expressions

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi, T), \quad (5.19)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi, T), \quad (5.20)$$

where the spatial gradients of the scalar field  $\phi$  were neglected, according to the assumption of homogeneity. In fact, the gradient terms in the energy density (5.19) and pressure (5.20) are of the form  $\sim (\nabla\phi)^2/a^2$ , so that even if they are initially present (at a perturbative level) they are redshifted away by the expansion of the Universe. When specialized to the RW geometry in a synchronous reference frame, the dynamics of the field is described by the Euler-Lagrange equations obtained from the Lagrangian density in Eq. (5.18)

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (5.21)$$

If the potential term is neglected, this equation gives  $\dot{\phi} \propto 1/a^3$  and hence Eq. (5.19) yields  $\rho_\phi \propto 1/a^6$ . This result is compatible with the analysis of Sec. 2.2.2, where it was shown how the potential-free scalar field is isomorphic to a perfect fluid with equation of state  $P = \rho$  (i.e.  $\gamma = 0$ , or  $w = 1$ ). For  $a \rightarrow 0$ , the energy density of the field is much larger than the radiation contribution, which scales as  $a^{-4}$ . Near the Big Bang ( $a \rightarrow 0$ ), the energy density  $\rho_\phi$  has a strong divergent behavior which is expected to dominate the typical potential terms of the inflationary paradigm. A qualitative constraint on the form of the potential close enough to the

singularity can be obtained by the Friedmann Eq. (3.46). Near the Big Bang one can neglect the spatial curvature and therefore it reduces to

$$H^2 = \frac{\kappa}{6} \dot{\phi}^2 \propto \frac{1}{a^6}. \quad (5.22)$$

From this equation, i.e. from  $\dot{a}^2/a^2 \propto 1/a^6$ , it follows that  $a \propto t^{1/3}$ . Substituting this behavior of the scalar field back into the same equation (5.21), we arrive at the explicit expression of  $\phi(t)$  as

$$\frac{\kappa}{6} \dot{\phi}^2 = \frac{1}{9t^2} \Rightarrow \phi(t) = \sqrt{\frac{2}{3\kappa}} \ln t + \phi_0, \quad (5.23)$$

where  $\phi_0$  is an integration constant. If the dependence of the potential term on the temperature is assumed to be weak, as argued in the next subsection (apart from the transition phase), in order to have a kinetic energy of the field dominant close to the singularity ( $t \rightarrow 0$ ) we get the condition

$$\lim_{\phi \rightarrow -\infty} \frac{dV/d\phi}{e^{-\sqrt{6\kappa}\phi}} = 0. \quad (5.24)$$

The scalar field is treated as classical because of its scalar character and of the high occupation numbers of its states, in agreement with its nature of source for the geometrodynamics of the Universe. However, the regime in which the kinetic term of this field dominates can overlap with the Planck era, where the quantum features of both the field and the scale factor are relevant (see Sec. 10.5).

As the Universe expands, the scalar field evolution is damped and, sooner or later, the potential term will become important. In fact, multiplying Eq. (5.21) by  $\dot{\phi}$  and recalling the form of the energy density (5.19), one gets the decay law (valid also in the presence of a potential term)

$$\dot{\rho}_\phi = -3H\dot{\phi}^2, \quad (5.25)$$

which, for an expanding Universe ( $H > 0$ ), implies the monotonic decrease of the energy density associated to the scalar field. After an initial regime, where the kinetic term of the scalar field dominates the Universe dynamics, the damping due to the expansion eventually makes the field fall into a minimum of the potential. When the scalar field is in a minimum, it behaves as a perfect fluid with  $\rho_\phi = \text{const.}$  and an equation of state parameter  $w = -1$ . When the energy density of the scalar field in the minimum is dominant, it gives rise to a de Sitter phase of exponential expansion. This situation is eventually going to happen because the density of any other component (e.g. radiation) decreases with the expansion (an increasing energy density

would require  $w < -1$ ).

The idea behind the original model of inflation (now called *old inflation*), proposed by Guth in 1980, is that, at some point in the past history of the Universe, it was dominated by the energy density of a scalar field in a minimum of its potential which started expanding exponentially. What happens before this time is not really relevant for the inflationary paradigm, because the effect of the exponential expansion is to get rid of the initial conditions from which inflation itself started. However, the minimum where the field is standing in is not the global minimum of the potential (the *true vacuum*) but a local minimum (a *false vacuum*). At high temperatures, the true vacuum is not accessible to the system (we will explain this in more detail later) but it becomes accessible as the Universe expands and cools down. When this happens, at some critical temperature  $T_c$ , the field can undergo a symmetry-breaking phase transition. At  $T < T_c$ , the two minima of the potential are separated by a barrier that the field has to overcome, for example via tunneling, in order to complete the phase transition and evolve toward the global minimum where  $\rho_\phi \simeq 0$ , with a first-order transition.

Guth realized that the de Sitter phase of expansion allows one to solve the shortcomings of the SCM, but the first-order character of the transition was still problematic. A modification of Guth's original model, called *new inflation*, was proposed by Linde and Albrecht & Steinhardt shortly thereafter. The basic idea behind new inflation is that the phase transition is a second-order transition, i.e. it happens smoothly. This can be realized by requiring that the potential is such that the de Sitter expansion does not happen when the field is trapped in the false vacuum, but instead when the field is slowly-rolling towards the true vacuum over a plateau (so that  $\dot{\phi} \ll V$  and  $w \simeq -1$ ).

Inflation is not necessarily associated to symmetry breaking and to phase transitions. For example, in the model of *chaotic inflation* proposed by Linde, the scalar field potential has a single minimum, but the value of the field is not homogeneous across the Universe. At the points where it is displaced from the minimum, inflation occurs.

Now we will describe in more detail the old and new inflationary models. At some early time, the Universe density was dominated by a scalar field in a minimum of its potential. The phase transition associated to inflation consists of the appearance of a second local minimum in the potential  $V(\phi, T)$ . Initially, above the critical temperature  $T_c$ , the energy



density associated to this second minimum is greater than that of the vacuum state. While the Universe expands and the temperature decreases as  $T \propto 1/a$ , the height of the second minimum decreases up to be degenerate with the original vacuum state, at  $T = T_c$ . Thus, if the potential is characterized by a fundamental symmetry  $\phi \rightarrow -\phi$ , the two (correspondingly symmetric) degenerate vacuum states realize a SSB scenario. Indeed, the SSB process is not strictly necessary (see the considerations below), but it has to be inferred because of the link between the inflaton (as the scalar field responsible for the inflation is named) and the Higgs field, emerging from a SSB framework. When the new minimum becomes lower than the original, the scalar field, and hence the whole Universe, remains trapped in the higher state, the false vacuum, in contrast to the real vacuum corresponding to the newly formed absolute local minimum. Since between the two minima there is a barrier, the false vacuum becomes a metastable state and the Universe can perform a phase transition from the false to the true vacuum state. The process underlying this transition is, in general, a quantum or thermal tunneling across the barrier, which has to take place independently on each causal region of the Universe. The crucial dynamical feature of the inflationary scenario is the de Sitter phase (see Sec. 3.2.3) driven by the constant energy density (corresponding to the gap between the two minima) that dominates the geometrodynamics. In old inflation, the constant energy density manifests its effects during the phase in which the Universe is trapped in the false vacuum, i.e. before the tunneling. This point of view requires that the barrier is high enough to get a sufficiently long de Sitter phase; a typical example is sketched in Fig. 5.2. However, in this scheme, the different causal regions are associated to *inflating bubbles*, which randomly undergo the transition from the false to the true vacuum.

When a bubble of true vacuum is created, the energy density corresponding to the gap between the vacua is stored in the walls of the bubble and is released when two bubbles collide. This provides the energy that reheats the Universe (that has supercooled after the inflationary phase of expansion) and allows the usual Friedmann evolution to start. This mechanism poses the problem of whether the bubbles of the new phase can appear fast enough to cover all the Universe, because the “voids” between the bubbles will be exponentially expanding, being still in the old, false vacuum phase. The answer is negative: the bubbles never percolate, so that bubbles of true vacuum continue to be created over a false vacuum background that expands too fast, and inflation never ends. This is called the *graceful exit* problem and is based on the bubble nucleation rate, i.e. the probability of

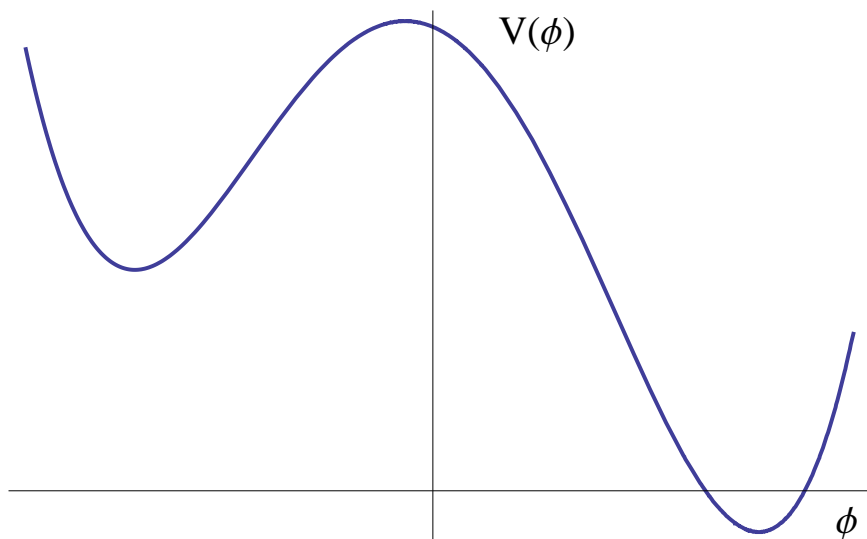


Figure 5.2 A typical potential able to induce inflation, as thought by Guth in its original work. The potential exhibits two different minima, separated by a barrier that does not allow the field  $\phi$  to evolve in the real vacuum through classical mechanisms.

undergoing the phase transition, which has to be small enough in order for inflation to last enough to solve the SCM paradoxes. Moreover, the collisions between bubbles tend to create topological defects (the transition is strongly first order) so that a nucleation rate too large would also generate too many topological defects.

The idea underlying the new inflation paradigm is instead that the de Sitter phase takes place after the Universe has left the false vacuum. This difference is crucial because the barrier between the two minima must no longer be particularly high (as required for having a sufficiently long de Sitter regime before the transition) and, moreover, the Universe undergoes the tunneling effect when it still has a rather smooth dynamics. These two combined effects, a lower barrier and a regular evolution of the scale factor, allow to overcome the graceful exit problem and to avoid the appearance of a large number of topological defects. Yet, it remains to be explained how a constant cosmological term is induced in the dynamics to get the desired de Sitter phase. A solution to this problem is offered by the idea that the barrier between the two minima has a long plateau, on which the scalar field classically performs a *slow-rolling evolution*. In such situation, the

kinetic energy of the scalar field is negligible with respect to the constant potential energy on the plateau (see Fig. 5.3).

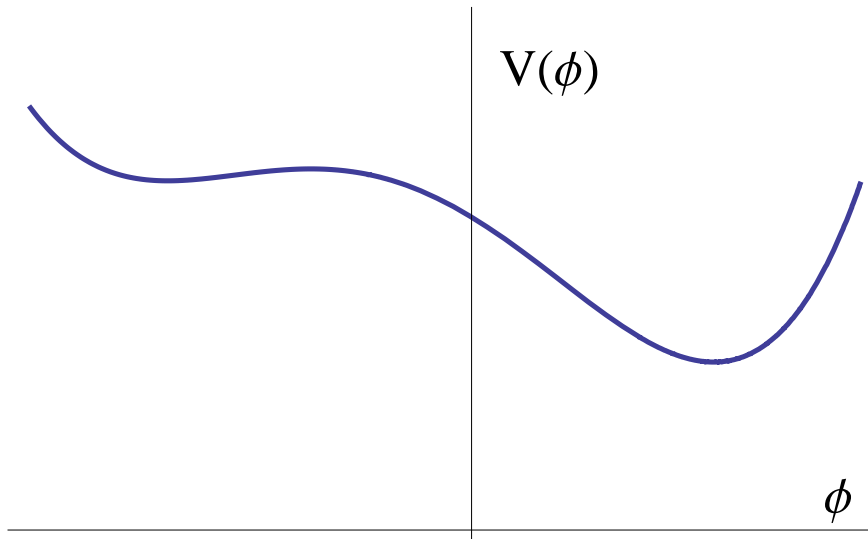


Figure 5.3 A possible potential able to induce slow-rolling. The slow-rolling corresponds to the phase when the field “falls” from the higher minimum on the left into the lower minimum on the right of the figure.

Thus, in the new inflation framework, the cosmological constant-like energy is relevant during the phase transition of the Universe from the false vacuum towards the real vacuum which is slowly approaching. Of course, when the scalar field falls into the second potential well, the de Sitter phase stops and the inflationary process ends with a different scenario (see Sec. 5.4.2). It is possible to argue (see the next Subsection) that, in agreement with the SSB paradigm, the new inflation model can take place without the existence of a quantum tunneling through the barrier. In fact, one could consider a simple transition in which a local minimum is transformed into a really flat local maximum standing up on two degenerate deep minima. In this case, the whole evolution remains essentially classical and the slow-rolling of the field coincides with the field falling into one of the two equivalent minima. The first phase of this falling is characterized by an almost flat potential around its value on the two degenerate vacua and its happening is ensured by the unstable nature of the maximum and

by the expectation that the transition from a minimum to a maximum perturbs the scalar field, even if originally at rest at the minimum.

The price to pay for this new inflation paradigm is in a more stringent fine tuning to be imposed on the form of the scalar field potential term at the end of the phase transition.

### 5.3.1 *Coupling of the scalar field with the thermal bath*

We will now discuss how the phase transition associated to the SSB can be triggered by the coupling of the scalar field with the underlying thermal bath, whose temperature decreases as the Universe expands.

The presentation below has no aim to fix a specific model, but provides a rather general paradigm, linking the true vacuum to the false vacuum configuration throughout the embedding of a self-interacting scalar field into an expanding background.

The main point is that the potential  $V$  fully determines the dynamics of the field only in the zero-temperature limit. At finite temperature  $T \neq 0$  one should take into account the presence of a thermal bath of particles. In this case the dynamics of the field is determined by the “finite-temperature effective potential”  $V_T(\phi)$ , that is nothing else but the free energy density  $F(\phi, T)$ . Of course  $F(\phi, T = 0) = V(\phi)$ . It can be shown that the effective potential for scalar particles of mass  $m$  in the ultrarelativistic limit ( $T \gg m$ ) is

$$V_T(\phi) = V(\phi) - \frac{\pi^2}{90}T^4 + \frac{m^2}{24}T^2 \left[ 1 + \mathcal{O}\left(\frac{m}{T}\right) \right], \quad (5.26)$$

where we recall that  $m^2 = m^2(\phi) = d^2V/d\phi^2$ . The first additional term is the free energy of a gas of spin 0 massless bosons and does not alter the dynamics since it does not depend on  $\phi$ . The effect of the mass-dependent terms can instead be interpreted like as introducing a temperature-dependent mass  $m_T = \sqrt{d^2V_T(\phi)/d\phi^2}$ . If we consider the potential  $V_H$  for a Higgs field as in Eq. (5.9) so that  $m^2 = -\mu^2 + 3\lambda\phi^2$ , the effective potential  $V_T$  in the high-temperature limit is

$$V_T(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda T^2}{8}\phi^2 + \frac{\lambda}{4}\phi^4 = \frac{m_T^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \quad (5.27)$$

where we have omitted terms that do not depend on  $\phi$ , and thus do not alter the field dynamics. The temperature-dependent effective mass  $m_T$  is

$$m_T = \sqrt{\frac{\lambda T^2}{4} - \mu^2}, \quad (5.28)$$

which is real at temperatures above the critical temperature  $T_c = 2\mu/\sqrt{\lambda}$ , while it is imaginary below  $T_c$ .

At sufficiently high temperatures, the Higgs field has an effective mass, associated to the fluctuation around the minimum for  $\phi = 0$ . However, when the temperature of the Universe decreases enough, the SSB configuration (with two degenerate minima) appears and the minimum is replaced by a local maximum of the potential (see Fig. 5.4). Thus, the phase tran-

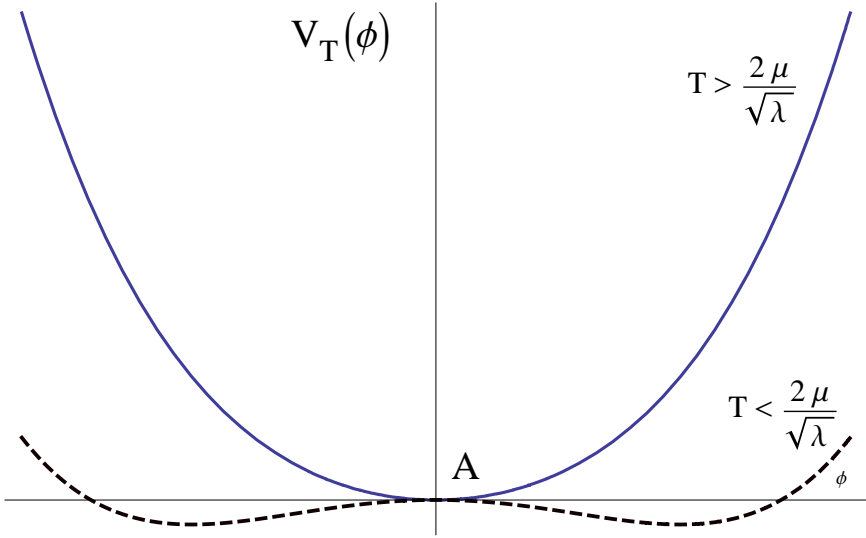


Figure 5.4 The potential (5.27) is depicted above for two different temperatures. When  $T > 2\mu/\sqrt{\lambda}$  (solid line) a unique true minimum in A exists and the field  $\phi$  lies there; when  $T < 2\mu/\sqrt{\lambda}$  (dashed line) the minimum in A becomes a local maximum, and two minima appear. This is a direct consequence of the cooling of the Universe during its evolution.

sition associated to the SSB process can be realized by taking into account the effects of the finite temperature and the presence of a thermal bath of particles. The presence of a cosmological background of interacting particles at a given temperature alters the absolute zero of the energy density and, when the Universe is very hot, it can hide the SSB configuration, maintaining the false vacuum always stable.

The simple scenario depicted above opens a new point of view about the nature of the inflationary paradigm. In the present framework, the de Sitter phase is generated as the slow-rolling of the scalar field on the plateau around the emerging maximum (requiring an appropriate fine tuning of the

parameters). Furthermore, no real barrier exists between false and true vacuum, and the scalar field falls into one of the two degenerate local minima in an essentially classical evolution, while the Universe remains smooth during the whole process. Such point of view appears as an intriguing and original interpretation of the SSB phase transition. The real inflationary picture can follow more general and not exactly symmetric features and therefore the scheme above must be thought of as the dominant component of a mixed framework where, for instance, a tiny barrier can arise between two non-perfectly equivalent vacua.

We can argue that, in a realistic scenario, the role played by the background temperature is relevant only in generating the SSB configuration, while the evolution on the plateau is well approximated by a slow-rolling on the following temperature-independent (and hence time-independent) profile

$$V_{\text{Plateau}}(\phi) \simeq \rho_{\Lambda} - \frac{\omega}{4}\phi^4, \quad (5.29)$$

where  $\rho_{\Lambda}$  and  $\omega$  are constant quantities describing the gap of the energy density between the local maximum and the minima, and describing the departure from a pure de Sitter phase, respectively. We will use this form of the potential to describe the slow-rolling phase and the dynamical paradigm solving the paradoxes outlined in Sec. 5.1.

## 5.4 Inflationary Dynamics

In this Section we analyze the specific features of the scalar field dynamics during the slow-roll, which give rise to the de Sitter phase of expansion. Intuitively, the evolution of the homogeneous scalar field  $\phi(t)$  over the plateau of its potential can be understood as the behavior of a point-particle moving on a horizontal potential profile. The scalar field coordinate will perform a slow-roll, characterized by a small  $\dot{\phi}$  and negligible  $\ddot{\phi}$  values. During this phase, the Universe is dominated by the potential energy of the field (the effective cosmological constant). We will discuss the implications of the resulting exponential expansion of the Universe (see Sec. 3.2.3), allowing to overcome the horizon and flatness paradoxes.

### 5.4.1 *Slow-rolling phase*

The conditions to be imposed on the system to get the desired de Sitter regime can be summarized as follows.

- (i) The effective cosmological constant energy density  $\rho_\Lambda \simeq \rho_\phi$  dominates the relativistic energy density of radiation  $\rho_{\text{rad}}$ , as well as any other contribution, i.e.

$$\rho_\Lambda \gg \rho_{\text{rad}}. \quad (5.30)$$

This implies that inflation cannot start before the temperature drops below a value  $T \sim \rho_\Lambda^{1/4}$ .

- (ii) The constant term of the potential of the scalar field dominates its correction depending on  $\phi$  over the interval  $(\phi_i, \phi_f)$ , corresponding to the initial and final stages of the slow-roll region, i.e.

$$\phi \in (\phi_i, \phi_f) : \rho_\Lambda \gg \frac{\omega}{4} \phi^4, \quad (5.31)$$

which can be easily satisfied requiring  $\omega \ll 1$ .

- (iii) The acceleration of the  $\phi$ -coordinate must be negligible in comparison to the velocity term associated to the damping due to the Universe expansion, i.e.

$$\ddot{\phi} \ll 3H\dot{\phi}. \quad (5.32)$$

Under these assumptions, the Friedmann Eq. (3.46) and the one for the scalar field (5.21) take the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa}{3} \rho_\Lambda \quad (5.33a)$$

$$3\frac{\dot{a}}{a}\dot{\phi} - \omega\phi^3 = 0. \quad (5.33b)$$

Such system can be solved with respect to the two unknowns  $a(t)$  and  $\phi(t)$ , giving the explicit expressions

$$a(t) = a_0 \exp[H^* t] \quad , \quad H^* \equiv \sqrt{\frac{\kappa \rho_\Lambda}{3}} = \text{const.}, \quad (5.34)$$

$$\phi(t) = \sqrt{\frac{3H^*}{2\omega(t^* - t)}}, \quad (5.35)$$

$a_0$  and  $t^*$  being two integration constants. It is immediate to check that all the requirements on the dynamics are satisfied as long as  $t \ll t^*$ , so that we impose  $t_f \ll t^*$ .

During the de Sitter phase, all the physical lengths (for instance the particle wavelengths) are stretched to much larger distances due to the exponential behavior of the scale factor. Similarly, the energy density of the relativistic species populating the early Universe is drastically decreased as well as the corresponding temperature ( $T \propto a^{-1} \propto \exp(-H^*t)$ ). The crucial point here is the constant behavior in time of the microphysical horizon  $L_H \equiv a/\dot{a} \simeq (H^*)^{-1} = \text{const.}$  As a consequence of the slow-rolling dynamics, the Universe profile is deeply modified. In fact, the matter that before the inflationary expansion was contained within a single Hubble radius is redistributed after the de Sitter regime over a much larger region containing many Hubble lengths. These regions would be causally disconnected according to the standard Friedmann evolution of the radiation dominated Universe where  $d_H \simeq L_H$  (see Sec. 3.1.5). Furthermore, the wavelengths of particles undergo a strong redshift, resulting in an extreme cooling of the cosmological fluid.

We emphasize that, despite the scalar field potential does not play any dynamical role in Eq. (5.33b) during the de Sitter phase, we address the evolution at an essentially classical level, instead of speaking of free self-gravitating bosons. This can be done because we are assuming that the energy density can be regarded as a classical object, hence ensuring the classical nature of  $\phi(t)$ . As already emphasized, this is exactly the same situation of an electromagnetic field, which appears as a classical entity in view of the high photon density (i.e. we deal with a boson state or an electromagnetic wave with an extremely high occupation number). Nevertheless, we will see in the following that both the free field character and the quantum nature of the inflaton field contribute to the achievement of a self-consistent inflationary paradigm.

#### 5.4.2 The reheating phase

When the slow-rolling of the scalar field ends, the evolution follows the profile of the potential term close to the true vacuum and we can reliably infer the fall of the scalar field into the well of the SSB configuration. This regime corresponds to a very fast decay of the scalar field into the local minimum describing the true vacuum configuration, as suggested by the analogy of a massive point-particle moving over the potential profile. After such fast transition, the evolution of the Universe is governed by the damped oscillations which take place around the true vacuum. In fact, near enough to the bottom of the potential well, the expression for  $V(\phi)$  admits



the usual quadratic representation

$$V(\phi) \simeq \frac{1}{2} \mu_0^2 (\phi - \sigma)^2 \quad , \quad \mu_0 \equiv \sqrt{\left( \frac{d^2 V}{d\phi^2} \right)_{\phi=\sigma}} \quad , \quad (5.36)$$

where we have considered a Taylor expansion around the value  $\phi = \sigma$ , corresponding to the minimum configuration, assuming that in the true vacuum  $V(\phi = \sigma) = 0$ . This is in agreement with the observations which indicate that the vacuum energy density is at most of the order of the present critical density, and thus very small with respect to any reasonable energy scale for inflation (see Sec. 5.7 for further details). Here,  $\mu_0^2$  denotes the effective mass acquired by the scalar field during its small oscillations.

In this approximation, the scalar field dynamics is that of a free massive boson living on an expanding background, i.e. Eq. (5.21) takes the form

$$\ddot{\phi}_\sigma + 3H\dot{\phi}_\sigma + \mu_0^2 \phi_\sigma = 0 \quad , \quad (5.37)$$

where  $\phi_\sigma \equiv \phi - \sigma$  but, for the sake of convenience, in what follows we will drop the subscript  $\sigma$ , i.e. we shift the minimum to the origin of the  $\phi$  axis. The physics underlying this equation is that of a super-cooled Bose-Einstein condensate, whose constituents are very massive scalar bosons, i.e. we assume that  $\mu_0 \gg H_f$ , where  $H_f \equiv H(t = t_f)$  is an estimate of the typical value of the Hubble function during this oscillatory period. The field evolves very rapidly on a cosmological time scale and the existence of the condensate is due to the very low temperature of the Universe after the de Sitter phase.

Such system of very cold spin 0 bosons is unstable, essentially because the particles should have decay channels into particles with a lower mass, which will turn out as relativistic components. In other words, the huge effective mass that these bosons acquire as an effect of their small oscillations around the true vacuum is transformed by the decay processes into energy of ultrarelativistic species, so that the Universe undergoes a strong reheating phase.

The decay processes can be phenomenologically described by an average characteristic time  $\tau_d = \text{const.}$ , acting as a friction term in Eq. (5.37), which has to be restated as

$$\ddot{\phi} + 3(H + H_d)\dot{\phi} + \mu_0^2 \phi = 0 \quad , \quad (5.38)$$

where we set  $H_d \equiv 1/(3\tau_d)$ . When  $(t - t_f) \ll \tau_d$ , the decay is very slow on a cosmological time scale and the damping of the field is due to the expansion. On the contrary, when  $(t - t_f) \gtrsim \tau_d$  the decay is fast on a cosmological time

scale and is responsible for the damping of the field. In the latter case, the evolution of the scalar field around the minimum acquires the behavior of a damped oscillator, due to the fundamental scalar particles instability, that is to say

$$\ddot{\phi} + 3H_d \dot{\phi} + \mu_0^2 \phi = 0. \quad (5.39)$$

This equation admits the damped oscillating solution

$$\phi(t) = A \exp\left(-\frac{3}{2}H_d t\right) \sin\left[\left(\mu_0^2 - \frac{9}{4}H_d^2\right)^{1/2} t + \phi_0\right], \quad (5.40)$$

where  $A$  and  $\phi_0$  are constant amplitude and phase, respectively. In the case when the expansion damping is dominant, as well as in the general case when both damping terms are relevant, the evolution of the field should be derived by solving Eq. (5.38) coupled to the Friedmann equation.

In general, the solution always shows a damped oscillatory behavior, although the damping is less severe than the exponential one found during the decay phase. For example, if the scale factor evolves with time as  $a \propto t^{2/3}$  (as it happens during reheating), the corresponding solution for  $\phi$  is

$$\phi(t) = \frac{A}{t} \sin(\mu_0 t + \phi_0). \quad (5.41)$$

However, since  $\mu_0$  is much larger than both  $H$  and  $H_d$ , oscillations are very rapid with respect to the damping timescale and thus the oscillatory behavior can be integrated out by averaging over many periods. Therefore we will generally write

$$\phi(t) = A(t) \sin(\mu_0 t + \phi_0) \quad (5.42)$$

where the amplitude  $A(t)$  should be thought of as a slowly-varying function of time, i.e. varying on a time scale much larger than  $\mu_0^{-1}$  and can be taken as constant over a single oscillation period, providing

$$\langle \rho_\phi \rangle \simeq \left\langle \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \mu_0^2 \phi^2 \right\rangle \simeq \frac{1}{2} (A \mu_0)^2 \simeq \langle \dot{\phi}^2 \rangle. \quad (5.43)$$

Multiplying Eq. (5.38) by  $\dot{\phi}$  and taking the average, we obtain the energy loss of the scalar field as

$$\langle \dot{\rho}_\phi \rangle = -3(H + H_d) \langle \rho_\phi \rangle, \quad (5.44)$$

which admits the solution

$$\langle \rho_\phi \rangle \simeq \frac{\bar{\rho}_\phi}{a^3} \exp\left\{-\frac{t - t_f}{\tau_d}\right\}, \quad (5.45)$$

where  $\bar{\rho}_\phi$  is an integration constant. For simplicity, in the following we will drop the brackets around  $\rho_\phi$ , even if we are always dealing with a period average. The decay process of these massive bosons results in an increase of the relativistic species and the Universe is reheated. The law underlying the increase of the relativistic component is dictated by energy conservation, amended for the expansion damping, i.e.

$$\dot{\rho}_{\text{rad}} = -4H\rho_{\text{rad}} + 3H_d\rho_\phi. \quad (5.46)$$

The evolution of  $\rho_\phi$  and  $\rho_{\text{rad}}$  is obtained by solving the two coupled Eqs. (5.44) and (5.46), where  $H$  is given by the Friedmann Eq. (3.46) as

$$H^2 = \frac{\kappa}{3}(\rho_\phi + \rho_{\text{rad}}), \quad (5.47)$$

taking into account that, after the de Sitter regime, the spatial curvature is negligible (see Sec. 5.5.1).

Although a full solution to the coupled system has to be obtained numerically, an analytical solution can however be found for  $t_f \leq t \ll \tau_d$ . In this interval, neglecting the exponential term the energy density of the scalar field is given by Eq. (5.45)

$$\rho_\phi \simeq \frac{\bar{\rho}_\phi}{a^3} \quad (5.48)$$

so that the scalar field behaves as non-relativistic matter. In fact, during the coherent oscillations, it can be seen that  $\langle \dot{\phi}^2 \rangle = \langle 2V(\phi) \rangle$  and thus  $P \simeq 0$  holds. Immediately after the de Sitter phase, the total energy density is provided by the massive bosons condensate (the radiation component has been redshifted away so it is safe to assume that  $\rho_{\text{rad}}(t_f) = 0$ ) thus the Universe behaves as matter-dominated. One can neglect  $\rho_{\text{rad}}$  on the right-hand side of Eq. (5.47) and get the familiar result for the evolution of the cosmic scale factor in a matter-dominated Universe  $a \propto t^{2/3}$  and  $H = 2/3t$ . Moreover, Eq. (5.47) also gives

$$\rho_\phi = 3H^2/\kappa = \frac{4}{3\kappa t^2} \quad (5.49)$$

and the radiation energy density  $\rho_{\text{rad}}$  obeys the equation<sup>5</sup>

$$\dot{\rho}_{\text{rad}} = -\frac{8}{3t}\rho_{\text{rad}} + 4\frac{H_d}{\kappa t^2}, \quad (5.50)$$

that admits the solution with initial condition  $\rho_{\text{rad}}(t_f) = 0$  as

$$\rho_{\text{rad}}(t) = \frac{12H_d}{5\kappa t} \left[ 1 - \left( \frac{t_f}{t} \right)^{5/3} \right]. \quad (5.51)$$

---

<sup>5</sup>We need to keep the term  $\propto H_d$  in Eq. (5.46), even if we neglected it in Eq. (5.44), because  $\rho_\phi \gg \rho_{\text{rad}}$ . In other words, the condition  $H \gg H_d$  does not imply  $H\rho_{\text{rad}} \gg H_d\rho_\phi$ ; in fact the solution shows that  $H\rho_{\text{rad}} \lesssim H_d\rho_\phi$ .

This expression reaches a maximum for  $t \simeq 1.8t_f$  and for  $t \gg t_f$  behaves as  $1/t$ . The radiation density has a sharp rise around  $t \sim t_f$  and then starts decreasing; however, it decreases more slowly than the matter density because  $\rho_{\text{rad}} \propto t^{-1} \propto a^{-3/2}$  while  $\rho_\phi \propto a^{-3}$ , so that at some point the radiation will end up dominating the Universe. This condition approximately realizes at the time  $t \simeq \tau_d$  and the approximations used to derive this solution break down. In fact, when the evolution enters the region  $t \gtrsim \tau_d$ , the scalar field energy density begins to exponentially decay and the radiation contribution drastically rises, eventually dominating the Universe dynamics. At such state reheating ends and the usual Friedmann expansion starts. In Fig. 5.5 we show the exact solution, obtained by the numerical integration of Eqs. (5.44) and (5.46), compared with the approximated analytic solution discussed so far.

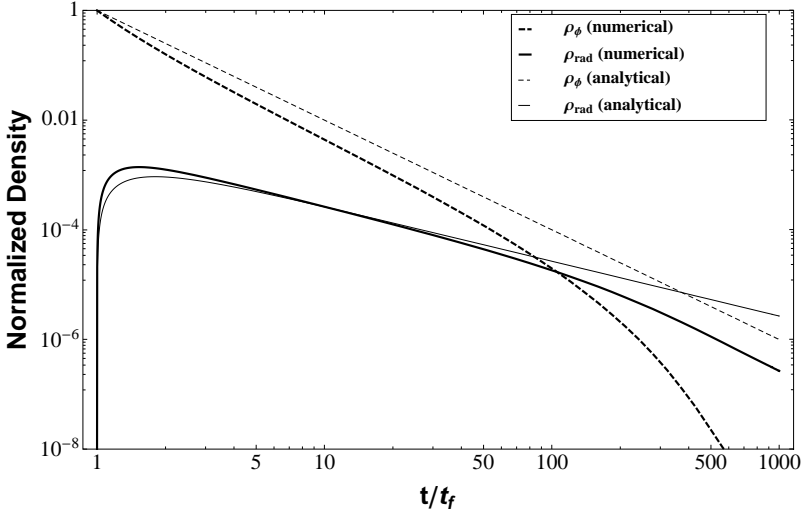


Figure 5.5 Behaviour of the densities of the scalar field  $\rho_\phi$  (dashed lines) and radiation  $\rho_{\text{rad}}$  (solid lines) during reheating, assuming  $\tau_d = 100t_f$ . The thick lines represent the numerical solutions to Eqs. (5.44) and (5.46), while the thin lines represent the approximated analytical solutions (5.48) and (5.51). The densities are normalized to the value of  $\rho_\phi$  at  $t_f$ .

Let us estimate the temperature at which the Universe is reheated. If the decay time is much smaller than the Hubble time  $H_f^{-1}$  at the end

of the de Sitter phase, the decay of the scalar field is instantaneous (on a cosmological time scale) and all the vacuum energy is instantly converted in radiation. In this limiting case, there is no matter-dominated regime after the exponential expansion. After reheating  $\rho_{\text{rad}} \simeq \rho_\phi(t_f) \simeq \rho_\Lambda$  and recalling the relation between the energy density and temperature of radiation we can estimate

$$T_{\text{rh}}^G \sim \left( \frac{30\rho_\Lambda}{g_*\pi^2} \right)^{1/4}, \quad (5.52)$$

where  $g_*$  is estimated at the end of the reheating phase and the superscript  $G$  stands for “good”. If  $g_*$  did not vary too much, according to this estimate the temperature of the Universe at the end of the inflation scenario is of the same order of magnitude it had when inflation started. The FRW standard evolution is recovered although with a strongly stretched geometry. This situation is not the most general possible one and for a discussion of the so-called *poor reheating*, as opposed to the case of a *good reheating* discussed below, in the next Section. In the general case with  $\tau > H_f^{-1}$ , the vacuum energy is not immediately converted and is partially redshifted away by the expansion during the coherent oscillations regime, so that the reheating temperature is smaller than the value given by Eq. (5.52).

We will now discuss the evolution of the entropy during reheating. The behavior of the entropy per comoving volume  $S$  during the coherent oscillations regime can be inferred recalling that the entropy density  $s$  is related to the radiation energy density by the relation  $s = 4\rho_{\text{rad}}/3T \propto \rho_{\text{rad}}^{3/4}$  (the power-law behavior stands as long as  $g_*$  is nearly constant). Recalling that during reheating  $\rho_{\text{rad}} \propto a^{-3/2}$ , the entropy  $S = sa^3$  increases as  $a^{15/8}$ . However, most of the entropy production during reheating happens close to  $t_f$ , when the radiation energy density sharply rises from zero to a finite value.

If the decay process of the  $\phi$  bosons violates the symmetry particles/antiparticles, at the end of the reheating phase we get a net baryon number density  $n_B$ . Assuming that the decay of a single boson of mass  $\mu_0$  produces  $\xi$  baryons, we can fix the relation  $n_B \simeq \xi n_\phi$ ,  $n_\phi$  denoting the boson number density. Since the  $\phi$ -particles behave as a non-relativistic species, we have

$$\rho_\phi \sim n_\phi \mu_0 \sim \rho_\Lambda \sim \frac{\pi^2}{30} g_* (T_{\text{rh}}^G)^4. \quad (5.53)$$

Combining these relations and recalling the expression of the entropy density in terms of the radiation one, we get the ratio of the net baryon density

number to the entropy density as

$$\frac{n_B}{s} = \frac{3\xi}{4\mu_0} T_{\text{rh}}^{\text{G}}. \quad (5.54)$$

This result allows to calculate the baryon asymmetry at the end of the reheating phase once  $\xi$  and  $\mu_0$  are provided by the details of the SSB process.

## 5.5 Solution to the Shortcomings of the Standard Cosmology

### 5.5.1 Solution to the horizon and flatness paradoxes

The feature of the de Sitter phase that allows to overcome the SCM paradoxes is the constant character of the Hubble length in comparison to the exponential behavior of physical scales. In particular, the value  $a_f$  of the cosmic scale factor at the end of the de Sitter phase, is related to the value  $a_i$  at the beginning of the slow-rolling phase by the relation

$$a_f = a_i \exp[H^*(t_f - t_i)] \equiv a_i \exp[\mathcal{E}], \quad (5.55)$$

where  $\mathcal{E} = \ln(a_f/a_i)$  is called the *e-folding* of the inflationary process. The exponential growth of the physical scales offers a proper framework to solve the horizon paradox. In fact, it can be seen that the particle horizon at the end of inflation is  $d_H \simeq \exp(\mathcal{E})/H^*$ , i.e. it is exponentially larger than the Hubble length at the same time, so that the numerical coincidence between the two quantities does not hold in the presence of an inflationary phase of expansion. We can thus explain the strong uniformity of the CMB in the sky simply by requiring that all the material we are looking at was initially contained within a single microphysical horizon before inflation started. According to this point of view, the different Hubble volumes at recombination were not in “local” causal contact (i.e. in the sense that they were one outside the Hubble radius of the other, as explained in Sec. 3.1.5) but nevertheless they were inside the respective particle horizons, meaning that they had the possibility to interact sometime in the early Universe (we recall that the particle horizon is an integral quantity, so it “has memory” of the past history of the Universe, while the Hubble radius does not). In other words, the scale corresponding to a Hubble length *before* inflation, containing matter in thermal equilibrium, has been stretched to a very large scale by the de Sitter phase of exponential expansion; in particular, this scale is much larger than the Hubble length *after* inflation. In order to see large inhomogeneities of the CMB, corresponding to the real causally

disconnected regions, we would have to wait a sufficiently long time (indeed, a huge one even on a cosmological scale), leaving time to the Hubble volume to incorporate many microcausal horizons of the primordial Universe at  $t_i$ .

Since the mismatch between the causal horizon (as estimated assuming a Friedmann-like expansion) at the time of recombination and the Hubble length today is not so severe, a modest amount of inflation (a small value of  $\mathcal{E}$ ) is required to solve the paradox in this form. However, we have seen that to explain the observed homogeneity of the Universe one has to require that this homogeneity was already present at the time when the Friedmann expansion started. Assuming that inflation started at the Planck time  $t_P$  gives the most severe constraint on the amount of inflation necessary to explain the observed homogeneity. In other words, one has to require that the present Hubble scale  $H_0^{-1} \sim \mathcal{O}(10^{28}\text{cm})$  was, at the beginning of the de Sitter phase, within the corresponding Hubble length. At the Planck time the maximum causal distance was roughly given by the Planck length  $l_P \simeq 10^{-33}\text{cm}$ . After inflation, the corresponding scale  $d$  is inflated to a value  $d(t_f) = e^{\mathcal{E}}l_P$ . Assuming that the reheating is maximally efficient, so that the Universe is reheated to a temperature corresponding to the Planck energy  $T_P \simeq 10^{19}\text{GeV}$ , the scale factor of the Universe between the end of inflation and today has increased by a factor<sup>6</sup>  $T_P/T_0 \sim 10^{32}$ . The scale corresponding to a Planck length today is thus  $d(t_0) \simeq 10^{32}e^{\mathcal{E}}l_P \simeq 0.1e^{\mathcal{E}}\text{cm}$ . Requiring at least such length to be equal to the present Hubble radius yields the inequality

$$0.1e^{\mathcal{E}}\text{cm} \gtrsim 10^{28}\text{cm} \quad (5.56)$$

that expresses the request that all the matter inside the present Hubble radius was inside a single causal horizon at the time inflation began (assumed at the Planck time). Equation (5.56) is equivalent to the following condition on the number  $\mathcal{E}$  of e-folds

$$\mathcal{E} \gtrsim \ln 10^{29} \simeq 67. \quad (5.57)$$

If inflation started later than the Planck time, or if the temperature after reheating was less than the temperature at the start of inflation, the minimum number of e-folds required to solve the horizon paradox is reduced. In the following, when needed we will use  $\mathcal{E} = 60$  as a typical value for the number of e-folds. In conclusion we can claim that, once the request on the number of e-foldings parameter has been satisfied, the inflationary paradigm is a convincing explanation for the deep conceptual problem underlying the horizon paradox.

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<sup>6</sup>In this and similar estimates, we will neglect variations of  $g_{*s}$ .

The solution of the flatness paradox is grounded to the dynamical behavior of the density parameter  $\Omega$ , described by Eq. (3.52) during the de Sitter evolution. As already noted, the Hubble function  $H$  remains fixed to its constant value  $H^*$  while the scale factor of the Universe inflates. As a result, the quantity  $\Omega_K \equiv \Omega - 1$  strikingly decreases, while the relation between the initial and final values of  $\Omega - 1$  is

$$\Omega_f - 1 = (\Omega_i - 1) \exp(-2\mathcal{E}). \quad (5.58)$$

Using the fact that  $\Omega_K$  scales as  $a^2$  during the radiation-dominated era and like  $a$  during the matter dominated era, and assuming again that the Friedmann phase of expansion starts at the Planck temperature, we have that it has increased by a factor  $\sim 10^{60}$  between the end of inflation and the present time. This yields

$$|\Omega_0 - 1| = 10^{60} |\Omega_i - 1| e^{-2\mathcal{E}} \lesssim 10^{-2} \quad (5.59)$$

where the last inequality is the observational constraint on  $|\Omega_0 - 1|$  and the minimum value of e-folds required to solve the flatness paradox is thus:

$$\mathcal{E} \gtrsim \ln 10^{31} + \frac{1}{2} |\Omega_i - 1| \simeq 71 + \frac{1}{2} |\Omega_i - 1|. \quad (5.60)$$

As it was for the horizon paradox, the minimum number of e-folds is smaller if inflation takes place later than the Planck time or if reheating is not maximally efficient. It is also interesting to note that, if the initial deviation from flatness is not too large, the amount of inflation required to solve the flatness and horizon paradoxes is approximately the same.

The solution to the flatness paradox relies on the fact that, starting from a generic value of  $\Omega_i$ , not fine-tuned to a (unphysical) huge value, at the end of the de Sitter phase  $\Omega_f$  is equal unity up to a very high degree of approximation (see Eq. (5.58)). After  $t_f$ , the critical parameter regains its standard evolution and, until today, it has increased but did not yet have the time to deviate from unity. Thus, in the framework of the inflationary scenario, the observation of a critical parameter very close to unity does not imply any fine-tuning on the initial conditions at the Planck era. On the contrary, the observation of this feature is a good indication in favor of the inflationary paradigm.

Summarizing, the de Sitter phase of the inflationary scenario has the effect to distribute thermalized matter on a large spatial scale and to stretch the spatial geometry up to be indistinguishable from the real case with zero space curvature  $K = 0$ . The behavior of the microcausal horizon and of the particle horizon are very different during the de Sitter regime associated to



the slow-rolling. In fact, while the former remains essentially constant, the latter drastically increases, so that one can say that the inflaton dynamics solves the horizon paradox because the cosmological horizon is exponentially increased with respect to the value it would have in the SCM.

### 5.5.2 *Solution to the entropy problem and to the unwanted relics paradox*

Both the entropy problem and the unwanted relics paradox are resolved by the decay of the scalar field during the reheating phase. The enormous value of the entropy per comoving volume  $S \simeq 10^{87}$  that is measured today is due to the heat produced during reheating. To illustrate this property, let us consider a volume  $V_i$  at the onset of the inflationary expansion, when the temperature of the Universe was  $T = T_c$ , while the entropy contained in  $V_i$  was

$$S_i \simeq T_c^3 V_i. \quad (5.61)$$

When the de Sitter phase ends, just before the onset of the oscillations, the entropy inside the volume is still equal to  $S_i$ , because no entropy is produced during the de Sitter phase. This implies that the linear size of the volume has increased by a factor  $\exp(\mathcal{E})$  and the temperature has decreased by the same factor. After an efficient reheating, the Universe is brought back to a temperature  $T_{\text{rh}} \simeq T_c$ . Assuming for simplicity that the reheating happens instantaneously, no further expansion occurs and so the initial volume has increased to a value  $V_f = \exp(3\mathcal{E})V_i$ . The entropy inside the volume at the beginning of the Friedmann phase is thus

$$S_f \simeq T_{\text{rh}}^3 V_f \simeq T_c^3 e^{3\mathcal{E}} V_i \simeq e^{3\mathcal{E}} S_i, \quad (5.62)$$

showing how, after reheating, the entropy inside the volume is increased by the huge value  $e^{3\mathcal{E}}$ . For the typical value  $\mathcal{E} = 60$ , this is  $e^{180} \simeq 10^{78}$ . The entropy inside a volume corresponding to the present Hubble radius before inflation was at most  $\sim 10^9$ , that is a far less impressive number with respect to  $10^{87}$ . Putting the argument the other way around, if we assume that the region with the size of the present Hubble radius had initially an entropy of order unity,  $S_i$ , then roughly  $87 \ln(10)/3 = 66$  e-folds of expansion are required to produce the observed entropy. We find again that the requirements on the number of e-folds imposed by the horizon, flatness and entropy paradoxes are remarkably close to each other.

For what concerns the unwanted relics paradox, the solution lies in the fact that the pre-inflationary abundance  $n_X$  of any particle species  $X$

is reduced by a factor  $\exp(3\mathcal{E})$  after the de Sitter phase. Of course this also holds for photons, so that the “ $X$  to photon ratio”  $n_X/n_\gamma$  is actually constant during the slow-roll. However, photons are produced copiously during reheating (in fact, practically all the photons observed today were produced at that time) while, if the reheating temperature is low enough, the  $X$ ’s are not, so that their abundance with respect to photons is greatly diluted. In particular, the final abundance of  $X$ ’s will be  $\exp(-3\mathcal{E})$  times their initial abundance. On the contrary, the abundance of photons just after reheating is  $n_\gamma \simeq T_{\text{rh}}^3 \simeq T_c^3$ , i.e. it is basically the same as it was before inflation. We stress again that it is fundamental, for this argument to work, that the reheating temperature is low enough so that the  $X$ ’s stay decoupled from the plasma and do not share the entropy transfer from the scalar field. If this is not true, they would rapidly thermalize and we would be back to the uncomfortable situation  $n_X \sim n_\gamma$ . Another way to see the solution to this paradox is to say that during reheating the specific entropy per  $X$  particle  $S/N_X$  increases enormously.<sup>7</sup> The connection between these two formulations is readily made by noting that  $n_X/n_\gamma \simeq n_x/s = N_X/S$  and that  $S$  increases by a factor  $e^{3\mathcal{E}}$  after reheating.

## 5.6 General Features

We now describe some general aspects of the inflationary scenario which do not rely on the specific form of the potential, but only on the SSB profile with a significant plateau.

### 5.6.1 Slow-rolling phase

Let us restate the equations of the coupled system (5.21) that describes the evolution of the scale factor  $a(t)$  and of the scalar field  $\phi(t)$  during the slow-rolling phase, associated to the zero-temperature potential  $V(\phi)$ , i.e.

$$\left(\frac{\dot{a}}{a}\right)^2 H^2 = \frac{\kappa}{3} V(\phi) \quad (5.63a)$$

$$3H\dot{\phi} = -\frac{dV}{d\phi}. \quad (5.63b)$$

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<sup>7</sup>The paradox itself can be restated in this form: the specific entropy per  $X$  particle is expected to be of order unity, but it turns out to be much larger than that.

From the definition of the Hubble function, the e-folding  $\mathcal{E}$  remains defined as

$$\mathcal{E} \equiv \ln \left( \frac{a_f}{a_i} \right) = \int_{t_i}^{t_f} H dt = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi, \quad (5.64)$$

with  $\phi_{i,f} \equiv \phi(t_{i,f})$ . Making use of Eqs. (5.63), the quantity  $\mathcal{E}$  rewrites as

$$\mathcal{E} = -3 \int_{\phi_i}^{\phi_f} \frac{H^2}{V'} d\phi = -\kappa \int_{\phi_i}^{\phi_f} \frac{V}{V'} d\phi, \quad (5.65)$$

where the prime denotes the derivative with respect to  $\phi$ . Let us assume that in the interval  $(\phi_i, \phi_f)$ , the ratio  $V/V'$  (i.e.  $H^2/V'$ ) is nearly constant and that  $V' \simeq V''(\phi_f - \phi_i)$ . The integral above (5.65) can be evaluated as

$$\mathcal{E} = \kappa \frac{V}{|V''|} = 3 \frac{H^2}{|V''|}, \quad (5.66)$$

where the modulus accounts for the negativity of  $V''$  resulting from the slow decreasing of the plateau from the maximum arising from the SSB scenario. From the relation (5.66) we can see that an efficient de Sitter phase, associated to a high value of the e-folding parameter (say  $\mathcal{E} \sim 60$ ), requires the constraint

$$\left| \frac{d^2 V}{d\phi^2} \right| \ll H^2. \quad (5.67)$$

This requirement characterizes the slow-rolling phase and ensures that the time evolution of the scalar field is very slow on a cosmological time scale.

### 5.6.2 Reheating phase

The subsequent stage of the scalar field dynamics is associated to a rapid fall of the scalar field into the true vacuum well. This evolution takes place on a scale smaller than the Hubble time and inequality (5.67) has to be reversed towards the opposite condition

$$\left| \frac{d^2 V}{d\phi^2} \right| \gg H^2. \quad (5.68)$$

As we have seen in Sec. 5.4.2, the second derivative of the potential term fixes the square of the boson mass  $\mu_0^2$  and therefore the condition (5.68) states that the period of the coherent oscillations of the scalar field is much smaller than the Hubble time. Indeed, the analysis in Sec. 5.4.2 of the reheating process is rather general since the main features do not depend on the form of the zero-temperature potential  $V(\phi)$ . There we noted that if

the decay time of the boson species is much smaller than the Hubble time at the end of inflation, a good reheating of the Universe is reached, in which the entire energy density of these particles is transferred to the radiation component. In fact, under the assumption  $H_d \gg H_f$ , the mass density of the Bose condensate is not significantly redshifted by the Universe expansion, before being transformed into reheating relativistic species. Since the fall into the potential well is very rapid, the energy density of the supercooled bosons is approximately the vacuum one  $\rho_\Lambda$ . If instead we are in the case  $H_d \ll H$ , i.e.  $\tau_d \gg t_H$ , the Universe expansion has the net effect of redshifting the value of the energy density which is going to reheat the causally connected regions. Such redshift can be estimated in the time interval from  $t_f$  (the beginning of the coherent oscillation phase) to  $t_f + \tau_d$  (i.e. when the conversion of the condensate into relativistic particles affects the evolution). We have the relations

$$\langle \rho_\phi \rangle_{t=t_f} \simeq \rho_\Lambda \quad (5.69)$$

$$\langle \rho_\phi \rangle_{t=t_f+\tau_d} \simeq \rho_\Lambda \left( \frac{a(t_f)}{a(t_f + \tau_d)} \right)^3 \simeq \rho_\Lambda \left( \frac{t_f}{\tau_d} \right)^2 \simeq \rho_\Lambda \left( \frac{H_d}{H_f} \right)^2 \ll \rho_\Lambda, \quad (5.70)$$

where we have taken into account that  $\tau_d \gg t_f$ . This value of the boson mass density is significantly decreased as effect of the Universe expansion and, when converted into the radiation component, we get the reheating temperature as

$$T_{\text{rh}}^{\text{P}} \simeq \left( \frac{H_d^2}{H_f^2} \frac{30\rho_\Lambda}{g_*\pi^2} \right)^{1/4} \simeq \sqrt{\frac{H_d}{H_f}} T_{\text{rh}}^{\text{G}}. \quad (5.71)$$

In this case, after inflation the Universe is characterized by a much smaller temperature than the case treated in Sec. 5.4.2 and we deal with a *poor reheating*.

In order to deal with a phase of coherent oscillations performed by the scalar field, the condition  $\mu_0 \gg H_d$  must hold even in a general case.

### 5.6.3 The Coleman-Weinberg model

One of the first proposals for new inflation is the potential of the  $SU(5)$  Coleman-Weinberg scenario. In this model, before the SSB process, the inflaton field is a 24-dimensional Higgs field, responsible for the decoupling of the  $SU(5)$  interaction of a GUT into the Standard Model of elementary particles  $SU(3)_G \otimes SU(2)_L \otimes U(1)_Y$ . The relic scalar field  $\phi$  is described

by the (one-loop zero-temperature) potential

$$V(\phi) = \beta + 2\beta u^4 \left[ \ln(u^2) - \frac{1}{2} \right], \quad (5.72)$$

where  $u = \phi/\Sigma$  ( $\Sigma \simeq 2 \times 10^{15}$  GeV being the energy scale of the SSB process) and  $\beta = B\Sigma^4/2$ , with  $B = 25\alpha_{\text{GUT}}/16 \simeq 10^{-3}$ ,  $\alpha_{\text{GUT}}$  denoting the coupling constant associated to the GUT interaction.

The dependence on the temperature of the full potential is characterized by a small barrier, having a height of about  $\mathcal{O}(T_c^4)$  and a critical temperature  $T_c$  running between  $10^{14}$  GeV and  $10^{15}$  GeV. Nonetheless, the false vacuum in  $\phi = 0$  remains metastable up to a temperature of about  $10^9$  GeV and the phase transition is due to the one-loop radiative corrections.

Near  $\phi = 0$ , i.e.  $\phi \ll \Sigma$ , the potential (5.72) admits a plateau in the quartic form (5.29), with  $\rho_\Lambda = \beta$  and  $\omega$  is provided by the logarithmic term, when calculated for a characteristic value  $\phi^*$  of the plateau such that  $\omega = 4B \ln(\phi^*/\Sigma)^2 \simeq 0.1$ .

The Coleman-Weinberg model is no longer a reliable candidate to provide the basis for inflation, because it is based on a  $SU(5)$  symmetry (almost abandoned in GUT because it violates the limits on the proton lifetime) and for its internal inconsistencies (unless the parameters of the model - essentially  $\omega$  - are fine-tuned). However, this scheme remains an important example, elucidating how the inflation proposal is strengthened from its crossmatch with the fundamental particle physics background.

#### 5.6.4 *Genesis of the seeds for structure formation*

Until now, our analysis was based on the idea that the scalar field is a function of time only, as required by the Universe homogeneity. This statement is valid on a classical level, because the energy density of the self-interacting bosons is, as a whole, the source for the spacetime geometry. Such situation does not hold when quantum fluctuations of the field are taken into account. We have implicitly made reference to this inhomogeneous features of the inflaton during the discussion of the quantum tunneling across the barrier between the false and the true vacua (see Sec. 5.2). In fact, the microphysical causal structure requires independent evolution over causally disconnected patches. The same concept is also applicable to the quantum fluctuations of the inflaton during the de Sitter phase associated to the slow-rolling regime. We will concentrate our attention on this stage of the inflationary paradigm because the inhomogeneous scales generated during

the exponential growth of the scale factor are stretched to super-horizon size, and can re-enter the Hubble scale at later times, consistently with the requested spectrum of structure formation.

In the inflationary paradigm, the initial seeds for structure formation are the quantum fluctuations of the scalar field. During the phase of inflationary expansion, these fluctuations are stretched well above the microcausal horizon and become classical curvature perturbations. When the Friedmann phase begins, the microcausal horizon starts growing faster than the perturbation scale so that, after a certain time, the perturbation will re-enter the horizon and will start to evolve as a density fluctuation. The idea we are tracing is apparently surprising: *galaxies originally arise from quantum disturbances that evolved until the present time via the Universe expansion and the gravitational instabilities*; the possibility for a different origin of the inhomogeneous fluctuations is forbidden because of the strong cancellation of the initial conditions that inflation produces during the de Sitter phase. For example, it could be thought that the origin of the primordial fluctuations is in the radiation component present before inflation. However, since the radiation energy density behaves as  $a^{-4}$ , it is strongly depressed by the exponential expansion of the slow-rolling evolution. The initial value of the radiation density would be suppressed by the huge factor  $\exp(-4\mathcal{E})$  before the end of the de Sitter phase. The radiation component of the Universe is suppressed so strongly that its spectrum could never be at the ground of the structure formation process. For a detailed discussion of the inhomogeneity behavior during an exponential expansion of the Universe, see the study on the quasi-isotropic inflation of Sec. 6.5. It is exactly this suppression of the energy density of all components of the cosmological fluid, apart from the scalar field, that leads to search in the quantum fluctuations of the scalar field itself the only reliable mechanism for the generation of a spectrum of primordial inhomogeneities.

In the following we will make use of the concepts introduced in Sec. 4.2.3 to characterize the fluctuations of a field. First of all, we split the field  $\phi$  in the sum of an unperturbed, homogeneous part with a small fluctuation as

$$\phi(\vec{x}, t) = \bar{\phi}(t) + \delta\phi(\vec{x}, t). \quad (5.73)$$

By a Fourier transform of the spatial dependence in  $\delta\phi$ , we can deal with the perturbation in  $k$ -space  $\delta\phi_k$ , where  $k$  is the co-moving wave-number, related to the physical wave-number by  $k_{\text{phys}} = k/a(t)$ . During the de Sitter phase the slow-roll condition ensures that  $V'' \ll H^2$ , so that the mass of the field  $m^2 = V''$  can be neglected, i.e. one deals with a free massless and

minimally coupled boson field in a de Sitter space. The variance of the field perturbations in  $k$ -space is given by

$$\langle |\delta\phi_k|^2 \rangle = \frac{H^2}{2k^3}, \quad (5.74)$$

so that the power spectrum  $\Delta_\phi^2$  is

$$\Delta_\phi^2 = \left( \frac{H}{2\pi} \right)^2. \quad (5.75)$$

In the last two equations,  $H$  is evaluated at the time when the mode  $k$  exits the horizon. The perturbation of a massless scalar field obeys the evolution equation in  $k$ -space

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + k^2\delta\phi_k = 0 \quad (5.76)$$

which shows how, when a mode is well outside the horizon ( $k \rightarrow 0$ ), the amplitude of the corresponding perturbation remains constant and the mode re-enters the horizon with roughly the same amplitude it had when it left.

Since it can be shown that in the region of (co-moving) wavelengths interesting for the structure formation (i.e. from 0.1 Mpc to 100 Mpc<sup>8</sup>) the Hubble constant during inflation remains nearly constant and we finally obtain the suggestive feature of a scale independent fluctuation spectrum. All the perturbations leave the physical horizon of the slow-rolling regime with the same amplitude, almost independently of their size. When they become super-horizon sized, the microphysics cannot affect any longer their evolution and a process of freezing takes place with the net result of reducing the quantum spectrum to a classical profile of curvature perturbations.

The perturbation of the energy density  $\delta\rho_k$  associated to the quantum fluctuations in the field  $\delta\phi_k$  is

$$\delta\rho_k = \frac{\partial\rho}{\partial\phi}\delta\phi_k = V'\delta\phi_k = -3H\dot{\phi}_k\delta\phi_k, \quad (5.77)$$

since during the de Sitter phase  $\rho_\phi \simeq V$  and  $3H\dot{\phi}_k + V' = 0$ .

After the end of inflation the Universe regains its standard Friedmann like evolution and the microphysical horizon increases faster than the physical scales. The perturbations generated according to the mechanism described above start to re-enter the Hubble horizon. To investigate the structure formation process, as predicted by the inflationary spectrum, we need to calculate the form of such spectrum when the perturbations become sub-horizon-sized again. The main difficulty in this task is the gauge-dependent

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<sup>8</sup>We recall that the value of the scale factor today is fixed, by convention, equal to 1, so that the comoving scale coincides with the present physical ones.

nature of the spectrum evolution; in particular, the density contrast  $\delta\rho/\rho$  is not gauge invariant. We can overcome this problem by using the gauge invariant quantity  $\zeta$  introduced by Bardeen, which has the key property to remain constant during the super-horizon evolution of the perturbations. The point to be addressed is the link between the energy density fluctuations and the value of  $\zeta$  in correspondence of the two horizon crossings. When the mode is not too much outside the horizon,  $\lambda \lesssim H^{-1}$ , so that the perturbations of the metric can be neglected and  $\zeta$  is given by

$$\zeta = \frac{\delta\rho}{\rho + P} \quad \text{for } \lambda \lesssim H^{-1}. \quad (5.78)$$

Recalling that during the de Sitter regime the term in the denominator of Eq. (5.78) is fixed by the scalar field energy density as  $\rho + P \simeq \dot{\phi}^2$ , at the first crossing (FC) we have

$$\delta_k^{\text{FC}} = \left( \frac{\rho + P}{\rho} \zeta \right)^{\text{FC}} = \frac{\dot{\phi}^2}{\rho_\Lambda} \zeta^{\text{FC}}, \quad (5.79)$$

where  $\delta_k \equiv \delta\rho_k/\bar{\rho}$ . On the other hand, at the second crossing (SC) of the horizon, i.e. at the re-entrance into the Hubble radius, when the Universe is either radiation- or matter-dominated, we have

$$\delta_k^{\text{SC}} = \left( \frac{\rho + P}{\rho} \zeta \right)^{\text{SC}} = (1 + w) \zeta^{\text{SC}}. \quad (5.80)$$

Since  $\zeta$  is time independent,  $\zeta^{\text{SC}} = \zeta^{\text{FC}}$  and thus Eqs. (5.79) and (5.80) together yield

$$\delta_k^{\text{SC}} = \frac{(1 + w)\rho_\Lambda}{\dot{\phi}^2} \delta_k^{\text{FC}} \simeq \frac{\rho_\Lambda}{\dot{\phi}^2} \delta_k^{\text{FC}}. \quad (5.81)$$

The density perturbation at horizon exit can be obtained combining Eqs. (5.74) and (5.77) to get

$$\delta_k^{\text{FC}} \simeq -\frac{H^2 \dot{\phi}}{k^{3/2} \rho_\Lambda}, \quad (5.82)$$

where we omitted a numerical factor of order unity. The perturbation at horizon re-entry is thus

$$\delta_k^{\text{SC}} \simeq -\frac{H^2}{k^{3/2} \dot{\phi}} \quad (5.83)$$

and the power spectrum of density perturbations  $\Delta_k^2 = k^3 |\delta k|^2 / 2\pi^2$  is finally

$$\Delta_k^2 \simeq \frac{1}{2\pi^2} \left( \frac{H^2}{\dot{\phi}} \right)^2. \quad (5.84)$$



The perturbation spectrum predicted by inflation as an initial condition for structure formation has a flat profile because it does not depend on  $k$ . This is called the *Harrison-Zeld'ovich* (HZ) spectrum.

The fact that the Hubble parameter is actually varying (albeit slowly) during the slow roll introduces a small scale dependence in the perturbation spectrum so that the spectrum has a power-law form as

$$\Delta_k^2 = A k^{n-1}, \quad (5.85)$$

where  $A$  is the amplitude,  $n$  is the spectral index, and we follow common usage in making the HZ spectrum correspond to  $n = 1$ . The spectral index is given by

$$n - 1 = -6\epsilon + 2\eta, \quad (5.86)$$

where the *slow-roll parameters*  $\epsilon$  and  $\eta$  are defined as

$$\epsilon = \frac{1}{2\kappa} \left( \frac{V'}{V} \right)^2 \quad (5.87)$$

$$\eta \equiv \frac{1}{\kappa} \frac{V''}{V}. \quad (5.88)$$

As the name suggests, these parameters are related to the slow-roll since the conditions for its occurrence are  $\epsilon \ll 1$  and  $|\eta| \ll 1$ . The existence of the slow-roll phase automatically implies that  $n - 1 \ll 1$ , i.e. that the spectrum is very close to the HZ one. A measurement of the spectral index is a powerful tool to constraint the possible models of inflation, since its value is related to the characteristics of the potential through the slow-roll parameters. The value of the spectral index has actually been measured to percent accuracy through the observations of the CMB anisotropy spectrum made by the WMAP satellite (see Sec. 4.4) and it has been found to be less than unity.

While the shape of the spectrum is quite a precise prediction of the theory of inflation, the same cannot be said for its amplitude which has to be fixed by requiring a satisfactory scheme of structure formation, with particular reference to the origin of galaxies. Perturbations at the galactic scale  $k_{\text{gal}} \sim 1 \text{ Mpc}^{-1}$  enter the horizon at  $z \simeq 10^5$ , in the radiation-dominated era. As discussed in Sec. 3.4.3, the perturbations can grow only logarithmically in that regime, so for the moment we neglect the growth of the perturbation between horizon entry and matter-radiation equality ( $z_{\text{eq}} \simeq 10^4$ ). After equality, the perturbation grows as  $a$  (see again Sec. 3.4.3). A reasonable condition is a density contrast of order unity, corresponding to the

beginning of the non-linear evolution, not latest that  $z \simeq 10$ , which yields

$$|\delta_k^{\text{SC}}|_{k=k_{\text{gal}}} = \left| \frac{H^2}{k_{\text{gal}}^{3/2} \dot{\phi}} \right| \sim 10^{-3}. \quad (5.89)$$

Taking into account the logarithmic growth between the horizon re-entry and the matter-radiation equality would alter the above estimate by a factor  $\ln(t_{\text{eq}}/t_{\text{SC}}) = 2 \ln(a_{\text{eq}}/a_{\text{SC}}) \simeq 5$ .

Let us conclude this Section by stressing how the inflationary paradigm is able to provide a natural mechanism for the generation of density inhomogeneities. The predicted spectrum has the striking feature to have a nearly scale-independent form at the time when the perturbations re-enter the horizon. This simple and convincing picture for the genesis of a clumpy Universe must be regarded as one of the most appealing issues of the inflation scenario, in addition to the solution of the SCM paradoxes.

## 5.7 Possible Explanations for the Present Acceleration of the Universe

In this Section we will discuss some possible explanations for the present observed acceleration of the Universe, introduced in Sec. 4.3, in order to provide an overall picture of the contemporary ongoing lines of research. In particular, we will first introduce the possibility that the acceleration is caused by *dark energy*, namely an exotic component of the cosmological fluid with an equation of state parameter  $w_{\text{de}} < -1/3$ . We will focus on quintessence models, in which the dark energy is a scalar field. Then, we will describe modifications to GR, the so-called  $f(R)$  theories. A third possibility, not treated here, is that the acceleration is just an artefact due to the inhomogeneous structure of the Universe at small scales.

The ideas underlying the quintessence and the  $f(R)$  theory are somewhat related to inflation. In quintessence models, the present acceleration is due to the presence of a scalar field slowly rolling on its potential, so that the ideal connection with inflation is evident.

In the case of  $f(R)$  theories, a phase of accelerated expansion can also be caused by modifications to the action of GR. In fact, the possibility of an early inflationary phase was firstly discussed by Starobinsky in 1980. On the other hand, since both the early, inflationary Universe and the present one are accelerating, it is natural to investigate whether the two phenomena could be related. Although an explanation of both inflation and the present

acceleration has been proposed, the results to date are not yet satisfying.

### 5.7.1 *Dark energy*

In the mentioned models, the acceleration is due to the presence of an exotic component, dubbed “dark energy”, with negative pressure (in particular,  $w_{\text{de}} < -1/3$ ). This leads to a repulsive gravity (as it can be naively seen from the fact that the quantity  $\rho + 3P$ , which is the source of the Poisson equation for the gravitational field in the weak field limit, becomes negative) and thus to acceleration. This kind of models invoke a modification of the right-hand side of the Einstein equations, i.e. of the energy-momentum tensor. The simplest candidate for a negative-pressure component is the energy density associated to the quantum vacuum. From a mathematical point of view, the vacuum energy is equivalent to a cosmological constant since both give rise to a contribution to the energy-momentum tensor with equation of state  $P = -\rho$ . However, the computation of the quantum zero-point energy leads to divergent or anyway very large values. In general, the vacuum energy density is of the order of  $k_{\text{max}}^4$ , where  $k_{\text{max}}$  is the ultraviolet cutoff imposed to avoid divergences. Taking the cutoff at the Planck scale, we get  $\rho_{\text{vac}} \simeq m_{\text{P}}^4 \simeq 10^{112} \text{ eV}^4$ . On the other hand, the present dark energy density is  $\rho_{\text{de}} \simeq \rho_{\text{c}} \simeq 10^{-5} h^2 \text{ GeV/cm}^3 \simeq 10^{-11} \text{ eV}^4$ , so that this simple estimate is wrong by  $\sim 120$  orders of magnitude. The situation does not get much better by considering a cutoff at the electroweak scale. In fact, the cosmological energy density corresponds to an energy of the order of  $10^{-3} \text{ eV}$ , a scale where it is unreliable to invoke any new physics. The very large value of the vacuum energy compared to the observed density of the Universe goes under the name of “cosmological constant problem”.

A subset of dark energy models are the so-called *dynamical dark energy*, or “quintessence” models, where a scalar field is responsible for the present acceleration. Differently from the case of vacuum energy, the scalar field is dynamical and then the equation of state parameter  $w$  is expected to vary with time, i.e.  $w = w(z)$ . The scalar field has to be homogeneous (at least to zeroth order) in order to satisfy the requirements of the cosmological principle. The evolution of the field in a cosmological setting is given by Eq. (5.21), while its density and pressure are given by (see Sec. 2.2.2)

$$\rho_{\phi} = \frac{\dot{\phi}^2}{2} + V(\phi); \quad P_{\phi} = \frac{\dot{\phi}^2}{2} - V(\phi), \quad (5.90)$$

where  $\dot{\phi}^2/2$  is the kinetic energy of the field and  $V(\phi)$  is its potential. The

equation of state parameter  $w_\phi$  for the scalar field is thus given by

$$w_\phi \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \quad (5.91)$$

When the energy of the field is dominated by its potential energy (i.e. the field is slowly varying,  $\dot{\phi}^2/V \ll 1$ ), then  $w_\phi \simeq -1$ , the energy density of the field is nearly constant (we recall that  $\rho_\phi \propto a^{-3(1+w_\phi)}$ ) and the field mimics a cosmological constant. This is exactly what happens during the slow-roll phase of inflation. On the other hand, when the field is rapidly varying,  $\dot{\phi}^2/V \gg 1$  and  $w_\phi \simeq +1$ , so that  $\rho_\phi \propto a^{-6}$ . The regime during which the energy of a field is dominated by its kinetic energy is called *kination*. In general, the equation of state parameter can take any value between  $-1$  and  $+1$ . When  $w_\phi < -1/3$ , this can possibly give rise to a phase of accelerated expansion (if this actually happens depends on the energy density of the other components of the cosmological fluid). By assuming a non-standard form of the kinetic energy term, it is also possible to obtain the so-called *phantom scenarios* when  $w_\phi < -1$ , meaning that the scalar field density increases with time. The phantom models, however, are typically unstable with respect to perturbations.

The evolution of the field and of its equation of state parameter strongly depend on the form of the potential. In general, two classes of model can be distinguished, depending on whether the velocity of the field increases with time or not, i.e.  $\ddot{\phi} \gtrless 0$ . When  $\ddot{\phi} > 0$ , the field rolls faster with time so that it starts as a cosmological constant-like component and then evolves away from  $w < -1$ . These models are called *thawing*. On the contrary, in models with  $\ddot{\phi} < 0$ , the field rolls more slowly with time, so that the cosmological constant-like behavior is recovered at late times. These models are called *freezing*. A peculiar feature of some freezing models is that they have a *tracking* behavior, meaning that they track the dominant component of the energy density of the Universe at early times, and then they dominate at late times. This can possibly solve the “coincidence problem”, namely the puzzling fact that although the cosmological densities of matter and vacuum energy vary very differently with time, nevertheless we happen to live in a time in which they are just a factor of two apart.

Quintessence models have to face some issues. First of all, they still suffer from the cosmological constant problem. The minimum  $V_0$  of the potential has to be very small or exactly zero in order to avoid it. Secondly, in order to be responsible for the expansion, the effective mass of the field  $m_\phi \equiv \sqrt{V''(\phi)}$  has to be very small, of order  $H^{-1} = 10^{-33}$  eV (i.e. its

Compton wavelength has to be of order of the Hubble radius), while its vacuum expectation value  $\langle\phi\rangle$  has to be of order of the Planck mass  $m_{\text{P}}$ . This also poses a *hierarchy problem*, i.e. it should be explained why  $m_\phi$  is 60 orders of magnitude smaller than  $\langle\phi\rangle$ .

### 5.7.2 Modified gravity theory

The geometrical and tensor structure of GR determines the kinematics of the gravitational field in a very consistent formulation, but its dynamics admits a wide class of different proposals. In fact, the Einstein-Hilbert action is only the most simple proposal in agreement with the experimental data and its most striking feature is the absence in the field equations of derivatives of order higher than the second. A more general formulation of the gravitational field dynamics is the replacement of the Ricci scalar in the Einstein-Hilbert action by a generic function  $f(R)$ , which reduces to  $f(R) \sim R$  in the weak field limit, when the spacetime curvature is small enough. In what follows, we will consider a gravitational action of the form

$$S_{\text{f}} = -\frac{1}{2\kappa} \int_{\mathcal{M}} \sqrt{-g} f(R) d^4x, \quad (5.92)$$

where the function  $f$  corresponds to  $\infty^1$  degrees of freedom. Such open choice for the geometrodynamics allows to interpret the Universe acceleration (which is intrinsically a dynamical “anomaly” of the Universe evolution) by means of the additional term entering the new Einstein equations (and hence the modified Friedmann ones too). In fact, the variation of the action (5.92) with respect to the contravariant metric  $g^{ij}$  implies the following set of field equations (having order of differentiation greater than two)

$$f' R_{ij} - \frac{1}{2} f(R) g_{ij} - \nabla_i \nabla_j f' + g_{ij} \nabla_l \nabla^l f' = \kappa T_{ij}, \quad (5.93)$$

where  $f' \equiv df/dR$ . These modified Einstein equations can be recast in the form

$$R_{ij} - \frac{1}{2} R g_{ij} = \kappa (T_{ij} + T_{ij}^{\text{Curv}}), \quad (5.94)$$

with the identification

$$T_{ij}^{\text{Curv}} = \left( \frac{1}{2} F(R) - \nabla_l \nabla^l f' \right) g_{ij} + \nabla_i \nabla_j f', \quad (5.95)$$

$F(R) \equiv f(R) - R$  denoting the deviation from the Einstein-Hilbert Lagrangian density. The generalized theory can be rewritten as an Einstein-like theory with a curvature term as a source. The interpretation of the Universe acceleration in this modified gravity approaches is that the additional

terms in the field equations do not affect significantly the early Universe thermal history but, in the late evolution, they are able to mimic a perfect fluid contribution, having a dark energy equation of state  $P < -\rho/3$ . Recalling that for a RW geometry the scalar of curvature reads as

$$R = -6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{K}{a^2} \right], \quad (5.96)$$

the (synchronous) Friedmann equation (3.46) is deeply modified and takes the form

$$3 \frac{\dot{a}}{a} \dot{R} f'' - 3 \frac{\ddot{a}}{a} f' - \frac{1}{2} f = \kappa \rho(a). \quad (5.97)$$

In Eq. (5.97)  $\rho(a)$  denotes the energy density as a function of the cosmic scale factor in the standard form  $\rho \propto 1/a^{3\gamma}$ , as it is guaranteed by the continuity equation (3.35). This equation, like in the Friedmann case, is the one determining the full system dynamics. Indeed many different approaches succeeded in deriving an acceleration of the late Universe, offering a rather consistent cosmological picture. Any significant modification of the dynamics on cosmological scales must also allow to reconcile the Solar system data with the deviations due to the non-Einsteinian terms. It is worth reminding how a modified  $f(R)$  theory of gravity must be consistent with the observations on all the length scales of physical interest. Such request and the problems of possible degeneracy of different theories provide the most challenging tasks of these revised dynamical approaches.

When dealing with the generalized gravitational action (5.92), we require  $f(R = 0) = 0$  to avoid a huge cosmological constant, otherwise a fine-tuning of the model parameters would be needed. Furthermore, in order to recover GR for low curvature values, we take the representation of  $f(R)$  in the form

$$f(R) = R + F(R) \quad , \quad \lim_{R \rightarrow 0} F(R) = 0, \quad (5.98)$$

where we can address both analytical and non-analytical expressions for  $F(R)$ .

The two prescriptions above are not always addressed in the literature and they must be intended as simplicity requests for the modified action.

**Scalar-tensor theory** The scalar-tensor representation is based on translating the scalar degree of freedom related to the function  $f(R)$  into a dynamical scalar field coupled to the Einstein-Hilbert dynamics. This result is achieved via a suitable conformal transformation on the original space-time metric.

By means of the two auxiliary fields (i.e. Lagrange multipliers)  $A$  and  $B$ , the action (5.92) can be rewritten as

$$S_{\text{ST}} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} [B(A - R) + f(A)] . \quad (5.99)$$

The variation with respect to  $B$  gives  $R = A$ , while the variation with respect to  $A$  provides  $B = -df/dA \equiv -f'(A)$ , corresponding to the so-called Jordan frame, so that the action (5.99) takes the form

$$S_{\text{ST}} = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} [f'(A)(R - A) + f(A)] . \quad (5.100)$$

Let us now redefine the metric tensor as  $g_{ij} = e^{\sqrt{\frac{2\kappa}{3}}\phi} \hat{g}_{ij}$  and obtain

$$\sqrt{-g} = e^{2\sqrt{\frac{2\kappa}{3}}\phi} \sqrt{-\hat{g}} \quad (5.101)$$

$$R = e^{-\sqrt{\frac{2\kappa}{3}}\phi} \left( \hat{R} - \kappa \hat{g}^{ij} \partial_i \phi \partial_j \phi \right) , \quad (5.102)$$

so that adopting the identification  $\phi \equiv -\sqrt{3/2\kappa} \ln f'(A)$ , the gravitational action (5.100) can be restated as the scalar-tensor one

$$S_{\text{ST}} = -\frac{1}{2\kappa} \int d^4x \sqrt{-\hat{g}} \hat{R} + \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{2} \hat{g}^{ij} \partial_i \phi \partial_j \phi - V(\phi) \right] , \quad (5.103)$$

where the potential term is defined via the relation<sup>9</sup>

$$V(\phi) \equiv \frac{f - Af'}{f'^2} , \quad (5.104)$$

once the field  $A$  is expressed in terms of  $\phi$  as

$$A = f'^{-1}(e^{\sqrt{2\kappa/3}\phi}) . \quad (5.105)$$

This restated scheme is called the *Einstein frame* because the standard geometrodynamics is recovered, though the theory is no longer in vacuum and a real self-interacting scalar field appears.

The scalar-tensor scenario associated to a modified  $f(R)$  theory of gravity offers a natural context to solve the acceleration puzzle in the same spirit traced in the previous subsection, i.e. the scalar field dynamics is responsible for the dark energy density and is minimally coupled to gravity.

We conclude by stressing that the equivalence between the Jordan (original) frame and the Einstein one has not been definitely established although the former approach is commonly preferred despite its high complexity.

<sup>9</sup>Our definition of the potential differs by a sign from the definition usually found in the literature on the subject. This is because we use the  $(+---)$  signature of the metric instead of the  $(-+++)$  signature commonly used in the literature on the scalar-tensor theory. The two choices can be related by letting  $f(R) \rightarrow -f(R)$ .

## 5.8 Guidelines to the Literature

The theory of inflation is treated in many books, like those by Kolb & Turner [290], Linde [323, 324], Lyth & Liddle [327] and Mukhanov [357]. A discussion of the standard model paradoxes, treated in Sec. 5.1, can be found in each of them.

The spontaneous symmetry breaking and the Higgs mechanism, discussed in Sec. 5.2, are described in most books on quantum field theory, like those by Mandl & Shaw [335] and Weinberg [463].

The idea behind the inflationary paradigm, as described in Sec. 5.2, was first proposed by Guth in 1980 [212], in his model now called *old inflation*. The idea of an early phase of inflationary expansion had been previously discussed by Starobinsky in the context of modifications to the action of GR [426], albeit with no reference to the possibility of solving the shortcomings of the SCM. New inflation and the slow-roll were introduced by Linde [320] and Albrecht & Steinhardt [2]. Other inflationary models that have been proposed include the chaotic inflation model by Linde [321], the double inflation model by Turner & Silk [419] and the power-law inflation by Lucchin & Matarrese [326]. The number of existing inflationary models is indeed extremely large: we refer the interested reader to the review [328] and to the book by Liddle & Lyth [327], Chap. 8.

The theory of phase transitions and tunneling, the effective potential and the coupling to the thermal bath discussed in Sec. 5.3.1 are treated in more detail in [290, 323, 324, 357].

The theory of reheating, discussed in Secs. 5.4.2 and 5.6.2 is treated in the books by Linde [323, 324] and Mukhanov [357]. In particular, the latter covers some more recent developments that we have left aside for pedagogical purposes.

The generation of primordial fluctuations, briefly discussed in Sec. 5.6.4, is the focus of the above-mentioned book [327].

The Coleman-Weinberg potential discussed in Sec. 5.6.3 was introduced in [124].

The observational evidences and the possible explanations for the present acceleration of the Universe are discussed in [178]. Quintessence models, discussed in Sec. 5.7.1 were introduced in [108]. A comprehensive review on dark energy can be found in [379]. For what concerns the  $f(R)$  theories discussed in Sec. 5.7.2 we refer the reader to the reviews



in [109, 109, 366, 425]. In particular, an interesting scenario able to interpret the inflation paradigm and the Universe acceleration into a unified picture by modified gravity, is provided by [365].

## Chapter 6

# Inhomogeneous Quasi-isotropic Cosmologies

In this Chapter, we analyze the so-called *quasi-isotropic* solution, firstly derived by Lifshitz and Khalatnikov in 1963 (KL). This model is a generalization of the FRW cosmology in which a certain degree of inhomogeneity, and hence of anisotropy, is introduced in dynamics of the Universe. The inhomogeneity of the space slices is reflected to the presence of free functions of the coordinates, which are available for the Cauchy problem. In particular, the original KL solution, corresponding to the radiation-dominated Universe where such a quasi-isotropic regime corresponds to a Taylor expansion of the metric tensor in time, deals with three physically independent spatial functions. Their presence ensures that the class of solutions arises from three independent spatial degrees of freedom able to freely fulfill initial conditions on a non-singular spatial hypersurface.

The Chapter is devoted to describe the behavior of the quasi-isotropic solution in different cosmological contexts, characterized by a suitable nature of the matter source. In particular, we analyze the inflationary behavior of a quasi-isotropic Universe in the cases of a dominant massless scalar field (the primordial inflaton) or of a cosmological constant (the slow-rolling phase). The nature of the quasi-isotropic solution when both a massless scalar field and an electromagnetic field are present, is compared with the corresponding context when ultrarelativistic matter replaces the vector component. The quasi-isotropic model is treated in the last section of this Chapter in a viscous framework, which generalizes the KL regime to the presence of out-of-equilibrium features, described by a bulk viscosity coefficient.

Concerning the inflationary scenario, the presence of the scalar field allows to deal with an arbitrary spatial distribution of the ultrarelativistic energy density, but the corresponding spectrum of inhomogeneities is

washed out later by the de Sitter phase. This issue clarifies the necessity of a quantum nature for the perturbations which originated the large-scale structure observed in the present Universe.

## 6.1 Quasi-Isotropic Solution

Many interesting results can be derived from the study of the general properties of the cosmological solution of the Einstein field equations, in particular regarding the chaoticity characterization (as discussed in details in Chaps. 7, 8, 9), and the existence of a singularity in a general framework.

In order to describe the present Universe – which appears homogeneous and isotropic from experimental observations at large scales, see Chap. 4, it is interesting to investigate its gravitational stability. Hence, the evolution backwards in time of small density perturbations is of particular relevance when considering cosmological models more general than the homogeneous and isotropic one, since the assumptions of uniformity and isotropy are justified only at an approximate level.

In 1963, Lifshitz and Khalatnikov first proposed the so-called quasi-isotropic solution discussed in this Chapter. This model is based on the idea that the space contracts maintaining linear distance changes with the same time dependence order by order (i.e. considering a Taylor-like expansion of the three-metric).

The Friedmann solution near the Big Bang, corresponding to the radiation dominated era, is a particular case of this class of solutions in which the space contracts in a quasi-isotropic way, providing a solution which exists only in a space filled with matter.

## 6.2 The Presence of Ultrarelativistic Matter

When considering the isotropic solution in a generic reference frame, isotropy and homogeneity imply the vanishing of the off-diagonal metric components  $g_{0\alpha}$ . The decrease of such functions is related to the equation of state when regarded in a co-moving frame. In fact, in the inhomogeneous case we may deal with a synchronous and co-moving frame only in the presence of a dust fluid (i.e.  $P = 0$ ).

For the ultrarelativistic matter the equation of state reads as  $P = \rho/3$  and the metric  $h_{\alpha\beta}$  for the isotropic case is linear in  $t$ . Hence, when searching for a quasi-isotropic extension of the Robertson-Walker geometry the

metric should be expandable in integer powers of  $t$ , asymptotically as  $t \rightarrow 0$ , following the Taylor-like expansion

$$h_{\alpha\beta}(t, x) = \sum_{n=0}^{\infty} a_{\alpha\beta}^{(n)}(x^\gamma) \left( \frac{t}{t_0} \right)^n, \quad (6.1)$$

where

$$a_{\alpha\beta}^{(n)}(x^\gamma) \equiv \left. \frac{\partial^n h_{\alpha\beta}}{\partial t^n} \right|_{t=0} t_0^n, \quad (6.2)$$

and  $t_0$  is an arbitrary time ( $t \ll t_0$ ), while the existence of the singularity implies  $a_{\alpha\beta}^{(0)} \equiv 0$ . In what follows, we will deal only with the first two terms of this expansion, i.e.

$$h_{\alpha\beta} = a_{\alpha\beta}^{(1)} \frac{t}{t_0} + a_{\alpha\beta}^{(2)} \left( \frac{t}{t_0} \right)^2. \quad (6.3)$$

After a suitable rescaling we get

$$h_{\alpha\beta} = t a_{\alpha\beta} + t^2 b_{\alpha\beta} + \dots, \quad (6.4)$$

whose inverse matrix to lowest order reads as

$$h^{\alpha\beta} = t^{-1} a^{\alpha\beta} - b^{\alpha\beta} + \dots. \quad (6.5)$$

The tensor  $a^{\alpha\beta}$  is the inverse of  $a_{\alpha\beta}$  and is used for the operations of rising and lowering indices as well as for the spatial covariant differentiation, i.e.  $a^{\alpha\beta} a_{\beta\gamma} = \delta_\gamma^\alpha$  and  $b_\beta^\alpha = a^{\alpha\gamma} b_{\gamma\beta}$  is ensured by the scheme of approximation.

We recall how the Einstein equations in the synchronous system (see Eqs. (2.98)) assume the form

$$R_0^0 = -\frac{1}{2} \partial_t k_\alpha^\alpha - \frac{1}{4} k_\alpha^\beta k_\beta^\alpha = \kappa \left( T_0^0 - \frac{1}{2} T \right) \quad (6.6a)$$

$$R_\alpha^0 = \frac{1}{2} (\nabla_\beta k_\alpha^\beta - \nabla_\alpha k_\beta^\beta) = \kappa T_\alpha^0 \quad (6.6b)$$

$$R_\alpha^\beta = -\frac{1}{2\sqrt{h}} \partial_t \left( \sqrt{h} k_\alpha^\beta \right) - P_\alpha^\beta = \kappa \left( T_\alpha^\beta - \frac{1}{2} T \delta_\alpha^\beta \right), \quad (6.6c)$$

where the tensor  $k_{\alpha\beta}$  and its contractions read as

$$k_{\alpha\beta} = \partial_t h_{\alpha\beta} = a_{\alpha\beta} + 2t b_{\alpha\beta} \quad (6.7a)$$

$$k_\alpha^\beta = h^{\beta\delta} k_{\alpha\delta} = t^{-1} \delta_\alpha^\beta + b_\alpha^\beta \quad (6.7b)$$

$$k = \partial_t \ln h = 3t^{-1} + b, \quad (6.7c)$$

where  $b = b_\alpha^\alpha$  and from which we get

$$h = \det(h_{\alpha\beta}) \sim t^3 (1 + tb) \det(a_{\alpha\beta}). \quad (6.8)$$

Let us note that the notation in terms of  $k_{\alpha\beta}$  is equivalent to the one introduced in Eq. (2.66), since we have  $k_{\alpha\beta} \equiv -2K_{\alpha\beta}$ .

We complete this scheme by observing how this framework is covariant with respect to a coordinate transformation of the form

$$t' = t + f(x^\gamma), \quad x^{\alpha'} = x^{\alpha'}(x^\gamma), \quad (6.9)$$

being  $f$  a generic space dependent function. Such property will hold for all paradigms treated throughout the current chapter.

We recall that the energy-momentum tensor for an ultrarelativistic perfect fluid takes the form

$$T_{ik} = \frac{\rho}{3} (4u_i u_k - g_{ik}), \quad (6.10)$$

which provides the following relations

$$T_0^0 = \frac{1}{3} \rho (4u_0^2 - 1) \quad (6.11a)$$

$$T_\alpha^0 = \frac{4}{3} \rho u_\alpha u^0 \quad (6.11b)$$

$$T_\alpha^\beta = -\frac{\rho}{3} (4u_\alpha u^\beta + \delta_\alpha^\beta) \quad (6.11c)$$

$$T = 0, \quad (6.11d)$$

where  $u^\beta = h^{\alpha\beta} u_\alpha$ . Consequently, the Einstein equations reduce to the partial differential system

$$\frac{1}{2} \partial_t k_\alpha^\alpha + \frac{1}{4} k_\alpha^\beta k_\beta^\alpha = -\kappa \frac{\rho}{3} (4u_0^2 - 1) \quad (6.12a)$$

$$\frac{1}{2} (\nabla_\beta k_\alpha^\beta - \nabla_\alpha k_\beta^\beta) = \frac{4}{3} \kappa \rho u_\alpha u^0 \quad (6.12b)$$

$$\frac{1}{2\sqrt{h}} \partial_t (\sqrt{h} k_\alpha^\beta) + {}^3R_\alpha^\beta = \kappa \frac{\rho}{3} (4u_\alpha u^\beta + \delta_\alpha^\beta), \quad (6.12c)$$

where, as stated earlier,  ${}^3R_\alpha^\beta = h^{\beta\gamma} {}^3R_{\alpha\gamma}$  represents the three-dimensional Ricci tensor obtained by the metric  $h_{\alpha\beta}$  and  $u_i$  denotes the matter four-velocity vector field.

Computing the left-hand side of (6.6a), (6.6b) up to zeroth-  $\mathcal{O}(1/t^2)$  and first-order  $\mathcal{O}(1/t)$ , we can rewrite them as

$$-\frac{3}{4t^2} + \frac{b}{2t} = \kappa \frac{\rho}{3} (-4u_0^2 + 1), \quad (6.13a)$$

$$\frac{1}{2} (\nabla_\alpha b - \nabla_\beta b_\alpha^\beta) = -\frac{4}{3} \kappa \rho u_\alpha u^0, \quad (6.13b)$$

respectively. Let us consider the identity  $1 \equiv u_i u^i \sim u_0^2 - t^{-1} a^{\alpha\beta} u_\alpha u_\beta$ . If we assume that its last term is negligible, so that  $u_0 \sim 1$ , a consistent

solution can be found for the system (6.13). In fact we get  $\rho \sim t^{-2}$  and  $u_\alpha \sim t^2$ . From Eq. (6.13a), one can find the first two terms of the energy density expansion, and, from Eq. (6.13b), the leading term of the velocity and they read as

$$\kappa\rho = \frac{3}{4t^2} - \frac{b}{2t} \quad (6.14a)$$

$$u_\alpha = \frac{t^2}{2} (\nabla_\alpha b - \nabla_\beta b^\beta{}_\alpha) . \quad (6.14b)$$

As a consequence, the density contrast  $\delta$  can be expressed as the ratio between the first and zeroth-order energy density terms, i.e.

$$\delta = -\frac{2}{3} b t . \quad (6.15)$$

This behavior implies that, as expected in the standard cosmological model, the zeroth-order term of the energy density diverges more rapidly than the perturbations and the singularity is naturally approached with a vanishing density contrast.

Besides the solutions for  $\rho$  and  $u_\alpha$ , one has to consider the pure spatial components of the gravitational equation (6.6c). To leading order, the Ricci tensor can be written as  ${}^3R^\beta_\alpha = A^\beta_\alpha/t$ , where  $A^\beta_\alpha$  is constructed in terms of the constant three-tensor  $a_{\alpha\beta}$ . The terms of order  $t^{-2}$  in Eq. (6.6c) identically cancel out, while those proportional to  $t^{-1}$  give

$$A^\beta_\alpha + \frac{3}{4}b^\beta_\alpha + \frac{5}{12}b\delta^\beta_\alpha = 0 . \quad (6.16)$$

Taking the trace of equation (6.16), the relation between the six arbitrary functions  $a_{\alpha\beta}$  and the coefficients  $b_{\alpha\beta}$  from the next-to-leading term of the expansion can be determined as

$$b^\beta_\alpha = -\frac{4}{3}A^\beta_\alpha + \frac{5}{18}A\delta^\beta_\alpha . \quad (6.17)$$

It is worth reminding that, in the asymptotic limit  $t \rightarrow 0$ , the matter distribution as in Eq. (6.14a) becomes dominantly homogeneous because  $\rho$  approaches a value independent of  $b$ .

From the tridimensional Bianchi identity  $\nabla_\beta A^\beta_\alpha = \frac{1}{2}\nabla_\alpha A$ , the relation  $\nabla_\beta b^\beta_\alpha = \frac{7}{9}\partial_\alpha b$  can be determined; this gives the final expression for the three-velocity distribution as

$$u_\alpha = \frac{t^2}{9}\partial_\alpha b . \quad (6.18)$$

This result implies that, in this approximation, the three-velocity is a gradient field of a scalar function. As a consequence, the curl of the velocity vanishes and no rotations take place in the fluid.

Finally, it must be observed that the metric (6.3) allows an arbitrary spatial coordinate transformation while the above solution contains only  $6 - 3 = 3$  arbitrary space functions arising from  $a_{\alpha\beta}$ . The particular choice of these functions, corresponding to the space of constant curvature ( $A_\alpha^\beta = \text{const.} \times \delta_\alpha^\beta$ ), can reproduce the pure isotropic and homogeneous model.

### 6.3 The Role of a Massless Scalar Field

In this Section we discuss the quasi-isotropic Universe dynamics in the presence of ultrarelativistic matter and a real self-interacting scalar field asymptotically close to the cosmological singularity, while in Sec. 6.5 we will see the opposite limit in the framework of an inflationary scenario, i.e. far from the singularity when a cosmological constant term arises.

In particular, the presence of the scalar field kinetic term allows the existence of a quasi-isotropic solution characterized by an arbitrary spatial dependence of the energy density associated to the ultrarelativistic matter. To leading order, there is no direct relation between the isotropy of the Universe and the homogeneity of the ultrarelativistic matter distributed in it. Indeed the matter energy density enters the equations to first order only.

In the presence of a perfect ultrarelativistic fluid and of a self-interacting scalar field  $\phi(t, x)$  described by a potential term  $V(\phi)$ , the Einstein equations reduce to the following partial differential system

$$\frac{1}{2}\partial_t k_\alpha^\alpha + \frac{1}{4}k_\alpha^\beta k_\beta^\alpha = \kappa \left[ -\frac{\rho}{3}(4u_0^2 - 1) - \frac{1}{2}(\partial_t \phi)^2 + V(\phi) \right] \quad (6.19a)$$

$$\frac{1}{2}(\nabla_\beta k_\alpha^\beta - \nabla_\alpha k_\beta^\beta) = \kappa \left( \frac{4}{3}\rho u_\alpha u_0 + \partial_\alpha \phi \partial_t \phi \right) \quad (6.19b)$$

$$\begin{aligned} \frac{1}{2\sqrt{h}}\partial_t (\sqrt{h}k_\alpha^\beta) + {}^3R_\alpha^\beta \\ = \kappa \left[ h^{\beta\sigma} \left( \frac{4}{3}\rho u_\alpha u_\sigma + \partial_\alpha \phi \partial_\sigma \phi \right) + \left( \frac{\rho}{3} + V(\phi) \right) \delta_\alpha^\beta \right]. \end{aligned} \quad (6.19c)$$

The partial differential equation describing the scalar field  $\phi(t, x)$  dynamics is coupled to the Einstein equations and reads as

$$\partial_{tt}\phi + \frac{1}{2}k_\alpha^\alpha \partial_t \phi - h^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi + \frac{dV}{d\phi} = 0 \quad (6.20)$$

and finally the hydrodynamic equations  $\nabla_j T_i^j = 0$ , introduced in Sec. 2.2.1,

accounting for the matter evolution read explicitly as

$$\frac{1}{\sqrt{h}}\partial_t\left(\sqrt{h}\rho^{3/4}u_0\right) + \frac{1}{\sqrt{h}}\partial_\alpha\left(\sqrt{h}\rho^{3/4}u^\alpha\right) = 0 \quad (6.21a)$$

$$4\rho\left(\frac{1}{2}\partial_t u_0^2 + u^\alpha\partial_\alpha u_0 + \frac{1}{2}k_{\alpha\beta}u^\alpha u^\beta\right) = (1 - u_0^2)\partial_t\rho - u_0 u^\alpha\partial_\alpha\rho \quad (6.21b)$$

$$4\rho\left(u_0\partial_t u_\alpha + u^\beta\partial_\beta u_\alpha + \frac{1}{2}u^\beta u^\gamma\partial_\alpha h_{\beta\gamma}\right) = -u_\alpha u_0\partial_t\rho + \left(\delta_\alpha^\beta - u_\alpha u^\beta\right)\partial_\beta\rho. \quad (6.21c)$$

The presence of the scalar field allows to relax the assumption of expandability in integer powers adopted in (6.1). Indeed, similarly to what happens for the FRW Universe in the presence of the scalar field, we will consider an expansion including also non-integer powers. In order to introduce a quasi-isotropic scenario (eventually inflationary, see below Sec. 6.5) considering small inhomogeneous corrections to leading order, we require a three-dimensional metric tensor having the following structure

$$\begin{aligned} h_{\alpha\beta}(t, x) &= a^2(t)\xi_{\alpha\beta}(x^\gamma) + b^2(t)\theta_{\alpha\beta}(x^\gamma) + \mathcal{O}(b^2) \\ &= a^2(t)\left[\xi_{\alpha\beta}(x^\gamma) + \eta(t)\theta_{\alpha\beta}(x^\gamma) + \mathcal{O}(\eta^2)\right] \end{aligned} \quad (6.22)$$

where we defined  $\eta \equiv \frac{b^2}{a^2}$  and suppose that  $\eta$  satisfies the condition

$$\lim_{t \rightarrow 0} \eta(t) = 0. \quad (6.23)$$

In the limit of the approximation (6.23), the inverse three-metric reads as

$$h^{\alpha\beta}(t, x) = \frac{1}{a^2(t)}\left(\xi^{\alpha\beta}(x^\gamma) - \eta(t)\theta^{\alpha\beta}(x^\gamma) + \mathcal{O}(\eta^2)\right), \quad (6.24)$$

where  $\xi^{\alpha\beta}$  denotes the inverse matrix of  $\xi_{\alpha\beta}$  and assumes a metric role, i.e. we have

$$\xi^{\beta\gamma}\xi_{\alpha\gamma} = \delta_\alpha^\beta, \quad \theta^{\alpha\beta} = \xi^{\alpha\gamma}\xi^{\beta\delta}\theta_{\gamma\delta}. \quad (6.25)$$

The covariant and contravariant three-metric expressions lead to the relations

$$k_\alpha^\beta = 2\frac{\dot{a}}{a}\delta_\alpha^\beta + \dot{\eta}\theta_\alpha^\beta \quad \Rightarrow \quad k_\alpha^\alpha = 6\frac{\dot{a}}{a} + \dot{\eta}\theta, \quad \theta \equiv \theta_\alpha^\alpha. \quad (6.26)$$

Since the equality  $\partial_t(\ln h) = k_\alpha^\alpha$  holds, we get

$$\begin{aligned} h = ja^6 e^{\eta\theta} &\quad \Rightarrow \quad \sqrt{h} = \sqrt{j}a^3 e^{\frac{1}{2}\eta\theta} \\ &\quad \sim \sqrt{j}a^3 \left(1 + \frac{1}{2}\eta\theta + \mathcal{O}(\eta^2)\right), \end{aligned} \quad (6.27)$$



once  $j \equiv \det \xi_{\alpha\beta}$  has been defined.

The Landau-Raychaudhuri theorem (see Sec. 2.4) applied to the present case implies the condition

$$\lim_{t \rightarrow 0} a(t) = 0. \quad (6.28)$$

The set of field equations (6.19) is thus solved under the following assumptions:

- (1) the validity of the limit (6.28);
- (2) to retain only terms linear in  $\eta$  and its time derivatives;
- (3) to neglect all terms containing spatial derivatives of the dynamical variables, in order to obtain asymptotic solutions in the limit  $t \rightarrow 0$ ;
- (4) finally checking the self-consistence of the approximation scheme.

Although the possibility to neglect the potential term  $V(\phi)$  is not ensured by the field equations, it is based on the idea that, in an inflationary scenario, the scalar field potential energy becomes dynamically relevant only during the slow-rolling phase, far from the singularity, while the kinetic term asymptotically dominates. Let us start from Eq. (6.20) to obtain the following

$$\partial_t \phi = \frac{d}{a^3} e^{-\frac{1}{2}\eta\theta} \sim \frac{d}{a^3} \left( 1 - \frac{1}{2}\eta\theta + \mathcal{O}(\eta) \right) \quad (6.29)$$

and therefore

$$(\partial_t \phi)^2 = \frac{d^2}{a^6} e^{-\eta\theta} \sim \frac{d^2}{a^6} (1 - \eta\theta + \mathcal{O}(\eta)), \quad (6.30)$$

where  $d$  is a constant.

In the same approximation, from (6.21a), we get

$$\sqrt{h} \rho^{3/4} u_0 = l(x^\gamma) \quad \Rightarrow \quad \rho \sim \frac{l^{4/3}}{j^{2/3} a^4 u_0^{4/3}} \left( 1 - \frac{2}{3}\eta\theta + \mathcal{O}(\eta) \right) \quad (6.31)$$

being  $l(x^\gamma)$  an arbitrary function of the spatial coordinates. Let us analyze the Einstein equations (6.19). Taking into account (6.30) to first order in  $\eta$ , Eq. (6.19a) reads as

$$3 \frac{\ddot{a}}{a} + \frac{\kappa d^2}{a^6} + \left( \frac{1}{2} \ddot{\eta} + \frac{\dot{a}}{a} \dot{\eta} - \frac{\kappa d^2}{a^6} \eta \right) \theta = -\kappa \frac{\rho}{3} (3 + 4u^2) \quad (6.32)$$

having set

$$u^2 \equiv \frac{1}{a^2} \xi^{\alpha\beta} u_\alpha u_\beta \quad \Rightarrow \quad u_0 = \sqrt{1 + u^2}. \quad (6.33)$$

Furthermore, Eq. (6.19c) reduces to

$$\begin{aligned} \frac{2}{3} (a^3)^{\cdot\cdot} \delta_\alpha^\beta + (a^3 \dot{\eta})^\cdot \theta_\alpha^\beta + \frac{1}{3} [(a^3)^\cdot \eta]^\cdot \theta \delta_\alpha^\beta \\ = \frac{2}{3} \kappa \rho \left( \delta_\alpha^\beta + \frac{4}{a^2} \xi^{\beta\gamma} u_\alpha u_\gamma \right) a^3 \left( 1 + \frac{1}{2} \eta \theta \right). \end{aligned} \quad (6.34)$$

In agreement with our assumptions, in Eq. (6.34) we neglected the spatial curvature term which, to leading order, reads as

$${}^3R_\alpha^\beta(t, x) = \frac{1}{a^2(t)} A_\alpha^\beta(x^\gamma) \quad (6.35)$$

where  $A_{\alpha\beta}(x^\gamma) = \xi_{\beta\gamma} A_\alpha^\gamma$  is the Ricci tensor corresponding to  $\xi_{\alpha\beta}(x^\gamma)$ . Taking the trace of Eq. (6.34) we get

$$2(a^3)^{\cdot\cdot} + (a^3 \eta)^{\cdot\cdot} \theta = \frac{2}{3} \kappa \rho (3 + 4u^2) a^3 \left( 1 + \frac{1}{2} \eta \theta \right). \quad (6.36)$$

The compatibility of Eqs. (6.32) and (6.36) is ensured by the solution to the following system

$$(a^3)^{\cdot\cdot} + 3a^2 \ddot{a} + \frac{\kappa d^2}{a^3} = 0 \quad (6.37a)$$

$$3(a^3 \eta)^{\cdot\cdot} + 3a^3 \ddot{\eta} + 2(a^3)^\cdot \dot{\eta} + 9a^2 \ddot{a} \eta - \frac{3\kappa d^2}{a^3} \eta = 0. \quad (6.37b)$$

Equation (6.37a) admits the solution

$$a(t) = \left( \frac{t}{t_0} \right)^{1/3} \quad (6.38)$$

in correspondence to the choice  $d \equiv \sqrt{\frac{2}{3\kappa}} \frac{1}{t_0}$ .

Substituting the expression (6.38) for  $a(t)$  in Eq. (6.37b) we get

$$3t\ddot{\eta} + 4\dot{\eta} - 2\frac{\eta}{t} = 0. \quad (6.39)$$

By setting

$$\eta(t) = \left( \frac{t}{t_0} \right)^x, \quad (6.40)$$

the differential equation (6.37b) reduces to the following algebraic equation

$$3x^2 + x - 2 = 0 \quad \Rightarrow \quad x = -1, \frac{2}{3}. \quad (6.41)$$

Since  $\eta(t)$  must vanish for  $t \rightarrow 0$ , we exclude the negative solution  $x = -1$ , obtaining thus

$$\eta(t) = \left( \frac{t}{t_0} \right)^{2/3}. \quad (6.42)$$

From these solutions for  $a(t)$  and  $\eta(t)$ , the consistence of the model provides  $u_\alpha$  expressed as

$$u_\alpha(t, x) = v_\alpha(x^\gamma) \left( \frac{t}{t_0} \right)^{1/3} + \mathcal{O} \left( \frac{t}{t_0} \right). \quad (6.43)$$

In conclusion, we get the following identification regarding the arbitrary function  $l(x^\gamma)$

$$l = \sqrt{j} \left( \frac{5\theta}{3\kappa(3 + 4v^2)t_0^2} \right)^{3/4} \sqrt{1 + v^2} \quad (6.44)$$

with

$$v^2 \equiv \xi^{\alpha\beta} v_\alpha v_\beta. \quad (6.45)$$

From these results and from Eq. (6.34) one obtains the tensor  $\theta_{\alpha\beta}(x^\gamma)$  as

$$\theta_{\alpha\beta} = \frac{\zeta}{3 + 4v^2} \left[ (1 - 2v^2) \xi_{\alpha\beta} + 10v_\alpha v_\beta \right] \Rightarrow \theta = \zeta, \quad (6.46)$$

where  $\zeta(x^\gamma)$  denotes an arbitrary function of the spatial coordinates. The energy density of the ultrarelativistic matter is found, to leading order, in the form

$$\rho(t, x) = \frac{5\zeta(x^\gamma)}{3\kappa \left[ 3 + 4v^2(x^\gamma) \right] t_0^{2/3} t^{4/3}} + \mathcal{O} \left( \frac{t}{t_0} \right), \quad (6.47)$$

allowing to integrate the scalar field equation (6.20) as

$$\phi(t, x) = \sqrt{\frac{2}{3\kappa}} \left[ \ln \left( \frac{t}{t_0} \right) - \frac{3}{4} \left( \frac{t}{t_0} \right)^{2/3} \zeta(x^\gamma) + \sigma(x^\gamma) \right] + \mathcal{O} \left( \frac{t}{t_0} \right) \quad (6.48)$$

where  $\sigma(x^\gamma)$  is an arbitrary function of the spatial coordinates.

Finally, Eq. (6.19c) yields the expression for the functions  $v_\alpha$  in terms of  $\zeta$  and of the spatial gradient  $\partial_\alpha \sigma$  as

$$v_\alpha = - \frac{3(3 + 4v^2)}{10\zeta\sqrt{1 + v^2}} t_0 \partial_\alpha \sigma, \quad (6.49a)$$

$$v^2 = \frac{24\tau^2 - 1 + \sqrt{1 - 12\tau^2}}{2(1 - 16\tau^2)} \quad (6.49b)$$

where  $\tau$  represents the quantity

$$\tau = \frac{3t_0}{10\zeta} \sqrt{\xi^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma}. \quad (6.49c)$$

The simple case  $\sigma = 0$ , in correspondence to which  $v^2 = 0$  ( $v_\alpha = 0$ ) leads to the solutions

$$\theta_{\alpha\beta} = \frac{1}{3}\zeta(x^\gamma)\xi_{\alpha\beta} \quad (6.50a)$$

$$\rho(t, x) = \frac{5}{9\kappa}\zeta(x^\gamma)\frac{1}{t_0^{2/3}t^{4/3}} + \mathcal{O}\left(\frac{t}{t_0}\right) \quad (6.50b)$$

$$\phi(t, x) = \sqrt{\frac{2}{3\kappa}}\left[\ln\left(\frac{t}{t_0}\right) - \frac{3}{4}\left(\frac{t}{t_0}\right)^{2/3}\zeta(x^\gamma)\right] + \mathcal{O}\left(\frac{t}{t_0}\right) \quad (6.50c)$$

$$u_\alpha(t, x) = \frac{3}{8}\partial_\alpha \ln(\zeta(x^\gamma))t + \mathcal{O}\left(\frac{t}{t_0}\right). \quad (6.50d)$$

Finally we obtain the three-dimensional metric tensor as

$$h_{\alpha\beta}(t, x^\gamma) = \left(\frac{t}{t_0}\right)^{2/3}\left[1 + \left(\frac{t}{t_0}\right)^{2/3}\frac{\zeta(x^\gamma)}{3}\right]\xi_{\alpha\beta}(x^\gamma) + \mathcal{O}\left(\frac{t}{t_0}\right). \quad (6.50e)$$

On the basis of Eqs. (6.50), the hydrodynamic equations (6.21) reduce to identities in the approximation considered here.

It is worth noting that, in correspondence to a fixed time  $t^*$  ( $t^* \ll t_0$ ), if we set

$$\rho^*(x^\gamma) \equiv \rho(t^*, x), \quad u_\alpha^*(x^\gamma) \equiv u_\alpha(t^*, x), \quad (6.51)$$

then, in terms of these time independent quantities, Eqs. (6.47) and (6.43) read in the more expressive form as

$$\rho = \rho^*\left(\frac{t}{t^*}\right)^{4/3} + \mathcal{O}\left(\frac{t}{t_0}\right), \quad u_\alpha = u_\alpha^*\left(\frac{t}{t^*}\right)^{1/3} + \mathcal{O}\left(\frac{t}{t_0}\right). \quad (6.52)$$

The solution shown here is completely self-consistent to the first two orders in time and contains five physically arbitrary functions of the spatial coordinates: three out of the six functions  $\xi_{\alpha\beta}$  (the remaining three of them can be fixed by pure spatial coordinates transformations), the spatial scalar  $\zeta(x^\gamma)$  and finally  $\sigma(x^\gamma)$ .

The independence of the functions  $\xi_{\alpha\beta}$ ,  $\zeta$  and  $\sigma$  implies the existence of a quasi-isotropic dynamics in correspondence to an arbitrary spatial distribution of ultrarelativistic matter. The kinetic term of the scalar field behaves, to leading order, as  $\sim a^{-6}$  and therefore, in the limit  $a \rightarrow 0$ , it dominates the ultrarelativistic energy density. The latter diverges only as  $\sim a^{-4}$  and therefore the spatial curvature term  $\sim a^{-2}$  is negligible.

Thus, these behaviors are at the ground of the possibility to deal with an unconstrained spatial dependence of the ultrarelativistic energy density. The homogeneity is ensured by the zeroth order contribution from the scalar field.

## 6.4 The Role of an Electromagnetic Field

Here we analyze the dynamical behavior, near the cosmological singularity, of a quasi-isotropic Universe in the presence of an electromagnetic field and a real massless scalar field. More precisely, we show how the presence, in spite of its vectorial nature, of an electromagnetic field on a quasi-isotropic background is allowed due to the dominant character of the scalar field kinetic term. In fact, as discussed in Sec. 2.2.3, the electromagnetic field has an anisotropic energy momentum tensor and it would be incompatible with the quasi-isotropic assumption, when treated to lowest order. On the other hand, near the singularity the scalar field dominates the quasi-isotropic metric to leading order.

We outline the complete equivalence existing between the dynamical effect produced, on a quasi-isotropic Universe containing a real massless scalar field, by the presence of an electromagnetic field and by an ultra-relativistic matter component, discussed in detail in Sec. 2.2.3. In both cases, the metric of the Universe and the scalar field acquire, to the first two orders of approximation, the same time dependence, and similarly their corresponding energy densities.

Since close enough to the singularity (i.e. at sufficiently high temperature) the presence of a potential term for the scalar field is dynamically negligible, then our evolutive scheme can be thought of as an inflationary scenario, yet far from the later slow-rolling phase (where the potential term provides the main dynamical contribution).

The general formulation of the cosmological problem describing the dynamics of a three-dimensional Universe with an electromagnetic field  $F_{ik}$  (see Sec. 2.2) and a real massless scalar one  $\phi$  is based on an action of the form

$$S_{g+EM} = -\frac{1}{2\kappa} \int \sqrt{-g} \left( R + \frac{\kappa}{8\pi} F_{ik} F^{ik} - \kappa g^{ik} \partial_i \phi \partial_k \phi \right) d^4x. \quad (6.53)$$

By varying the action (6.53) with respect to these three fields, we get the dynamical equations for  $F_{ik}$  as

$$R_i^k = \kappa \left[ \frac{1}{4\pi} \left( -F_{il} F^{kl} + \frac{1}{4} F_{lm} F^{lm} \delta_i^k \right) + g^{kl} \partial_i \phi \partial_l \phi \right] \quad (6.54a)$$

$$\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} F^{ik})}{\partial x^k} = 0 \quad (6.54b)$$

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0 \quad (6.54c)$$

and for the scalar field  $\phi$  in the covariant form as

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ik} \frac{\partial \phi}{\partial x^k} \right) = 0. \quad (6.54d)$$

In Eq. (6.54c) the ordinary partial derivatives can be equivalently replaced by the covariant ones.

Since the energy momentum tensor of the electromagnetic field is traceless, from Eq. (6.54a) we get

$$R - \kappa g^{ik} \partial_i \phi \partial_k \phi = 0. \quad (6.54e)$$

Due to the real character of the scalar field (i.e. it does not bring any electric charge), the two matter fields interact only through the space-time curvature.

In a synchronous reference frame, the dynamical equations (6.54) reduce to the partial differential system

$$\frac{1}{2} \partial_t k_\alpha^\alpha + \frac{1}{4} k_\alpha^\beta k_\beta^\alpha = -\frac{\kappa}{8\pi} \left( E_\alpha E^\alpha + \frac{1}{2} \tilde{B}_{\alpha\beta} \tilde{B}^{\alpha\beta} \right) - \kappa (\partial_t \phi)^2 \quad (6.55a)$$

$$\frac{1}{2} (\nabla_\beta k_\alpha^\beta - \nabla_\alpha k_\beta^\beta) = \frac{\kappa}{4\pi} \tilde{B}_{\alpha\beta} E^\beta + \kappa \partial_\alpha \phi \partial_t \phi \quad (6.55b)$$

$$\begin{aligned} \frac{1}{2\sqrt{h}} \partial_t (\sqrt{h} k_\alpha^\beta) + P_\alpha^\beta = \\ -\frac{\kappa}{4\pi} \left( E_\alpha E^\beta - \frac{1}{2} E_\gamma E^\gamma \delta_\alpha^\beta - \tilde{B}_{\alpha\gamma} \tilde{B}^{\beta\gamma} + \frac{1}{4} \tilde{B}_{\gamma\delta} \tilde{B}^{\gamma\delta} \delta_\alpha^\beta \right) \\ + \kappa h^{\beta\gamma} \partial_\alpha \phi \partial_\gamma \phi, \end{aligned} \quad (6.55c)$$

which is coupled to the Maxwell equations

$$\frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} E^\alpha) = 0 \quad (6.56a)$$

$$\frac{1}{\sqrt{h}} \left[ \partial_t (\sqrt{h} E^\alpha) + \partial_\beta (\sqrt{h} \tilde{B}^{\alpha\beta}) \right] = 0 \quad (6.56b)$$

$$\partial_t \tilde{B}_{\alpha\beta} + \partial_\beta E_\alpha - \partial_\alpha E_\beta = 0 \quad (6.57a)$$

$$\partial_\gamma \tilde{B}_{\alpha\beta} + \partial_\beta \tilde{B}_{\gamma\alpha} + \partial_\alpha \tilde{B}_{\beta\gamma} = 0, \quad (6.57b)$$

and finally to the equations describing the scalar field

$$\frac{1}{\sqrt{h}} \left[ \partial_t (\sqrt{h} \partial_t \phi) - \partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta \phi) \right] = 0 \quad (6.58a)$$

$$\frac{1}{2} \partial_t k_\alpha^\alpha + \frac{1}{4} k_\alpha^\beta k_\beta^\alpha + \frac{\partial_{tt}(\sqrt{h})}{\sqrt{h}} + P + \kappa [(\partial_t \phi)^2 - h^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi] = 0. \quad (6.58b)$$

Above, according to Sec. 2.2.3, we adopted the definitions

$$E_\alpha \equiv F_{0\alpha} \quad E^\alpha = h^{\alpha\beta} E_\beta = -F^{0\alpha} \quad (6.59)$$

$$\tilde{B}_{\alpha\beta} \equiv F_{\alpha\beta} \quad \tilde{B}_\alpha^\beta = h^{\beta\gamma} \tilde{B}_{\alpha\gamma} = -F_\alpha{}^\beta. \quad (6.60)$$

We are interested to a quasi-isotropic solution of the system (6.55)–(6.58), i.e. according to a three-dimensional metric tensor of the form as in Eq. (6.22).

In what follows we shall analyze the field equations (6.55)–(6.58), retaining only terms linear in  $\eta$  and its time derivatives. The analysis of Eqs. (6.55) is based on the construction of an asymptotic power-law solution in the limit  $t \rightarrow 0$ , and by verifying *a posteriori* the self-consistence of the approximation scheme. In particular, since the action of the time derivative operator on a power-law expression provides terms of lower order in time, we can neglect the contribution of the spatial derivatives.

In the limit of this approximation, from Eqs. (6.56b) we get

$$E^\alpha = \frac{\hat{d}\mathcal{E}^\alpha(x^\gamma)}{a^3} + \mathcal{O}\left(\frac{\eta}{a^3}\right) \quad (6.61a)$$

where  $\hat{d}$  a generic constant and  $\mathcal{E}^\alpha(x^\gamma)$  an arbitrary vector field of the spatial coordinates. By substituting this expression for  $E^\alpha$  in Eq. (6.56a), we get the constraint for  $\mathcal{E}^\alpha$

$$\nabla_\alpha \mathcal{E}^\alpha = \frac{1}{\sqrt{h}} \partial_\alpha \left( \sqrt{h} \mathcal{E}^\alpha \right) = 0. \quad (6.61b)$$

For what regards the scalar field dynamics, the approximation (6.30) holds. When it is included in the Einstein Eq. (6.58b), we get the system obtained by the two (zeroth- and first-order) components as

$$3a^2\ddot{a} + \ddot{a}^3 + \frac{\kappa p^2}{a^3} = 0 \quad (6.62a)$$

$$a^3\ddot{\eta} + 2a^2\dot{a}\dot{\eta} + (a^3\eta)^{\cdot\cdot} - \left( \ddot{a}^3 + 2\frac{\kappa p^2}{a^3} \right) \eta = 0, \quad (6.62b)$$

where  $p$  is a constant of integration related to the scalar field. Repeating the same steps performed in the previous Section and choosing the constant  $p = \sqrt{\frac{2}{3\kappa}} \frac{1}{t_0}$ , we get the same time dependence for  $a(t)$  and  $\eta(t)$  as in (6.38) and (6.42), respectively. When such expressions under approximation (6.61) are inserted in Eq. (6.57a) we get

$$\tilde{B}_{\alpha\beta} = \hat{d}\sqrt{2}\mathcal{B}_{\alpha\beta}(x^\gamma) + \mathcal{O}\left(\left(\frac{t}{t_0}\right)^{2/3}\right), \quad (6.63a)$$

where

$$\mathcal{B}_{\alpha\beta} \equiv \partial_\alpha \mathcal{B}_\beta - \partial_\beta \mathcal{B}_\alpha, \quad (6.63b)$$

being  $\mathcal{B}_\alpha(x^\gamma)$  an arbitrary spatial vector. So far, it is straightforward to realize how Eq. (6.57b) is identically verified by Eq. (6.38) and Eq. (6.42).

Using the expressions (6.61), (6.30), (6.38), (6.42), and (6.63), Eqs. (6.55a) and (6.55c) turn out to be automatically satisfied to zeroth-order approximation, while, to first order, they require the identifications

$$d = \pm \sqrt{\frac{40\pi}{9\kappa t_0^2}} \quad (6.64a)$$

$$\theta = \mathcal{E}_\alpha \mathcal{E}^\alpha + \mathcal{B}_{\alpha\beta} \mathcal{B}^{\alpha\beta} \quad (6.64b)$$

$$\theta_{\alpha\beta} = -5\mathcal{E}_\alpha \mathcal{E}_\beta + 10\mathcal{B}_{\alpha\gamma} \mathcal{B}_\beta{}^\gamma + (2\mathcal{E}_\gamma \mathcal{E}^\gamma - 3\mathcal{B}_{\gamma\delta} \mathcal{B}^{\gamma\delta}) \xi_{\alpha\beta} \quad (6.64c)$$

where we set

$$\mathcal{E}_\alpha = \xi_{\alpha\beta} \mathcal{E}^\beta, \quad \mathcal{B}_\alpha{}^\beta = \xi^{\beta\gamma} \mathcal{B}_{\alpha\gamma}. \quad (6.65)$$

In agreement with our approximation, the spatial curvature term, having the form as in Eq. (6.35), is still negligible. By integrating the equation corresponding to Eq. (6.30) we get

$$\phi(t, x) = \sqrt{\frac{2}{3\kappa}} \left[ \ln \left( \frac{t}{t_0} \right) - \frac{3}{4} \left( \frac{t}{t_0} \right)^{2/3} \theta(x^\gamma) + \sigma(x^\gamma) \right] + \mathcal{O} \left( \left( \frac{t}{t_0} \right)^{2/3} \right) \quad (6.66)$$

being  $\sigma(x^\gamma)$  an arbitrary function of the spatial coordinates.

Finally, in terms of all the expressions above obtained, we observe that the leading order ( $\mathcal{O}(1/t)$ ) of Eq. (6.55b) reduces to the differential constraint

$$\partial_\alpha \sigma = -\frac{5}{3} \sqrt{2} \mathcal{B}_{\alpha\beta} \mathcal{E}^\beta. \quad (6.67)$$

It is worth noting how an exact evaluation of the terms to next order in Eq. (6.55c) ( $\mathcal{O}(1/t^{1/3})$ ) involves higher order terms in the expansions of  $E^\alpha$  and  $B_{\alpha\beta}$ , which contain contributions from nonlinear terms in  $\eta(t)$  and its time derivatives.

From the whole scheme, we see that the spatial tensor  $\xi_{\alpha\beta}$  can be arbitrarily assigned, while the quantities  $\mathcal{E}^\alpha$ ,  $\mathcal{B}_\alpha$  and  $\sigma$  are subjected to the constraint (6.67) only. By assigning the functions  $\mathcal{B}_\alpha$  (i.e.  $\mathcal{B}_{12}$ ,  $\mathcal{B}_{23}$  and  $\mathcal{B}_{31}$ ) the set of equations in (6.67) reduces to an algebraic inhomogeneous system in the three unknowns  $\mathcal{E}^\alpha$  which, being  $\det \mathcal{B}_{\alpha\beta} = 0$ , in order to be



solved requires the validity of the following partial differential equation for the function  $\sigma$

$$\mathcal{B}_{12}\partial_3\sigma + \mathcal{B}_{31}\partial_2\sigma + \mathcal{B}_{23}\partial_1\sigma = 0. \quad (6.68)$$

Due to its linear homogeneous structure, this equation always admits a solution in correspondence to any choice of  $\mathcal{B}_\alpha$ . Once solved, the algebraic system (6.67) allows us to express two of the components  $\mathcal{E}^\alpha$  in terms of the third one, and the quantities  $\mathcal{B}_\alpha$  and the spatial gradients in terms of the function  $\sigma$  (solution of Eq. (6.68)).

Taking into account the three allowed general transformations of the spatial coordinates, which remove three degrees of freedom among the ten free functions (six from  $\xi_{\alpha\beta}$ , three  $\mathcal{B}_\alpha$  and one from  $\mathcal{E}^\alpha$ ), the number of physically arbitrary functions of the spatial coordinates available for the Cauchy problem reduces to seven.

By comparing the analysis here developed with the one in Sec. 6.3, where the ultrarelativistic matter replaces the role of the electromagnetic field, the complete dynamical equivalence between these two cases arises. Indeed, a quasi-isotropic Universe in which a real scalar field lives (whose dynamics asymptotically has a dominant character) receives the same dynamical contribution from the ultrarelativistic matter (described by a perfect fluid with an ultrarelativistic equation of state), as well as from an electromagnetic field. The solutions for  $a(t)$  and  $\eta(t)$  take, in both cases, the same power-law expressions together with the correspondence among the two sets of spatial functions as

$$\zeta \Leftrightarrow \mathcal{E}_\alpha \mathcal{E}^\alpha + \mathcal{B}_{\alpha\beta} \mathcal{B}^{\alpha\beta} \quad (6.69)$$

$$v_\alpha = -\frac{3(3+4v^2)}{10\zeta\sqrt{1+v^2}}t_0\partial_\alpha\sigma \Leftrightarrow \frac{(3+4v^2)t_0}{\sqrt{2}\zeta\sqrt{1+v^2}}\mathcal{B}_{\alpha\beta}\mathcal{E}^\beta, \quad (6.70)$$

being  $v^2$  as defined in Eq. (6.49b). Similarly, for  $\tau$  defined as in Eq. (6.49c), we have the correspondence

$$\tau \Leftrightarrow \frac{t_0}{\sqrt{2}(\mathcal{E}_\alpha \mathcal{E}^\alpha + \mathcal{B}_{\alpha\beta} \mathcal{B}^{\alpha\beta})} \sqrt{\xi^{\alpha\beta} \mathcal{B}_{\alpha\gamma} \mathcal{E}^\gamma \mathcal{B}_{\beta\delta} \mathcal{E}^\delta}. \quad (6.71)$$

Here  $v_\alpha$  denote the spatial distribution of the fluid four-velocity spatial components, as in Eqs. (6.43) and (6.33). A complete correspondence exists in the two cases with respect to the form taken by the scalar field and by the energy densities too. The ultrarelativistic matter, in the absence of a scalar field, can yet survive on a quasi-isotropic background unlike the electromagnetic one.

The microwave background radiation, whose dynamical contribution is well described by the ultrarelativistic equation of state (pure radiation component), indeed corresponds to a completely incoherent (disordered) electromagnetic field.

## 6.5 Quasi-Isotropic Inflation

In this Section we find a solution for a quasi-isotropic inflationary Universe which allows to introduce a certain degree of inhomogeneity. We consider a model which generalizes the flat FRW model by introducing a first-order inhomogeneous term, whose dynamics is induced by an effective cosmological constant. As above, the three-metric tensor consists of a dominant term, corresponding to an isotropic-like component, while the amplitude of the first-order term is controlled by the higher order function  $\eta(t)$ .

In a Universe filled with ultrarelativistic matter and a real self-interacting scalar field, we analyze the dynamics up to first order in  $\eta$ , when the scalar field performs a slow roll on a plateau of a symmetry breaking configuration and induces an effective cosmological constant.

We show how the spatial distributions of the ultrarelativistic matter and of the scalar field admit an arbitrary form but nevertheless, due to the required inflationary e-folding, it cannot play a significant dynamical role in the process of structure formation (via the Harrison–Zeldovich spectrum). As a consequence, we reinforced the idea that the inflationary scenario is incompatible with a classical origin of the cosmological structures.

As seen in Chap. 5, the inflationary model is, up to now, the most natural and complete scenario able to solve the problems appearing in the Standard Cosmological Model, like the horizon and flatness paradoxes; indeed, such a dynamical scheme, on the one hand is able to justify the high isotropy of the cosmic microwave background radiation (and in general the large-scale homogeneity of the Universe) and, on the other hand, provides a mechanism for the generation of a nearly scale-invariant spectrum of inhomogeneous perturbations (via the quantum fluctuations of the scalar field). Moreover, as it will be shown in detail in Chap. 8, a slow-rolling phase of the scalar field allows to connect the Mixmaster dynamics with a later quasi-isotropic Universe evolution, in principle compatible with the standard cosmological picture.

In Sec. 6.3, the quasi-isotropic solution has been discussed in the pres-

ence of a kinetic energy-dominated real scalar field, which leads to a power-law solution for the three-metric, and predicts interesting features for the dynamics of ultrarelativistic matter.

In this section the opposite dynamical scheme, i.e. when the scalar field undergoes a slow-rolling phase due to the dominance of the effective cosmological constant over its kinetic energy, is analyzed. A detailed description is provided, in a synchronous reference frame, of the three-metric, of the scalar field and of the ultrarelativistic matter dynamics up to the first two orders of approximation, showing that the volume of the Universe exponentially expands and induces a corresponding exponential decay (as the inverse fourth-power of the cosmic scale factor), both of the three-metric corrections, and of the ultrarelativistic matter. The spatial dependence of this component is described by a function which remains an arbitrary degree of freedom, nevertheless there is no chance that, after the de Sitter phase, the relic perturbations survive enough to trace the large scale structures formation. This behavior suggests that the spectrum of inhomogeneous perturbations cannot directly arise by the classical field nature, but only by its quantum dynamics (see Sec. 5.6.4).

The presence of the scalar field kinetic term, considered negligible here, induces, near enough to the singularity, a deep modification to the general cosmological solution, leading to the appearance of a dynamical regime during which, point by point in space, the three spatial directions behave monotonically, as discussed in Sec. 8.7.1 for the homogeneous Mixmaster model.

### 6.5.1 *Geometry, matter and scalar field equations*

As done before, let us describe the matter field as a perfect fluid with ultrarelativistic equation of state  $P = \frac{\rho}{3}$  together with a scalar field  $\phi(t, x)$  with a potential term  $V(\phi)$ . In what follows, we write the Einstein equations as

$$R_i^k = \kappa \sum_{z=m, \phi} \left[ (T^z)_i^k - \frac{1}{2} \delta_i^k (T^z)_l^l \right] \quad (6.72)$$

where the label  $m$  and  $\phi$  are adopted to distinguish between the matter and scalar field energy momentum tensors.

The set of interactions (6.72), similarly to what described in Sec. 6.3, explicitly reduces to the system (6.19), coupled with the dynamics of the scalar field  $\phi(t, x)$  described by the partial differential Eq. (6.20). The

hydrodynamic equations, taking into account the matter evolution, in a synchronous reference and for the ultrarelativistic case, have the structure of Eqs. (6.21). This scheme is consistent with the invariance under the transformation as in Eq. (6.9).

### 6.5.2 Inflationary dynamics

In order to introduce small inhomogeneous corrections to the leading order in a quasi-isotropic inflationary scenario, we consider a three-dimensional metric tensor having the structure outlined in Eqs. (6.22)–(6.27).

Then the field equations (6.19) are analyzed retaining only terms linear in  $\eta$  and its time derivatives, and verifying *a posteriori* the self-consistency of the approximation scheme.

In the quasi-isotropic approach, we assume that the scalar field dynamics in the plateau region (see Sec. 5.4.1) is governed by a potential term of the form

$$V(\phi) = \rho_\Lambda + K(\phi), \quad \rho_\Lambda = \text{const.}, \quad (6.73)$$

where  $\rho_\Lambda$  is the dominant term and  $K(\phi)$  is a small correction. The role of  $K$ , as shown in the following, is to drive inhomogeneous corrections via the  $\phi$ -dependence; the functional form of  $K$  can be any of the most common inflationary potentials, as they appear near slow-roll region.

What follows remains valid, for example, in the relevant cases of the quartic and Coleman-Weinberg potentials introduced in Sec. 5.4.1, where the corrections to the constant  $\rho_\Lambda$  term are

$$K(\phi) = \begin{cases} -\frac{\omega}{4}\phi^4, & \omega = \text{const.} \\ B\phi^4 \left[ \ln\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2} \right], & B, \sigma = \text{const.} \end{cases} \quad (6.74)$$

In the following, explicit calculations are developed only for the first case. The inflationary solution is obtained under the usual requirements

$$\frac{1}{2} (\partial_t \phi)^2 \ll V(\phi) \quad (6.75a)$$

$$|\partial_{tt}\phi| \ll |k_\alpha^\alpha \partial_t \phi|. \quad (6.75b)$$

The above approximations and the substitution of Eq. (6.26) reduce the scalar field Eq. (6.20) to the form

$$\left( 3\frac{\dot{a}}{a} + \frac{1}{2}\dot{\eta}\theta \right) \partial_t \phi - \omega \phi^3 = 0, \quad (6.76)$$

where the contribution of the spatial gradient of  $\phi$  has been assumed to be negligible.

Similarly, the quasi-isotropic approach (in which the inhomogeneities become relevant only in the next-to-leading order), once the spatial derivatives in Eq. (6.21) have been neglected, leads to

$$\sqrt{h}\rho^{3/4}u_0 = l(x^\gamma) \Rightarrow \rho \sim \frac{l^{4/3}}{j^{2/3}a^4u_0^{4/3}} \left(1 - \frac{2}{3}\eta\theta + \mathcal{O}(\eta^2)\right), \quad (6.77)$$

where  $l(x^\gamma)$  denotes an arbitrary function of the spatial coordinates.

Let us now face, in the same approximation scheme, the analysis of the Einstein Eqs. (6.19). Taking into account Eq. (6.75a), to first order in  $\eta$ , Eq. (6.19a) reads as

$$3\frac{\ddot{a}}{a} + \left(\frac{1}{2}\ddot{\eta} + \frac{\dot{a}}{a}\dot{\eta}\right)\theta - \kappa\rho_\Lambda = -\kappa\frac{\rho}{3}(3 + 4u^2), \quad (6.78)$$

where  $u^2$  and  $u_0$  are given by Eq. (6.33). Equation (6.19c) reduces to the form

$$\begin{aligned} & \frac{2}{3}(a^3)'' \delta_\alpha^\beta + (a^3 \dot{\eta})' \theta_\alpha^\beta + \frac{1}{3}[(a^3)' \eta]' \theta \delta_\alpha^\beta + a A_\alpha^\beta \\ &= \kappa \left[ \frac{1}{a^2} (\xi^{\beta\gamma} - \eta \theta^{\beta\gamma}) \frac{4}{3} \rho u_\alpha u_\gamma + \left(\frac{\rho}{3} + \rho_\Lambda\right) \delta_\alpha^\beta \right] 2a^3 \left(1 + \frac{\eta\theta}{2}\right), \end{aligned} \quad (6.79)$$

where the spatial curvature term is expressed, to leading order, as in Eq. (6.35). The trace of Eq. (6.79) yields the additional relation

$$2(a^3)'' + (a^3\eta)'' \theta + a A_\alpha^\alpha = \kappa \left[ \frac{\rho}{3}(3 + 4u^2) + 3\rho_\Lambda \right] 2a^3 \left(1 + \frac{\eta\theta}{2}\right). \quad (6.80)$$

Comparing Eq. (6.78) with the trace Eq. (6.80), via their common term  $(3 + 4u^2)\rho/3$ , and equating the different orders of approximation, we get the following equations

$$(a^3)'' + 3a^2\ddot{a} - 4\kappa\rho_\Lambda a^3 = 0 \quad (6.81a)$$

$$A_{\alpha\beta} = 0 \quad (6.81b)$$

$$3(a^3\eta)'' + 3a^3\ddot{\eta} + 2(a^3)' \dot{\eta} + 9a^2\eta\ddot{a} - 12\kappa\rho_\Lambda a^3\eta = 0. \quad (6.81c)$$

Since Eq. (6.81b) implies the vanishing of the three-dimensional Ricci tensor and this condition corresponds to the vanishing of the Riemann tensor, we can conclude that the Universe described by this solution is flat up to leading order, i.e.

$$\xi_{\alpha\beta} = \delta_{\alpha\beta} \quad \Rightarrow \quad j = 1. \quad (6.82)$$

Equation (6.81a) admits the accelerating solution

$$a(t) = a_0 \exp \left( \frac{\sqrt{3\kappa\rho_\Lambda}}{3} t \right), \quad (6.83)$$

where  $a_0$  is an integration constant.

Expression (6.83) for  $a(t)$ , when substituted in Eq. (6.81c) yields the following differential equation for  $\eta$

$$\ddot{\eta} + \frac{4}{3} \sqrt{3\kappa\rho_\Lambda} \dot{\eta} = 0, \quad (6.84)$$

whose only solution, satisfying the condition expressed by Eq. (6.23), reads as

$$\eta(t) = \eta_0 \exp \left( -\frac{4}{3} \sqrt{3\kappa\rho_\Lambda} t \right) \quad \Rightarrow \quad \eta = \eta_0 \left( \frac{a_0}{a} \right)^4, \quad (6.85)$$

and we require  $\eta_0 \ll a_0$ .

Equations (6.77) and (6.78), in view of the solutions (6.83) for  $a(t)$  and (6.85) for  $\eta(t)$ , are matched by posing

$$\begin{aligned} u_\alpha(t, x) &= v_\alpha(x^\gamma) + \mathcal{O}(\eta^2) \\ (u_0)^2 &= 1 + \mathcal{O}\left(\frac{1}{a^2}\right) \approx 1, \end{aligned} \quad (6.86)$$

and

$$\rho = -\frac{4}{3} \rho_\Lambda \eta \theta \quad (6.87)$$

respectively. Equation (6.87) implies  $\theta < 0$  for all values of the spatial coordinates. The comparison of Eq. (6.77) with Eq. (6.87) leads to an explicit expression for  $l(x^\gamma)$  in terms of  $\theta$  as

$$l(x^\gamma) = \left( \frac{4}{3} \rho_\Lambda \eta_0 a_0^4 \right)^{3/4} (-\theta)^{3/4}. \quad (6.88)$$

Defining the auxiliary tensor with unit trace  $\Theta_{\alpha\beta}(x^\gamma) \equiv \theta_{\alpha\beta}/\theta$ , the above analysis allows, from Eq. (6.79), to obtain the expression

$$\Theta_\alpha^\beta = \frac{\delta_\alpha^\beta}{3}. \quad (6.89)$$

From Eq. (6.76), the explicit form for  $a$ , once expanded in powers of  $\eta$ , yields the first two orders of approximation for the scalar field as

$$\begin{aligned} \phi(t, x) &= \mathcal{C} \sqrt{\frac{t_r}{t_r - t}} \left( 1 - \frac{1}{4\sqrt{3\kappa\rho_\Lambda}} \frac{\eta}{t_r - t} \theta \right), \\ t_r &= \frac{\sqrt{3\kappa\rho_\Lambda}}{2\mathcal{C}^2\omega}, \end{aligned} \quad (6.90)$$

where  $\mathcal{C}$  is an integration constant. Finally, Eq. (6.19c) provides  $v_\alpha$  in terms of  $\theta$  as

$$v_\alpha = -\frac{3}{4} \frac{1}{\sqrt{3\kappa\rho_\Lambda}} \partial_\alpha \ln |\theta|. \quad (6.91)$$

On the basis of Eqs. (6.89)-(6.91), the hydrodynamic Eqs. (6.21) reduce to an identity, to the leading order of approximation; in fact, such equations contain the energy density of the ultrarelativistic matter, which is known only to first order (the higher one of the Einstein equations) so that higher order contributions cannot be taken into account.

As long as  $(t_r - t)$  is sufficiently large, it can be checked that the solution here constructed is completely self-consistent at least up to the order of approximation considered here and contains one physically arbitrary function of the spatial coordinates  $\theta(x^\gamma)$ . This function, being a three-scalar, is not affected by spatial coordinate transformations. In particular, the quadratic terms in the spatial gradients of the scalar field are of order

$$(\partial_\alpha \phi)^2 \approx \mathcal{O} \left( \frac{\eta^2}{a^2} \frac{1}{(t_r - t)^3} \right) \quad (6.92)$$

and therefore can be neglected with respect to the other inhomogeneous terms. This solution instead fails when  $t$  approaches  $t_r$ , therefore its validity requires that the de Sitter phase ends (with the fall of the scalar field in the true potential vacuum) when  $t$  is still much smaller than  $t_r$ .

### 6.5.3 Physical considerations

The peculiar feature of the solution constructed above lies in the free character of the function  $\theta$  which, from a cosmological point of view, implies the existence of a quasi-isotropic inflationary solution together with an arbitrary spatial distribution of ultrarelativistic matter and of the scalar field.

The Universe emerging from such inflationary picture has the appropriate standard features, but with the presence of a suitable spectrum of *classical* perturbations, due to the small inhomogeneities. The power spectrum of fluctuations can be modeled in the form of a Harrison-Zeldovich spectrum. In fact, expanding the function  $\theta$  in Fourier series as

$$\theta(x^\gamma) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \tilde{\theta}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d^3k, \quad (6.93)$$

we can impose a Harrison-Zeldovich spectrum by requiring

$$|\tilde{\theta}|^2 = \frac{Z}{|k|^3}, \quad Z = \text{const.} \quad (6.94)$$

However, let us complete our picture considering that:

- (i) limiting our attention to leading order, the validity of the slow-rolling regime is ensured by the natural conditions

$$\mathcal{O}(\sqrt{\kappa\rho_\Lambda}(t - t_r)) \ll 1, \quad \omega \gg \mathcal{O}(\kappa^2\rho_\Lambda), \quad (6.95)$$

which translate Eq. (6.75b) and Eq. (6.75a), respectively;

- (ii) denoting by  $t_i$  and  $t_f$  the beginning and the end of the de Sitter phase, respectively, we should have  $t_r \gg t_f$  and the validity of the solution is guaranteed if

- (a) the flatness of the potential is preserved, i.e.  $\omega\phi^4 \ll \rho_\Lambda$ : such a requirement coincides with the second inequality of (6.95);
- (b) denoting with  $\Delta$  the width of the flat region of the potential, we require that the de Sitter phase ends before  $t$  becomes comparable with  $t_r$ , i.e.

$$\phi(t_f) - \phi(t_i) \sim \frac{(\kappa\rho_\Lambda)^{1/4}}{\omega^{1/2}} \frac{t_f - t_i}{t_r^{3/2}} \sim \mathcal{O}(\Delta), \quad (6.96)$$

where the solution has been expanded to first order in  $t_{i,f}/t_r$ ; via the usual position  $(t_f - t_i) \sim \mathcal{O}(10^2)/\sqrt{\kappa\rho_\Lambda}$ , the relation (6.96) becomes a constraint for the integration constant  $t_r$ .

- (iii) The exponential expansion should last long enough in order to solve shortcomings present in the SCM. It has been shown in Sec. 5.5 that the minimum number  $\mathcal{E}$  of e-folds necessary to solve the shortcomings is typically  $\mathcal{E} \simeq 60$  so that  $a_f/a_i \sim e^{60} \sim \mathcal{O}(10^{27})$ . Thus any perturbation that could be present before inflation would be reduced by a factor  $\sim (\eta_f/\eta_i) \sim (a_i/a_f)^4 \sim \mathcal{O}(10^{-108})$ . Though these inhomogeneities increase as  $a^2$  when they are at scale greater than the horizon, they should start with an enormous amplitude in order to play a role in the process of structure formation. This result supports the idea that the spectrum of inhomogeneous perturbations cannot have a classical origin in the presence of an inflationary scenario.

So far, we have estimated  $\rho_f/\rho_i$  i.e. the ratio of the inhomogeneous terms  $\rho_f$  and  $\rho_i$ . In fact, after the reheating the Universe is dominated by a homogeneous (apart from the quantum fluctuations) relativistic energy density  $\rho_r$  to which the relic  $\rho_f$  is superimposed after inflation and therefore we have

$$\frac{\rho_f}{\rho_r} = \frac{\rho_f}{\rho_i} \frac{\rho_i}{\rho_r} = \left( \frac{a_i}{a_f} \right)^4 \frac{\rho_i}{\rho_r}, \quad (6.97)$$



where the inhomogeneous relativistic energy density before the inflation  $\rho_i$  and the uniform one  $\rho_r$ , generated by the reheating process, are of the same order of magnitude.

When referred to a homogeneous and isotropic FRW model, the de Sitter phase of the inflationary scenario rules out the small inhomogeneous perturbations so strongly to prevent them from seeding the later structures formation. This picture emerges sharply within the inflationary paradigm and it is at the basis of the statement that the cosmological perturbations arise from the quantum fluctuations of the scalar field.

Though this argument is well settled down and is very attractive even because the predicted quantum spectrum of inhomogeneities takes the Harrison-Zeldovich form, nevertheless the question remains open whether, in more general contexts, it is possible that early classical inhomogeneities can survive to be relevant for the origin of the cosmological structures.

## 6.6 Quasi-Isotropic Viscous Solution

In order to generalize the quasi-isotropic solution of the Einstein equations in the presence of dissipative effects, we consider a power law extension of the three-metric generalizing Eq. (6.4) as

$$h_{\alpha\beta} = t^x a_{\alpha\beta} + t^y b_{\alpha\beta}, \quad h^{\alpha\beta} = t^{-x} a^{\alpha\beta} - t^{y-2x} b^{\alpha\beta}. \quad (6.98)$$

Here, the constraints for the space contraction toward the singularity (i.e.  $x > 0$ ), and for the internal consistency of the perturbative scheme (i.e.  $y > x$ ) have to be imposed for the proper development of the model. In this approach, the extrinsic curvature and its contractions read as

$$k_{\alpha\beta} = x t^{x-1} a_{\alpha\beta} + y t^{y-1} b_{\alpha\beta}, \quad (6.99a)$$

$$k_{\alpha}^{\beta} = x t^{-1} \delta_{\alpha}^{\beta} + (y - x) t^{y-x-1} b_{\alpha}^{\beta}, \quad (6.99b)$$

$$k = 3x t^{-1} + (y - x) t^{y-x-1} b. \quad (6.99c)$$

The following relation also holds

$$\partial_t \ln \sqrt{h} = \frac{1}{2} k = \frac{3}{2} x t^{-1} + \frac{1}{2} (y - x) t^{y-x-1} b. \quad (6.100)$$

The aim is to obtain constraints and relations for the exponents  $x$ ,  $y$  in order to guarantee the existence of the solutions of this model. We can thus write down the final form of the components of the Ricci tensor contained in

the Einstein equations (6.6). These new expressions allow us to generalize the original quasi-isotropic approach and explicitly read as

$$R_0^0 = -\frac{3x(2-x)}{4t^2} + (y-x)(y-1)\frac{b}{2t^{2-y+x}}, \quad (6.101a)$$

$$R_\alpha^0 = (\nabla_\alpha b - \nabla_\beta b_\alpha^\beta)\frac{y-x}{2t^{1-y+x}}, \quad (6.101b)$$

$$R_\alpha^\beta = \frac{x(3x-2)}{4t^2}\delta_\alpha^\beta + \frac{(y-x)(2y+x-2)}{4t^{2-y+x}}b_\alpha^\beta \\ + \frac{(y-x)x}{4t^{2-y+x}}b\delta_\alpha^\beta + \frac{A_\alpha^\beta}{t^x} + \frac{B_\alpha^\beta}{t^{2x-y}}. \quad (6.101c)$$

We note that in Eq. (6.101c),  $A_\alpha^\beta$  represents the three-dimensional Ricci tensor built from the metric  $a_{\alpha\beta}$ , as in Sec. 6.2. On the other hand, the higher-order term  $B_\alpha^\beta$  denotes the part of  $P_\alpha^\beta$  containing the three-tensor  $b_{\alpha\beta}$ .

### 6.6.1 Form of the energy density

In this Section, we treat the immediate generalization of the non-viscous LK scheme. We consider the presence of dissipative processes affecting the fluid dynamics, as it is expected in the early phases of the Universe, especially at temperatures above  $\mathcal{O}(10^{16} \text{ GeV})$ . As discussed in Sec. 3.3 this extension is represented by an additional term in the expression of the energy momentum tensor (6.10) and it can be derived from thermodynamical properties of the fluid. The restated tensor reads as

$$T_{ij} = (\tilde{P} + \rho)u_i u_j - \tilde{p} g_{ij} \\ = \frac{\rho}{3}(4u_i u_j - g_{ij}) - \zeta \nabla_l u^l (u_i u_j - g_{ij}), \quad (6.102a)$$

$$\tilde{P} = P - \zeta \nabla_l u^l, \quad (6.102b)$$

where  $P = \rho/3$  denotes the usual thermostatic pressure in correspondence of an ultrarelativistic equation of state and  $\zeta$  is the bulk viscosity coefficient, introduced in Sec. 3.3. In what follows, we neglect the shear viscosity for consistency with the quasi-isotropic cosmological evolution (see Sec. 6.6.2).

The coefficient  $\zeta$  has to be expressed in terms of the thermodynamical parameters of the fluid. In particular, as in Sec. 3.3.1, this quantity is assumed to be a power-law function of the energy density fluid

$$\zeta = \zeta_0 \rho^s, \quad (6.103)$$

where  $\zeta_0$  is a constant and  $s$  is a dimensionless parameter whose behavior in correspondence to large values of  $\rho$  is constrained in the range  $0 \leq s \leq \frac{1}{2}$ .

Let us write the expressions of the mixed components of the tensor (6.102a) up to higher-order corrections as

$$T_0^0 = \frac{\rho}{3}(4u_0^2 - 1) - \zeta_0 \rho^s \nabla_i u^i (u_0^2 - 1) \quad (6.104a)$$

$$T = -3\zeta_0 \rho^s \nabla_i u^i \quad (6.104b)$$

$$T_\alpha^\beta = -\frac{\rho}{3}(4u_\alpha u^\beta + \delta_\alpha^\beta) - \zeta_0 \rho^s \nabla_i u^i (u_\alpha u^\beta + \delta_\alpha^\beta) \quad (6.104c)$$

$$T_\alpha^0 = \frac{4}{3}\rho u_\alpha u^0 - \zeta_0 \rho^s \nabla_i u^i u_\alpha u^0, \quad (6.104d)$$

where the divergence of the four-velocity reads as

$$\nabla_i u^i = \partial_t \ln \sqrt{h} = \frac{3}{2}xt^{-1} + \frac{1}{2}(y-x)t^{y-x-1}b. \quad (6.105)$$

Here we assume valid, as in the non-viscous case, the relation  $u_0^2 \simeq 1$ , whose consistence must be verified *a posteriori* comparing the time behavior of the quantities involved in the model. Taking into account the expressions (6.104a) and (6.104b), we can recast the Einstein equation (6.6a) in the form

$$-\frac{3x(2-x)}{4t^2} + (y-x)(y-1)\frac{b}{2t^{2-y+x}} = \kappa \left[ -\rho + \frac{9x}{4t}\zeta_0 \rho^s + \frac{3(y-x)}{4t^{1-y+x}}\zeta_0 \rho^s b \right]. \quad (6.106)$$

In what follows, as in Sec. 3.3.1, we fix the value  $s = \frac{1}{2}$  in order to deal with the maximum effect that bulk viscosity can have without dominating the dynamics of the cosmological fluid.

Since we are interested in the asymptotic limit  $t \rightarrow 0$ , such choice for  $s$  is the appropriate one to include dissipative effects in the primordial dynamics. Thus, from Eq. (6.106) the energy density  $\rho$  can be expanded as

$$\rho = \frac{e_0}{t^2} + \frac{e_1 b}{t^{2-y+x}}, \quad \sqrt{\rho} \simeq \frac{\sqrt{e_0}}{t} \left( 1 + \frac{e_1 b}{2e_0} t^{y-x} \right), \quad (6.107)$$

where the constants  $e_0$  and  $e_1$  will be determined combining the 0 – 0 gravitational equation with the hydrodynamical equations, comparing all terms order by order. We remark that only for the case  $s = \frac{1}{2}$  all terms of Eq. (6.106) have the same time behavior up to first order because of Eq. (6.107).

### 6.6.2 Comments on the adopted paradigm

In this Section, we discuss in some details the hypotheses at the ground of our analysis of the quasi-isotropic viscous Universe dynamics.

As introduced in Sec. 3.1.5, the *microphysical horizon*, i.e. the Hubble length, plays a crucial role as far as the thermodynamical equilibrium is concerned. In the isotropic Universe, this quantity is fixed by the inverse of the expansion rate,  $H^{-1} = (a/\dot{a})$  and provides the characteristic scale below which the particle interactions can preserve the thermal equilibrium of the system. Therefore, if the mean free path of the particles  $\ell$  is greater than the microphysical horizon (i.e.  $\ell > H^{-1}$ ), no notion of thermal equilibrium can be recovered at the microcausal scale. If we denote the number density of particles as  $n$  and the average cross section of the interactions as  $\sigma$ , the mean free path of the ultrarelativistic cosmological fluid (in the early Universe the particle velocity is very close to the speed of light) takes the form  $\ell \sim 1/n\sigma$ . Interactions mediated by massless gauge bosons are in general characterized by a cross section  $\sigma \sim \alpha^2 T^{-2}$  ( $\alpha = \frac{g^2}{4\pi}$ , where  $g$  is the coupling constant of the corresponding interaction) and the physical estimate  $n \sim T^3$  provides  $\ell \sim 1/\alpha^2 T$ . During the radiation dominated era  $H \sim T^2/m_P$ , so that

$$\ell \sim \frac{T}{\alpha^2 m_P} H^{-1}. \quad (6.108)$$

Thus, in the case  $T \gtrsim \alpha^2 m_P \sim \mathcal{O}(10^{16} \text{ GeV})$ , i.e. during the earliest epoch of the pre-inflating Universe, the interactions are “frozen out” and they are not able to establish or to maintain the thermal equilibrium. At temperatures greater than  $\mathcal{O}(10^{16} \text{ GeV})$ , the contributions to the estimate (6.108) due to the mass term of the gauge bosons can be ruled out for all known and so far proposed perturbative interactions.

As a consequence of this non-equilibrium configuration of the causal regions characterizing the early Universe, most of the well-established results about the kinetic theory concerning the cosmological fluid nearby equilibrium are not directly applicable. Indeed, the kinetic analysis is generally based on the assumption of a finite mean free path of the particles and, in particular, the viscosity is characterized by simply retaining pure collisions among the particles. However, when the mean free path is greater than the microcausal horizon,  $\ell$  can be regarded as infinite for any physical purpose.

The original analysis of the viscous cosmology is due to the Landau school which, aware of these difficulties for a consistent kinetic theory, treated the problem on the basis of a hydrodynamical approach. A notion of the *hydrodynamical description* can be provided by assuming that an arbitrary state is adequately specified by the particle flow vector and the energy momentum tensor alone. In particular, the entropy flux has to be expressed as a function of these two dynamical variables without

additional parameters. Thus, the viscosity effects are treated through a thermodynamical description of the fluid, i.e. the viscosity coefficients are fixed by the macroscopic parameters which govern the system evolution. The most natural choice is to take these viscosity coefficients as a power law of the energy density of the fluid. Such phenomenological assumption can be reconciled, for some simple cases, with a relativistic kinetic theory approach, especially in the limits of small and large energy densities.

Considering the hydrodynamical point of view, we retain the same equation of state which characterizes the corresponding ideal fluid. This is supported by the idea that viscosity effects provide only small corrections to the thermodynamical setting of the system. Since we are treating an ultrarelativistic thermodynamical system, very weakly interacting on the micro-causal scale, it is appropriately described by the equation of state  $P = \rho/3$ .

Let us note that the *shear viscosity*  $\eta$  is not included in the present scheme. Indeed, this kind of viscosity accounts for the friction forces acting between different portions of the viscous fluid. Therefore, as far as the isotropic character of the Universe is conserved, the shear viscosity must not provide any contribution, as discussed in Sec. 3.3.1. On the other hand, the rapid expansion of the early Universe suggests that an important contribution comes out from the bulk viscosity as an average effect of a quasi-equilibrium evolution.

Our analysis deals with small inhomogeneous corrections to the background FRW metric. In principle, to first order in our solution, shear viscosity should be included in the dynamics as well. In that case, if the bulk viscosity coefficient behaves as  $\zeta \sim \rho^s$ , correspondingly the shear is  $\eta \sim \rho^r$ , where  $r$  must satisfy the constraint condition  $r \geq s + \frac{1}{2}$ . As discussed in Sec. 3.3.1, we treat the case  $s = \frac{1}{2}$ , thus getting  $r \geq 1$  for the  $\eta$  coefficient. Such constraint implies that the shear viscosity is no longer a first order correction in this solution. For smaller values of  $s$ , the shear viscosity can be included without leading to unphysical solutions. In fact, the shear viscosity provides, among others, an equivalent contribution to the bulk one, since the energy-momentum tensor of the viscous fluid contains the term

$$T_{ij} \sim \dots - (\zeta - \frac{2}{3}\eta) \nabla_l u^l (u_i u_j + g_{ij}) + \dots \quad (6.109)$$

Let us observe that, to zeroth order,  $\nabla_l u^l \sim \mathcal{O}(1/t)$ , while the first-order correction to the energy density behaves as  $\mathcal{O}(1/t^x)$  and we have shown the relation  $1 \leq x < 2$  in Sec. 3.3.1. The request  $x \geq 1$  comes out from

the zeroth-order analysis which, due to the isotropy, is independent of the shear contribution. Since the estimate

$$\mathcal{O}(\eta) \sim \mathcal{O}(\rho^r) \sim \mathcal{O}\left(\frac{1}{t^{xr}}\right) \quad (6.110)$$

holds, we can conclude that the shear viscosity would produce the inconsistency associated to the term

$$\eta \nabla_l u^l \sim \mathcal{O}\left(\frac{1}{t^{xr+1}}\right). \quad (6.111)$$

The request  $rx + 1 \geq 2$  would make the contribution in (6.111) dominant in the model, despite the basic assumption that the shear viscosity must be negligible to the leading isotropic order. Thus, to include the shear viscosity in a quasi-isotropic model, we should consider the case  $s < \frac{1}{2}$  but it is not appropriate for analyzing the asymptotic limit towards the singularity because the corresponding contribution vanishes with no influence on the dynamics.

Let us discuss the implementation of a causal thermodynamics for this cosmological model. The hydrodynamical theory of a viscous fluid is applicable only when the spatial and temporal derivatives of the matter velocity are small, i.e. the characteristic rate of the fluid reaction is negligible with respect to the speed of light. This condition is expectedly violated in the asymptotic limit near the cosmological singularity. In this way, the viscous fluid would be described by a relaxation equation similar to the Maxwell equations in the theory of viscoelasticity. In that scheme, the energy-momentum tensor assumes the form (3.85). In the very early Universe, the relation between  $\Pi_v$  and the relaxation time  $\tau_0$  reads as

$$\Pi_v + \dot{\Pi}_v \tau_0 = \zeta \nabla_i u^i. \quad (6.112)$$

The relaxation time can be expressed as  $\tau_0/\zeta \sim 1/\rho$ : this physical assumption follows from the transverse wave velocity in matter which has a finite (non-zero) magnitude in the case of large values of  $\rho$ .

The time dependence of  $\tau_0$  follows from the fact that  $\rho \sim 1/t^2$  to leading order and then, using Eq. (6.103), the relaxation time behaves as  $\tau_0 \sim t^{2-2s}$ . Since  $s = \frac{1}{2}$  and thus  $\tau_0 \sim t$ , if we assume a power law dependence for  $\Pi_v$  (according to the structure of the solution) such as  $\dot{\Pi}_v \sim \Pi_v/t$ , the relation (6.112) rewrites as

$$\Pi_v = \tilde{\zeta}_0 \rho^s \nabla_l u^l. \quad (6.113)$$

From this analysis we recover the standard expression for the bulk viscous hydrodynamics, provided by the reparametrization  $\zeta_0 \rightarrow \tilde{\zeta}_0$  of the bulk coefficient. This is compatible with the paradigm of causal thermodynamics,

since it would affect only qualitative details (i.e. rescaling some coefficients), without altering the validity of the solution.

### 6.6.3 Solutions of the 00-Einstein and hydrodynamical equations

Thus far, we exploited Eq. (6.6a) in order to obtain the qualitative expression for the energy density  $\rho$  when the matter filling the space was described by a viscous fluid energy-momentum tensor. Let us consider Eq. (6.106) rewritten as

$$\begin{aligned} & \left[ -\frac{3}{4}x(2-x) + \kappa e_0 - \frac{9}{4}\zeta_0 x \sqrt{\kappa e_0} \right] t^{-2} \\ & + \left[ \frac{1}{2}(y-x)(y-1) + \kappa e_1 - \frac{9}{8}\sqrt{\kappa}\zeta_0 x e_1 e_0^{-1/2} \right. \\ & \quad \left. - \frac{3}{4}(y-x)\zeta_0 \sqrt{\kappa e_0} \right] b t^{y-x-2} = 0, \end{aligned} \quad (6.114)$$

coupled to the hydrodynamical ones  $\nabla_j T_i^j = 0$ . In the non-viscous case ( $\zeta_0 = 0$ ), the energy density solution is determined without exploiting the hydrodynamical equations, since  $\rho$  comes directly from the 00-gravitational equation. To the order of approximation considered here ( $u_\alpha$  being negligible with respect to  $u_0$ ), the energy-momentum tensor conservation law provides the equation

$$\partial_t \rho + \partial_t \left( \ln \sqrt{h} \right) \left[ \frac{4}{3} \rho - \zeta_0 \rho^s \partial_t \left( \ln \sqrt{h} \right) \right] = 0, \quad (6.115)$$

which rewrites as

$$\begin{aligned} & \left[ 2\kappa e_0 (x-1) - \frac{9}{4}\zeta_0 x^2 \sqrt{\kappa e_0} \right] t^{-3} \\ & + \left[ \kappa e_1 \left( b(y-x-2) + 2xb - \frac{9}{8}\zeta_0 x^2 b (\kappa e_0)^{-1/2} \right) \right. \\ & \quad \left. + \frac{2}{3}(y-x)be_0 - \frac{3}{2}x(y-x)\zeta_0 b \sqrt{\kappa e_0} \right] t^{y-x-3} = 0. \end{aligned} \quad (6.116)$$

When Eq. (6.114) is coupled to Eq. (6.116) it provides a polynomial expression in  $t$  and must be solved order by order in  $1/t$  (in the asymptotic limit  $t \rightarrow 0$ ). Since for the consistency of the solution  $y > x$  (as detailed when we discussed Eq. (6.98)), applying the polynomial identity principle we get the unique values

$$x = \frac{1}{1 - \frac{3\sqrt{3}}{4}\zeta_0}, \quad \kappa e_0 = \frac{3}{4}x^2. \quad (6.117)$$

The parameter  $\zeta_0$  is constrained as  $\zeta_0 \leq \frac{4}{3\sqrt{3}}$  in order to satisfy the condition  $x > 0$ . In this way, the exponent of the metric power law  $x$  runs from 1 (which corresponds to the non-viscous limit  $\zeta_0 = 0$ ) to infinity. Such constraint on  $\zeta_0$  arises from a zeroth-order analysis and defines the existence of a viscous Friedmann-like model, in which the early Universe expands with a power law in time.

Comparing the two first-order identities (involving terms proportional to  $t^{y-x-2}$  and  $t^{y-x-3}$ ), we get an algebraic equation for  $y$

$$y^2 - y(x+1) + 2x - 2 = 0, \quad (6.118)$$

whose solutions are  $y = 2$ ,  $y = x-1$ . The latter does not fulfill the condition  $y > x$ , thus the first order correction to the three-metric is characterized by the following parameters

$$y = 2, \quad \kappa e_1 = -\frac{1}{2}x^3 + 2x^2 - 2x. \quad (6.119)$$

In the non-viscous case ( $\zeta_0 = 0$ ) we get  $x = 1$ ,  $\kappa e_0 = \frac{3}{4}$ ,  $\kappa e_1 = -\frac{1}{2}$ , which reproduce the energy density solution (6.14).

The consistency of the model is ensured by constraining the parameter  $x$  to values  $x < y$ . Thus, from Eq. (6.117), the quasi-isotropic solution emerges only for

$$\zeta_0 < \zeta_0^* = \frac{2}{3\sqrt{3}}, \quad (6.120)$$

i.e. for small enough viscosity. When the viscous parameter  $\zeta_0$  overcomes the critical value  $\zeta_0^*$ , the quasi-isotropic expansion in the asymptotic limit as  $t \rightarrow 0$  cannot be considered, since the perturbations would grow more rapidly than the zeroth-order terms. The perturbation dynamics in a pure isotropic picture yields a very similar asymptotic behavior when including viscous effects. The Friedmann singularity scheme is preserved only if we deal with limited values of the viscosity parameter, in particular obtaining the condition  $\zeta_0^{\text{iso}} < \zeta_0^*/3$ : this constraint is physically motivated considering that the Friedmann model is a particular case of the quasi-isotropic solution.

The solution of the unperturbed dynamics gives rise to the expression of the metric exponent  $x$  in terms of the viscous parameter  $\zeta_0$  and to the zeroth-order expression of the energy density which reads as

$$\kappa\rho = \frac{3x^2}{4t^2} + \dots \quad (6.121)$$



In order to characterize the effective expansion of the early Universe, let us recall the expression of the total pressure  $\tilde{P}$  (6.102b) to leading order as

$$\tilde{P} = \frac{1}{3}\rho + \frac{3}{2t}\zeta_0\sqrt{\rho}x, \quad (6.122)$$

obtained from the four-divergence (6.105) truncated to zeroth order. From these relations, the condition  $\tilde{P} \geq 0$  yields the inequality

$$\zeta_0 \leq \zeta_0^*/2, \quad (6.123)$$

which strengthens the constraint (6.120) and restricts the  $x$ -domain to  $[1, \frac{4}{3}]$ .

The request of a positive (at most zero) total pressure is consistent with the idea that the bulk viscosity must not affect too much the standard dynamics of the isotropic Universe. In this respect, we consider the domain (6.123) as a physical restriction to the initial conditions for the existence of a well-grounded quasi-isotropic solution.

Let us rewrite the expression of the energy density to analyze the evolution of the density contrast. In the presence of the bulk viscosity,  $\rho$  assumes the form

$$\kappa\rho = \frac{3x^2}{4t^2} - \frac{(x^3/2 - 2x^2 + 2x)b}{t^x}, \quad (6.124)$$

and, hence, the density contrast  $\delta$  (defined in Sec. 3.4) can be written as

$$\delta = -\frac{8}{3} \left( \frac{x}{4} + \frac{1}{x} - 1 \right) b^{2-x}. \quad (6.125)$$

Since  $x$  runs from 1 to 2 as the viscosity increases towards its critical value, the density contrast evolution is strongly damped by the presence of dissipative effects which act on the perturbations. In this sense, the viscosity can damp the evolution of the perturbations forward in time. This behavior of the density contrast toward the singularity ( $\delta \rightarrow 0$ ) is characterized by a weaker power law in time in comparison to  $\zeta_0$  approaching  $\zeta_0^*$ . In correspondence to such threshold value, the density contrast remains constant in time.

### 6.6.4 The velocity and the three-metric

While the 00-Einstein equation provides a solution for the energy density, let us complete the dynamical scenario analyzing the quasi-isotropic model, to verify the consistency of our approximations considering the solutions to the whole system of gravitational equations.

Imposing the condition  $s = \frac{1}{2}$ , the Einstein equation (6.6b) reads as

$$\frac{y-x}{2t^{1-y+x}}(\nabla_\alpha b - \nabla_\beta b_\alpha^\beta) = \frac{4}{3}\kappa\rho u_\alpha - \zeta_0\sqrt{\kappa\rho}u_\alpha\left(\frac{3x}{2t} + \frac{(y-x)b}{2t^{1-y+x}}\right). \quad (6.126)$$

Substituting Eq. (6.124) into Eq. (6.126), we get the expression for the velocity, which to leading order reads as

$$u_\alpha = \frac{2-x}{2x}(\partial_\alpha b - \nabla_\beta b_\alpha^\beta)t^{3-x}, \quad (6.127)$$

where we neglected terms of order  $\mathcal{O}(t^{-1})$  and  $\mathcal{O}(t^{1-x})$ . The assumption  $u_0^2 \simeq 1$  is verified since  $u_\alpha u^\alpha \sim t^{6-3x}$  and can be neglected in the four-velocity expression  $u_l u^l = 1$ . The approximated hydrodynamical equation (6.116) is still self-consistent using Eq. (6.127) for  $u_\alpha$ .

Let us address Eq. (6.6c): the first two leading order terms of the right-hand side are  $\mathcal{O}(t^{-2})$  and  $\mathcal{O}(t^{-x})$ , respectively, only if  $x < 2$ , as in our scheme. Hence,  $u_\alpha u^\beta$  can be neglected and terms  $\mathcal{O}(t^{-2})$  cancel out, while those proportional to  $t^{-x}$  give the equation

$$A_\alpha^\beta + \mathcal{A}b_\alpha^\beta + \mathcal{B}b\delta_\alpha^\beta + \mathcal{C}\delta_\alpha^\beta = 0, \quad (6.128)$$

which generalizes Eq. (6.16), and the quantities  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$  are defined as

$$\mathcal{A} = \frac{1}{4}(4-x^2) \quad (6.129a)$$

$$\mathcal{B} = \frac{1}{6}(2x-1)(x-2)^2 - \frac{1}{4}x(x-2) \quad (6.129b)$$

$$\mathcal{C} = -\frac{1}{6}(2-x)(x-1), \quad (6.129c)$$

respectively. Taking the trace of Eq. (6.128), we obtain the relation

$$(\mathcal{A} + 3\mathcal{B})b = -A - 3\mathcal{C} \quad (6.130)$$

which, combined with the Ricci three-tensor relation  $\nabla_\beta A_\alpha^\beta = \frac{1}{2}\nabla_\beta A$ , provides the equation

$$2\mathcal{A}\nabla_\beta b_\alpha^\beta = (\mathcal{A} + \mathcal{B})\partial_\alpha b. \quad (6.131)$$

Let us write down the three-velocity in terms of the trace of the perturbed metric tensor  $b$  as

$$u_\alpha = \frac{2-x}{4x}\frac{\mathcal{A}-\mathcal{B}}{\mathcal{A}}t^{3-x}\partial_\alpha b. \quad (6.132)$$

The solution constructed here matches the non-viscous solution (6.18) if we set  $\zeta_0 = 0$  and it is completely self-consistent up to the first two orders in time. The present model contains three physically arbitrary functions of

the spatial coordinates only, i.e. the six functions  $a_{\alpha\beta}$  minus three degrees of freedom that can be eliminated by fixing suitable space coordinates. The only remaining free parameter of the model is the viscosity  $\zeta_0$ .

The quasi-isotropic solution exists for particular values of the bulk viscosity coefficient  $\zeta_0$  only. When the dissipative effects become too relevant, we are not able to construct the solution following the line of the LK model. In fact, when  $\zeta_0$  approaches the threshold value  $\zeta_0^* = \frac{2}{3\sqrt{3}}$ , the approximation scheme fails and the model becomes not self-consistent.

By requiring the viscosity parameter  $\zeta_0$  to be smaller than its critical value, the behavior of the density contrast is influenced by the presence of the bulk viscosity. As far as dissipative effects are taken into account, the density contrast contraction ( $\delta \rightarrow 0$  as  $t \rightarrow 0$ ) is damped out, or remains constant if  $\zeta_0$  equals its critical value.

## 6.7 Guidelines to the Literature

With reference to Sec. 6.1, for a discussion of the Einstein equations in a synchronous reference, as introduced in Sec. 2.4, we refer to the classical textbook [301].

The original derivation of the quasi-isotropic solution for the radiation dominated Universe, discussed in Sec. 6.2, was provided by Lifshitz & Khalatnikov in [312]. For a generalization of this solution to a generic equation of state, see [272].

When introducing the scalar field in the quasi-isotropic model, the resulting features near the cosmological singularity, as detailed in Sec. 6.3, were derived in [349]. An interesting related analysis, starting from the long-wavelength approximation can be found in [439].

The presentation in Sec. 6.4 of a quasi-isotropic solution in the presence of a massless scalar field and an electromagnetic component is stated in the article [350].

The description of the quasi-isotropic inflation in Sec. 6.5 refers to [259] with reference also to [439].

For a general discussion regarding the inflationary scenario and the origin of a perturbation spectrum, as introduced in Sec. 5.6.4, we refer to the following textbooks [155, 290, 327, 370].

The analysis of the quasi-isotropic solution in the presence of the bulk viscosity presented in Sec. 6.6 is inspired by the derivation of [112]. The comparison of these results with the behavior of the inhomogeneous per-

turbations in a viscous FRW Universe is allowed by reading [111].

For the original analysis of the influence of the bulk viscosity on the early Universe dynamics see [61, 63, 67]. Such line of research treated the description of bulk viscosity within a hydrodynamical approach, as it is addressed in Sec. 6.6.

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## PART 3

# Mathematical Cosmology

In these Chapters, the evolution of the Universe near the singularity is discussed under more general hypotheses than homogeneity and isotropy. The analysis of the homogeneous models of the Bianchi classification is presented in details, both in the field equations and the Hamiltonian frameworks.

Chapter 7 is dedicated to the investigation of the geometry and the dynamics of the homogeneous, but anisotropic cosmologies, as prescribed by the Einstein geometrodynamics. The chaotic nature of the Belinskii-Khalatnikov-Lifshitz oscillatory regime is outlined.

Chapter 8 concerns the Hamiltonian description of the Bianchi type VIII and IX models near the singularity (the so-called Mixmaster Universe). The question inherent the proper characterization of the Mixmaster chaos and of its covariant nature is analyzed with accuracy.

Chapter 9 presents a picture of the generic cosmological solution in the asymptotic limit to the singularity (the so-called inhomogeneous Mixmaster Universe). The piecewise nature of this solution is analyzed by means of the Einstein equations, as well as of the Hamiltonian representation. The parametric role of the space coordinates is outlined and the existence of a space-time foam is argued.

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## Chapter 7

# Homogeneous Universes

In this Chapter we analyze the dynamics and the morphology of the homogeneous but anisotropic cosmological models. This set of Universes has been classified by Bianchi in 1898 into nine different types, corresponding to the independent groups of isometries for the three-dimensional space.

We provide a precise definition of homogeneity for a space manifold and outline the underlying Lie algebra and the Jacobi identities which lead to the Bianchi classification. Via the projection technique on the triad setting of the three-manifold, we show how the Einstein equations reduce to an ordinary differential system in time. In fact, the homogeneity constraint implements the dynamical equivalence of all the spatial points, so that spatial gradients cannot enter the Universe evolution. The space geometry remains fixed by the 1-forms describing the specific model, while the dynamics is summarized by three scale factors, associated to the evolution of the three independent spatial directions, according to the Einstein prescription.

The analysis of the simple Bianchi I model, derived by Kasner in 1921, has a peculiar role in the construction of the asymptotic regime to the cosmological singularity of all the other Bianchi Universes.

Then, we discuss the Bianchi type II and we outline how its evolution is characterized by two distinct Kasner regimes (*Kasner epochs*) related by a specific map which accounts for the effect of the spatial curvature inducing the transition process.

The Bianchi VII model emerges as consisting of a *Kasner era*, i.e. a sequence of Kasner epochs during which two space directions oscillate toward the singularity and the third one decays monotonically, plus a final stable Kasner regime associated to a re-increase of the decaying direction.

The behavior of the Bianchi type VIII and IX models is discussed in detail as far as the Universe approaches the singularity and in both cases,



we outline the existence of an *oscillatory* regime of the solution, consisting of an infinite sequence of Kasner eras. The statistical properties of this piecewise solution are addressed and the stationary distribution functions of the metric parameters are derived. Thus, the most general models (types VIII and IX) allowed by the homogeneity constraint are still characterized by a cosmological singularity, but the corresponding asymptotic regime is characterized by an anisotropic morphology and outlines significant features of a chaotic dynamics. Such a chaotic solution admits, as we shall see in Chap. 9 a natural inhomogeneous extension, which provides the evolution to the singularity of a generic model, i.e. having, in vacuum, four arbitrary degrees of freedom (four free space functions) to address a generic Cauchy problem on a non-singular spatial hypersurface.

## 7.1 Homogeneous Cosmological Models

A space is said to be homogeneous if its metric tensor admits an isometry group that maps the space onto itself. Such group results to be generated by the Killing vector fields forming a Lie algebra. The corresponding simply transitive Lie group can be identified, via its orbits, with the Cauchy surfaces filling the space-time manifold. Considering the Maurer-Cartan 1-forms (7.28), the homogeneous (and in general anisotropic) metric tensor is written as in Eq. (7.37). The matrix  $\eta_{ab}$  thus depends on the time coordinate only, is position independent on each Cauchy surface, and the Einstein equations reduce to a system of ordinary differential equations.

### 7.1.1 Definition of homogeneity

Let us consider a group of transformations

$$x^\mu \rightarrow \bar{x}^\mu = f^\mu(x, \tau) \equiv f_\tau^\mu(x) \quad (7.1)$$

on a space  $\Sigma$  (eventually a manifold), where  $\{\tau^a\}_{a=1,\dots,r}$  are  $r$  independent parameters that characterize the group. Furthermore, let  $\tau_0$  correspond to the identity

$$f^\mu(x, \tau_0) = x^\mu. \quad (7.2)$$

Let us take an infinitesimal transformation close to the identity  $\tau_0 + \delta\tau$  so that

$$\begin{aligned} x^\mu &\rightarrow \bar{x}^\mu = f^\mu(x, \tau_0 + \delta\tau) \\ &\approx f^\mu(x, \tau_0) + \left( \frac{\partial f^\mu}{\partial \tau^a} \right) (x, \tau_0) \delta\tau^a \equiv x^\mu + \xi_a^\mu(x) \delta\tau^a \\ &= (1 + \delta\tau^a \xi_a) x^\mu. \end{aligned} \quad (7.3)$$

Here the  $r$  first-order differential operators  $\{\xi_a\}$  are defined by  $\xi_a = \xi_a^\mu \partial_\mu$  and correspond to the  $r$  vector fields with components  $\{\xi_a^\mu\}$ , constituting the *generating vector fields*. This way, under the infinitesimal transformation (7.3) all points of the space  $\Sigma$  are translated by a distance  $\delta x^\mu = \delta\tau^a \xi_a^\mu$  in the coordinates  $\{x^\mu\}$  and thus

$$\bar{x}^\mu \approx (1 + \delta\tau^a \xi_a) x^\mu \approx e^{\delta\tau^a \xi_a} x^\mu. \quad (7.4)$$

The finite transformations (which have the structure of a group) may be represented as

$$\bar{x}^\mu \rightarrow \bar{\bar{x}}^\mu = e^{\theta^a \xi_a} x^\mu \quad (7.5)$$

where  $\{\theta^a\}$  ( $a = 1, \dots, r$ ) are  $r$  parameters on the group. In particular, if the group is a Lie group, the generators  $\xi_a$  form a Lie algebra, i.e. a real  $r$ -dimensional vector space where  $\{\xi_a\}$  is a basis which is closed under commutation relations

$$[\xi_a, \xi_b] = \xi_a \xi_b - \xi_b \xi_a = C_{ab}^c \xi_c, \quad (7.6)$$

where  $C_{ab}^c$  are called the structure constants of the group.

Let us consider a Lie group acting on a manifold  $\Sigma$  as a group of transformations (7.1), and let us define the orbit of  $x$  as

$$f_{\mathfrak{G}}(x) = \{f_\tau(x) \mid \tau \in \mathfrak{G}\} \quad (7.7)$$

i.e. the set of all points that can be reached from  $x$  under the action of the transformations.

**Definition 7.1.** *The group of isometry at  $x$  is given by the subgroup of  $\mathfrak{G}$  which leaves  $x$  fixed, i.e.*

$$\mathfrak{G}_x = \{f_\tau(x) = x \mid \tau \in \mathfrak{G}\}. \quad (7.8)$$

Suppose  $\mathfrak{G}_x = \{\tau_0\}$  and  $f_{\mathfrak{G}}(x) = \Sigma$ , i.e. every transformation of  $\mathfrak{G}$  moves the point  $x$  and every point in  $\Sigma$  can be reached from  $x$  by a unique transformation. Since  $\mathfrak{G}/\mathfrak{G}_x = \{\tau/\tau_0 \mid \tau \in \mathfrak{G}\}$ , the group  $\mathfrak{G}$  is isomorphic to the manifold  $\Sigma$  and one may identify the two objects. The transformations

leaving invariant the metric  $h_{\alpha\beta}$  are called isometries, particular cases of diffeomorphisms. In the particular case of an isometry group, the  $\{\xi_a^\mu\}$  satisfy the so-called Killing equation

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0, \quad (7.9)$$

and for this reason, are called *Killing vector fields*. They satisfy a Lie algebra and generate the groups of motions via infinitesimal displacements, yielding conserved quantities and allowing a classification of homogeneous spaces.

We now introduce the concept of an invariant basis. Suppose  $\{e_a\}$  is a basis of the Lie algebra  $\mathfrak{g}$  of a group  $\mathfrak{G}$

$$[e_a, e_b] = C_{ab}^c e_c \quad (7.10)$$

and define  $\gamma_{ab} = C_{ad}^c C_{bc}^d = \gamma_{ba}$ . This quantity is symmetric by definition and provides a natural inner product on  $\mathfrak{g}$

$$\gamma_{ab} \equiv e_a \cdot e_b = \gamma(e_a, e_b). \quad (7.11)$$

When  $\det(\gamma_{ab}) \neq 0$ , Eq. (7.11) is non-degenerate and the group is called *semi-simple*. The  $r$  vector fields  $\{e_a\}$  form a frame, because they may be used instead of the coordinate basis  $\{\frac{\partial}{\partial x^a}\}$  to express an arbitrary vector field on  $\mathfrak{G}$ . The basis  $\{e_a\}$  is called invariant<sup>1</sup> if

$$\mathcal{L}_{\xi_a} e_b = [\xi_a, e_b] = 0. \quad (7.12)$$

This means that  $e_a$  are vector fields invariant under the action of the Killing vectors  $\xi_a$ .

Summarizing, a manifold  $\Sigma$  is said to be invariant under a group  $\mathfrak{G}$  if there are  $m$  (where  $m = \dim \mathfrak{G}$ ) Killing vector fields  $\xi_a$  which satisfy the commutation relation (7.6). As soon as one identifies  $\mathfrak{G}$  to  $\Sigma$ , the metric tensor  $h_{\alpha\beta}$  on  $\Sigma$  is invariant under the action of the group  $\mathfrak{G}$ . Thus,  $h_{\alpha\beta}$  corresponds to an invariant tensor on such a group, and it is completely specified by the inner product (7.11) of the invariant vector fields  $e_a$ .

Let us consider the case of a space-time whose metric  $g_{ij}$  is invariant under a three-dimensional isometry group. Given an invariant basis  $\{e_a\}$ , the spatial metric at each moment of time can be specified by the spatially constant inner products

$$e_a \cdot e_b = \eta_{ab}(t). \quad (7.13)$$

---

<sup>1</sup>To be more precise,  $e_a$  are called “left-invariant”, while  $\xi_a$  “right-invariant”.

Here, the tetradic projection  $\eta_{ab}$  of  $h_{\alpha\beta}$  (see Sec. 2.5) has been identified with  $\gamma_{ab}$  of Eq. (7.11).

We can give the following definition of a spatially homogeneous space-time:

**Definition 7.2.** *A space-time  $(\mathcal{M}, g_{ij})$  is spatially homogeneous if a family of space-like surfaces  $\Sigma_t$  exists such that for any two points  $p, q \in \Sigma_t$ , there is a unique element  $\tau : \mathcal{M} \rightarrow \mathcal{M}$  of a Lie group  $\mathfrak{G}$  such that  $\tau(p) = q$ .*

Because of the uniqueness of the group element  $\tau$ ,  $\mathfrak{G}$  is said to act *simply transitively* on each  $\Sigma_t$ . Such condition implies also that the Killing vectors are linearly independent, and that<sup>2</sup>  $\dim \mathfrak{G} = \dim \Sigma_t = 3$ . Spatially homogeneous models are those for which the symmetry group  $\mathfrak{G}$  acts simply transitively on each spatial manifold, i.e. the space-time topology is given by

$$\mathcal{M} = \mathbb{R} \otimes \mathfrak{G}. \quad (7.14)$$

It is worth noting that, because of the identification of  $\mathfrak{G}$  with  $\Sigma_t$ , the action on  $\Sigma_t$  of the isometry  $\tau$  corresponds to a left multiplication by  $\tau$  on  $\mathfrak{G}$ . This way, tensor fields invariant under the isometries correspond to the left-invariant ones on  $\mathfrak{G}$ .

### 7.1.2 Application to Cosmology

For a spatially homogeneous space-time, one needs only to consider a representative group from each class of equivalence of isomorphic Lie groups of dimension three. The classification of inequivalent three-dimensional Lie groups is called the Bianchi classification and determines the various possible symmetry types for homogeneous three-spaces, just as  $K = 0, \pm 1$  classify the possible symmetry types for homogeneous *and* isotropic FRW three-spaces.

The space-time metric  $g_{ij}$  in the homogeneous models must reflect that the metric properties are the same in all space points. Under the isometry  $\tau : x \rightarrow x'$ , the spatial line element

$$dl^2 = h_{\alpha\beta}(t, x) dx^\alpha dx^\beta, \quad (7.15)$$

has to be invariant (as discussed above it is also left-invariant), which implies

$$dl^2 = h_{\alpha\beta}(t, x') dx'^\alpha dx'^\beta, \quad (7.16)$$

---

<sup>2</sup>In general, given an  $n$ -dimensional manifold,  $\dim \mathfrak{G} \leq n(n+1)/2$ .

where  $h_{\alpha\beta}$  has the same form in the old and in the new coordinates. The metric tensor for a homogeneous space-time is obtained by choosing a basis of dual vector fields  $\omega^a$  which are preserved under the isometries. This basis is dual to the left-invariant one (7.12), i.e.  $\omega^a(e_b) = \delta_b^a$ .

In the general case of a non-Euclidean homogeneous three-dimensional space, there are three independent differential forms which are invariant under the transformations of the group of motions. These forms, however, do not represent the total differential of any function of the coordinates. We shall write them as  $\omega^a = e_\alpha^a dx^\alpha$  and hence the spatial line element is re-expressed as  $dl^2 = \eta_{ab}(e_\alpha^a dx^\alpha)(e_\beta^b dx^\beta)$  so that the triadic representation of the metric tensor reads as

$$h_{\alpha\beta}(t, x) = \eta_{ab}(t) e_\alpha^a(x^\gamma) e_\beta^b(x^\gamma), \quad (7.17)$$

where  $\eta_{ab}$  is a symmetric matrix depending on time only. Differently from the general case discussed in Sec. 2.5, the homogeneity condition allows one to encode the time dependence of the triads into  $\eta_{ab}$ . In contravariant components we have

$$h^{\alpha\beta}(t, x) = \eta^{ab}(t) e_\alpha^a(x^\gamma) e_\beta^b(x^\gamma), \quad (7.18)$$

where  $\eta^{ab}$  denote the components of the matrix inverse of  $\eta_{ab}$ . All the results obtained in Sec. 2.5 apply straightforwardly also here.

The relationships between the covariant and contravariant expressions for the three basis vectors are

$$e_1 = \frac{1}{v} [e^2 \wedge e^3], \quad e_2 = \frac{1}{v} [e^3 \wedge e^1], \quad e_3 = \frac{1}{v} [e^1 \wedge e^2], \quad (7.19)$$

where  $e^a$  and  $e_a$  must be formally considered as Cartesian vectors with components  $e_\alpha^a$  and  $e_a^\alpha$ , while  $v$  represents

$$v = |e_\alpha^a| = e^1 \cdot [e^2 \wedge e^3]. \quad (7.20)$$

The determinant of the metric tensor (7.17) is given by  $h = \eta v^2$ , being  $\eta \equiv \det(\eta_{ab})$ .

The invariance of the line element (7.16) under the action of the symmetry group  $\mathfrak{G}$  implies that (for easier notation we denote hereafter  $x = x^\gamma$ )

$$e_\alpha^a(x) dx^\alpha = e_\alpha^a(x') dx'^\alpha, \quad (7.21)$$

where  $e_\alpha^a$  on both sides of Eq. (7.21) are the same functions expressed in terms of the old and the new coordinates, respectively. The algebra of the triads permits one to rewrite Eq. (7.21) as

$$\frac{\partial x'^\beta}{\partial x^\alpha} = e_a^\beta(x') e_\alpha^a(x). \quad (7.22)$$

This is a system of differential equations which defines the change of coordinates  $x'^\beta(x)$  in terms of given basis vectors. Integrability of the system (7.22) is expressed in terms of the Schwartz condition

$$\frac{\partial^2 x'^\beta}{\partial x^\alpha \partial x^\gamma} = \frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\alpha} \quad (7.23)$$

which, explicitly, leads to

$$\begin{aligned} \left[ \frac{\partial e_a^\beta(x')}{\partial x'^\delta} e_b^\delta(x') - \frac{\partial e_b^\beta(x')}{\partial x'^\delta} e_a^\delta(x') \right] e_\gamma^b(x) e_\alpha^a(x) \\ = e_a^\beta(x') \left[ \frac{\partial e_\gamma^a(x)}{\partial x^\alpha} - \frac{\partial e_\alpha^a(x)}{\partial x^\gamma} \right]. \end{aligned} \quad (7.24)$$

Multiplying both sides of (7.24) by  $e_d^\alpha(x) e_c^\gamma(x) e_\beta^f(x')$  simple algebra provides the following form for the left-hand side of the equation

$$\begin{aligned} e_\beta^f(x') \left[ \frac{\partial e_d^\beta(x')}{\partial x'^\delta} e_c^\delta(x') - \frac{\partial e_c^\beta(x')}{\partial x'^\delta} e_d^\delta(x') \right] \\ = e_c^\beta(x') e_d^\delta(x') \left[ \frac{\partial e_\beta^f(x')}{\partial x'^\delta} - \frac{\partial e_\delta^f(x')}{\partial x'^\gamma} \right] \end{aligned} \quad (7.25)$$

and the right-hand side gives the same expression but in terms of  $x$ . Since  $x$  and  $x'$  are generic coordinates, such an equality implies that both sides must be equal to the same set of constants, and Eq. (7.25) reduces to

$$\left( \frac{\partial e_\alpha^c}{\partial x^\beta} - \frac{\partial e_\beta^c}{\partial x^\alpha} \right) e_a^\alpha e_b^\beta = C_{ab}^c, \quad (7.26)$$

which gives the relations of the vectors with the group structure constants  $C_{ab}^c$ . Let us note that such constants coincide with the three-dimensional analogous of Eq. (2.111b). Multiplying Eq. (7.26) by  $e_c^\gamma$ , we finally have

$$e_a^\alpha \frac{\partial e_b^\gamma}{\partial x^\alpha} - e_b^\beta \frac{\partial e_a^\gamma}{\partial x^\beta} = C_{ab}^c e_c^\gamma. \quad (7.27)$$

Such expression states that the homogeneity condition reduces to a constraint on the left-invariant 1-forms  $\omega^a = e_a^\alpha dx^\alpha$  which have to satisfy the Maurer-Cartan structure equation

$$d\omega^a + \frac{1}{2} C_{bc}^a \omega^b \wedge \omega^c = 0. \quad (7.28)$$

By construction, we have the antisymmetry property  $C_{ab}^c = -C_{ba}^c$  from the formula (7.26). In terms of the vector fields  $e_a = e_a^\alpha \partial_\alpha$ , Eq. (7.27) rewrites as Eq. (7.10). Homogeneity is then expressed as the Jacobi identity

$$[[e_a, e_b], e_c] + [[e_b, e_c], e_a] + [[e_c, e_a], e_b] = 0 \quad (7.29)$$

and is eventually stated as

$$C_{ab}^f C_{cf}^d + C_{bc}^f C_{af}^d + C_{ca}^f C_{bf}^d = 0, \quad (7.30)$$

or, in the language of forms, as

$$d^2 \omega^a = 0. \quad (7.31)$$

Introducing the two-indices structure constants as  $C_{ab}^c = \varepsilon_{abd} C^{dc}$  (or, equivalently,  $C^{dc} = \varepsilon^{abd} C_{ab}^c$ ), where  $\varepsilon_{abc} = \varepsilon^{abc}$  is the totally anti-symmetric three-dimensional Levi-Civita tensor, the Jacobi identity (7.30) becomes  $\varepsilon_{bcd} C^{cd} C^{ba} = 0$ . We also note that Eq. (7.26) can be restated as

$$C^{ab} = -\frac{1}{v} e^a \wedge e^b, \quad (7.32)$$

where the vectorial operations are to be performed as if coordinates  $x^\gamma$  were Cartesian ones. The problem of classifying all homogeneous spaces thus reduces to finding out all inequivalent sets of structure constants of a three-dimensional Lie group.

As we will see, by this formalism the Einstein equations for a homogeneous Universe can be written as a system of *ordinary* differential equations which involve functions of time only, provided the use of projections of 3-tensors on the tetradic basis.<sup>3</sup>

### 7.1.3 Bianchi Classification and Line Element

The list of all three-dimensional Lie algebras was first accomplished by Bianchi in 1897 such that each algebra uniquely determines the local properties of a three-dimensional group. Given a homogeneous space-time with its symmetry group being the “Bianchi Type  $N$ ” ( $N = \text{I}, \dots, \text{IX}$ ), the corresponding structure constants can be written as

$$C_{bc}^a = \varepsilon_{bcd} n^{da} + \delta_c^a a_b - \delta_b^a a_c, \quad (7.33)$$

or, equivalently,

$$C^{ab} = n^{ab} + \varepsilon^{abc} a_c, \quad (7.34)$$

where  $n^{ab} = n^{ba}$  and  $a_a = C_{ba}^b$ . From Eq. (7.33), the Jacobi identity (7.30) reduces to the condition

$$n^{ab} a_b = 0, \quad (7.35)$$

---

<sup>3</sup>The explicit coordinate dependence of the basis vectors is not necessary for the equations ruling the dynamics. In fact, such choice is not unique as  $e_a = A_{ba} e^b$  yields again a set of basis vectors, for any constant matrix  $A_{ba}$ .

and a three-dimensional Lie group (algebra) is then determined by assigning a dual vector  $a_c$  and a symmetric matrix  $n^{ab}$  satisfying the constraint (7.35). Without loss of generality (i.e. with a global rotation of the triad vectors), we can set  $a_c = (a, 0, 0)$  and reduce the matrix  $n^{ab}$  to its diagonal form  $n^{ab} = \text{diag}(n_1, n_2, n_3)$ . The condition (7.35) reduces to  $an_1 = 0$ , i.e. either  $a$  or  $n_1$  has to vanish. The sub-classification as class A and class B models refers to the case  $a_c = 0$  or  $a_c \neq 0$ , respectively. The Jacobi identity can also be restated in terms of the vector fields  $e_a$  as

$$\begin{aligned} [e_1, e_2] &= -a e_2 + n_3 e_3 \\ [e_2, e_3] &= n_1 e_1 \\ [e_3, e_1] &= n_2 e_2 + a e_3 \end{aligned} \quad (7.36)$$

where the set of parameters  $a \geq 0$  and  $n_1, n_2, n_3$  can be rescaled to unity by a corresponding constant re-scaling of the triad. Therefore, all three-dimensional Lie algebras can be classified (according to the Bianchi classification) into nine types, six of class A and three of class B, according to Table 7.1.

It is worth noting that the Bianchi type I is isomorphic to the three-dimensional translation group  $\mathbb{R}^3$ , for which the flat FRW model is a particular case (once isotropy is restored). Analogously, the Bianchi type V contains, as a particular case, the open FRW space. Finally, the closed FRW spatial line-element can be obtained in the isotropic limit of type IX model whose symmetry group is  $SO(3)$  (see Sec. 8.1). Bianchi IX is then the most general model in which the topology of the spatial surfaces is given by the three-sphere  $S^3$ . It has been shown by Lin and Wald in 1990 that all these models first expand and, after reaching a turning point, start to collapse. Of course, the closed FRW Universe belongs to such a class.

The metric tensor  $g_{ij}$  can be immediately written by considering a basis of dual vector fields  $\omega^a$  preserved under the isometries. Recalling Eq. (7.17), the four-dimensional line element is then expressed as

$$ds^2 = N^2(t) dt^2 - \eta_{ab}(t) \omega^a \omega^b, \quad (7.37)$$

parametrized by the proper time, where the one-forms  $\omega^a = \omega^a(x^\gamma)$  obey the Maurer-Cartan equations (7.28). Notice that  $\omega^a$  take value in the Lie algebra  $\mathfrak{g}$  of the symmetry group  $\mathfrak{G}$  and (in the case of<sup>4</sup>  $\mathfrak{G} = GL_n$ ) can be written as

$$\omega(\tau) = \tau^{-1} d\tau, \quad \tau \in \mathfrak{G}. \quad (7.38)$$

---

<sup>4</sup>The general linear group  $GL_n$  denotes the set of real  $n \times n$  matrices with non-vanishing determinant. Of course,  $SO(n)$  and  $SU(n)$  are subgroups of  $GL_n$ .



Table 7.1 Inequivalent structure constants corresponding to the Bianchi classification.

Type	a	n <sub>1</sub>	n <sub>2</sub>	n <sub>3</sub>
I	0	0	0	0
II	0	1	0	0
VII	0	1	1	0
VI	0	1	-1	0
<b>IX</b>	<b>0</b>	<b>1</b>	<b>1</b>	<b>1</b>
<b>VIII</b>	<b>0</b>	<b>1</b>	<b>1</b>	<b>-1</b>
V	1	0	0	0
IV	1	0	0	1
VII <sub>a</sub>	a	0	1	1
III (a = 1)	a	0	1	-1
VI <sub>a</sub> (a ≠ 1)				

Let us now write the Einstein equations for a homogeneous Universe. In the tetradic basis  $\omega^a$ , the vierbein components (see Sec. 2.5) can be written in the form of a system of ordinary differential equations which involve functions of time only

$$R_0^0 = \frac{\partial}{\partial t} K_a^a - K_a^b K_b^a = \kappa \left( T_0^0 - \frac{1}{2} T \right) \quad (7.39a)$$

$$R_a^0 = K_b^c (C_{ca}^b - \delta_a^b C_{dc}^d) = \kappa T_a^0 \quad (7.39b)$$

$$R_b^a = \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial t} (\sqrt{\eta} K_b^a) - {}^3R_b^a = \kappa \left( T_b^a - \frac{1}{2} \delta_b^a T \right), \quad (7.39c)$$

where the relation  $K_{ab} = -\partial_t \eta_{ab}/2$  holds. The triad components of the Ricci tensor  ${}^3R_{ab}$  becomes (2.112b)

$$\begin{aligned} {}^3R_{ab} = -\frac{1}{2} \Big( & C_{cb}^d C_{da}^c + C_{db}^c C_{ca}^d - \frac{1}{2} C_b^{cd} C_{acd} \\ & - C_{cd}^c C_{ab}^d + C_{cd}^c C_{ba}^d \Big). \end{aligned} \quad (7.40)$$

The dynamics of the homogeneous (but anisotropic) Universes will be investigated in detail in the following Sections.

## 7.2 Kasner Solution

The simplest solution of the Einstein equations (7.39) in the framework of the Bianchi classification is the so-called *Kasner solution*, i.e. the vacuum type I model.

The simultaneous vanishing of the three structure constants and of the parameter  $a$  implies the vanishing of the three-dimensional Ricci tensor

$$\left. \begin{array}{l} e^a_\alpha = \delta^a_\alpha \\ C^c_{ab} \equiv 0 \end{array} \right\} \Rightarrow {}^3R_{ab} = 0. \quad (7.41)$$

Furthermore, since the three-dimensional metric tensor does not depend on space coordinates, also  $R_{0\alpha} = 0$  holds, as confirmed by Eq. (7.39b); we stress that this model contains the standard Euclidean space as a particular case. The system (7.39) describes a uniform space and reduces to

$$\dot{K}^a_a + K^b_a K^a_b = 0, \quad (7.42a)$$

$$\frac{1}{\sqrt{\eta}} \frac{\partial}{\partial t} (\sqrt{\eta} K^b_a) = 0. \quad (7.42b)$$

From Eq. (7.42b) we get the first integral

$$\sqrt{\eta} K^b_a = \zeta^b_a = \text{const.}, \quad (7.43)$$

and contraction of indices  $a$  and  $b$  leads to

$$K^a_a = \frac{\dot{\eta}}{2\eta} = \frac{\zeta^a_a}{\sqrt{\eta}}, \quad (7.44)$$

and finally

$$\eta = (\zeta^a_a)^2 t^2. \quad (7.45)$$

Without loss of generality, a simple rescaling of the coordinates  $x^\alpha$  allows one to set

$$\zeta^a_a = 1. \quad (7.46)$$

Substituting Eq. (7.43) into Eq. (7.42a), one obtains the relations among the constants  $\zeta^b_a$

$$\zeta^a_b \zeta^b_a = 1, \quad (7.47)$$

and lowering index  $b$  in Eq. (7.43) one gets a system of ordinary differential equations in terms of  $\eta_{ab}$

$$\dot{\eta}_{ab} = \frac{2}{t} \zeta^c_a \eta_{cb}. \quad (7.48)$$

The set of coefficients  $\zeta^c_a$  can be considered as the matrix of a certain linear transformation, reducible to its diagonal form. In such a case, denoting its eigenvalues as  $(p_l, p_m, p_n) \in \mathbb{R}$ , and its eigenvectors as  $\mathbf{l}, \mathbf{m}, \mathbf{n}$  the solution of (7.48) writes as

$$\eta_{ab} = t^{2p_l} l_a l_b + t^{2p_m} n_a n_b + t^{2p_n} m_a m_b. \quad (7.49)$$

If we choose the frame of eigenvectors as the triad basis (recall that  $e_\alpha^a = \delta_\alpha^a$ ) and label the coordinates as  $x^1, x^2, x^3$ , then the spatial line element reduces to

$$dl^2 = t^{2p_l}(dx^1)^2 + t^{2p_m}(dx^2)^2 + t^{2p_n}(dx^3)^2. \quad (7.50)$$

Here  $p_l, p_m, p_n$  are the so-called *Kasner indices*, satisfying the two relations

$$p_l + p_m + p_n = 1 \quad (7.51a)$$

$$p_l^2 + p_m^2 + p_n^2 = 1, \quad (7.51b)$$

and therefore there is only one independent parameter characterizing the solution. Equation (7.51a) follows from relation (7.46), while Eq. (7.51b) latter comes from Eq. (7.47). Except for the cases  $(0, 0, 1)$  and  $(-1/3, 2/3, 2/3)$ , such indices are never equal to each other, and one of them is negative while two are positive; in the peculiar case  $p_l = p_m = 0, p_n = 1$ , the metric is reducible to a Minkowskian form by the transformation

$$t \sinh x^3 = \xi, \quad t \cosh x^3 = \tau. \quad (7.52)$$

It is worth noting that in this particular case, the singularity in  $t = 0$  is a fictitious one.

Once that Kasner indices have been ordered according to

$$p_1 < p_2 < p_3, \quad (7.53)$$

their corresponding variation ranges are

$$-\frac{1}{3} \leq p_1 \leq 0, \quad 0 \leq p_2 \leq \frac{2}{3}, \quad \frac{2}{3} \leq p_3 \leq 1. \quad (7.54)$$

This ordered set of indices admits the following parametrization

$$p_1(u) = \frac{-u}{1+u+u^2} \quad p_2(u) = \frac{1+u}{1+u+u^2} \quad p_3(u) = \frac{u(1+u)}{1+u+u^2} \quad (7.55)$$

as the parameter  $u$  varies in the range (see Fig. 7.1)

$$1 \leq u < +\infty. \quad (7.56)$$

The values  $u < 1$  lead to the same range by following the inversion property

$$p_1\left(\frac{1}{u}\right) = p_1(u), \quad p_2\left(\frac{1}{u}\right) = p_3(u), \quad p_3\left(\frac{1}{u}\right) = p_2(u). \quad (7.57)$$

The line element from Eq. (7.49) describes an anisotropic space where volumes linearly grow with time, while linear distances grow along two directions and decrease along the third one, different from the Friedmann solution where all distances contract towards the singularity with the same behavior. This metric has only one non-eliminable singularity in  $t = 0$  with the single exception of the case  $p_1 = p_2 = 0, p_3 = 1$  mentioned above, corresponding to the standard Euclidean space.

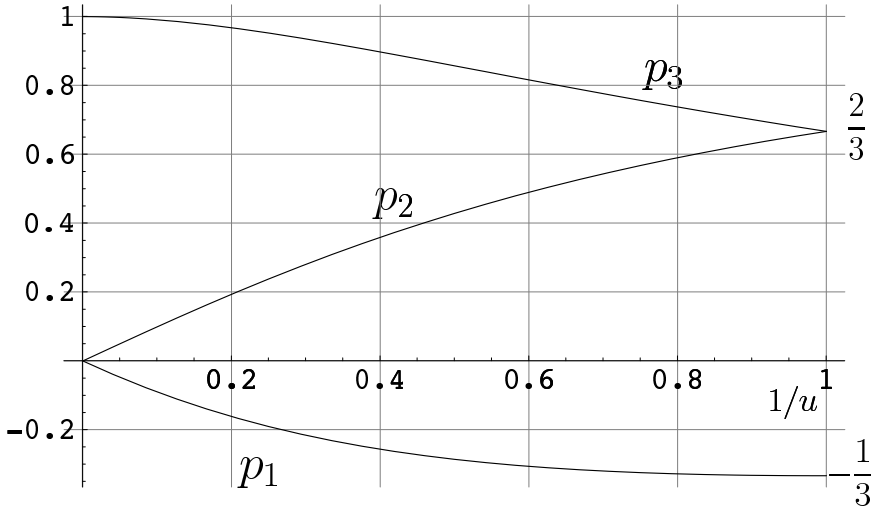


Figure 7.1 Behavior of Kasner indices in terms of the parameter  $1/u$ . The domain of  $u$  is  $[1, \infty)$ ; for lower values of  $u$  the inversion property (7.57) holds.

### 7.2.1 The role of matter

Here we discuss the temporal evolution of a uniform distribution of matter in the Bianchi type I space near the singularity; it will result in behaving as a test fluid and thus not affecting the properties of the solution.

Let us take a uniform distribution of matter and assume that its influence on the gravitational field can be neglected in a certain region of evolution. The hydrodynamics equations that describe its evolution in terms of a perfect fluid are (2.18) and the spatial components of (2.19) (see Sec. 2.2.1) read as

$$\nabla_k [(\rho + P) u^k] = u^i \partial_i P. \quad (7.58)$$

$$u^k \nabla_k u_\alpha = \frac{1}{(\rho + P)} (\partial_\alpha P - u_\alpha u^k \partial_k P). \quad (7.59)$$

In the neighborhood of the singularity, it is necessary to use the ultra-relativistic equation of state  $P = \rho/3$ , and then we get that Eq. (7.58) becomes

$$\nabla_k (\rho^{3/4} u^k) = 0. \quad (7.60)$$

By using the metric (7.50), and imposing that all the functions depend on the time variable only, we get

$$\nabla_k (\rho^{3/4} u^k) = \frac{1}{\sqrt{-g}} \partial_k (\sqrt{-g} \rho^{3/4} u^k) = \frac{1}{t} \frac{d}{dt} (t \rho^{3/4} u^0) = 0 \quad (7.61)$$

coupled to Eq. (7.59); the final system stands as

$$\frac{d}{dt} \left( tu^0 \rho^{3/4} \right) = 0, \quad (7.62a)$$

$$4\rho \frac{du_\alpha}{dt} + u_\alpha \frac{d\rho}{dt} = 0. \quad (7.62b)$$

From Eq. (7.62), we obtain the two integrals of motion

$$tu^0 \rho^{3/4} = \text{const.}, \quad u_\alpha \rho^{1/4} = \text{const.} \quad (7.63)$$

From Eq. (7.63) we see that all the covariant components  $u_\alpha$  are of the same order of magnitude. Among the contravariant ones, when  $t \rightarrow 0$  the greatest is  $u^3 = u_3 t^{-2p_3}$  ( $p_1 < p_2 < p_3$ ). Retaining only the dominant contribution in the identity  $u_i u^i = 1$ , we have  $u_0^2 \approx u_3^2 t^{-2p_3}$  and, from Eq. (7.63),

$$\rho \sim t^{-2(p_1+p_2)} = t^{-2(1-p_3)}, \quad u_\alpha \sim t^{(1-p_3)/2}. \quad (7.64)$$

As expected,  $\rho$  diverges as  $t \rightarrow 0$  for all values of  $p_3$ , except for  $p_3 = 1$  (this is due to the non-physical character of the singularity in this case). The validity of the test fluid approximation is verified from a direct evaluation of the components of the energy-momentum tensor  $T_i^k$ , whose dominant terms are (2.14)

$$T_0^0 \sim \rho u_0^2 \sim t^{-(1+p_3)}, \quad T_1^1 \sim \rho \sim t^{-2(1-p_3)}, \quad (7.65a)$$

$$T_2^2 \sim \rho u_2 u^2 \sim t^{-(1+2p_2-p_3)}, \quad T_3^3 \sim \rho u_3 u^3 \sim t^{-(1+p_3)}. \quad (7.65b)$$

As  $t \rightarrow 0$ , all the components grow slower than  $t^{-2}$ , which is the behavior of the dominant terms in the Kasner analysis. Thus, the fluid contribution can be disregarded in the Einstein equations near the singularity.

This test character of a perfect fluid on a Kasner background remains valid even in the Mixmaster scenario, both in the homogeneous as well as in the inhomogeneous case. The reason for the validity of such an extension relies on the piecewise Kasner behavior of the oscillatory regime.

### 7.3 The Dynamics of the Bianchi Models

The Kasner solution properly approximates the cases when the Ricci tensor  ${}^3R_{\alpha\beta}$  appearing in the Einstein equations is of order higher than  $1/t^2$  and thus negligible. However, since one of the Kasner exponents is negative, terms dominant with respect to  $t^{-2}$  appear in the tensor  ${}^3R_{\alpha\beta}$  and make the Kasner solution unstable toward the initial singularity.

Let us introduce three spatial vectors  $e^a = \{l(x^\gamma), m(x^\gamma), n(x^\gamma)\}$  satisfying the homogeneity conditions (7.32) and take the matrix  $h_{\alpha\beta}$  in the diagonal form

$$h_{\alpha\beta} = a^2(t)l_\alpha l_\beta + b^2(t)m_\alpha m_\beta + c^2(t)n_\alpha n_\beta. \quad (7.66)$$

Such vectors are called *Kasner vectors*, while  $a(t), b(t), c(t)$  are three different cosmic scale factors. Consequently, the Einstein equations in a synchronous reference and for a generic homogeneous cosmological model in empty space are given by the system

$$-R_l^l = \frac{(\dot{a}bc)}{abc} + \frac{1}{2a^2b^2c^2} \left[ \lambda_l^2 a^4 - (\lambda_m b^2 - \lambda_n c^2)^2 \right] = 0 \quad (7.67a)$$

$$-R_m^m = \frac{(\dot{a}\dot{b}c)}{abc} + \frac{1}{2a^2b^2c^2} \left[ \lambda_m^2 b^4 - (\lambda_l a^2 - \lambda_n c^2)^2 \right] = 0 \quad (7.67b)$$

$$-R_n^n = \frac{(\dot{a}b\dot{c})}{abc} + \frac{1}{2a^2b^2c^2} \left[ \lambda_n^2 c^4 - (\lambda_l a^2 - \lambda_m b^2)^2 \right] = 0 \quad (7.67c)$$

$$-R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0 \quad (7.68)$$

where the other off-diagonal components of the four-dimensional Ricci tensor identically vanish due to the diagonal form of  $\eta_{ab}$  as in Eq. (7.66). Eventually, the  $0\alpha$  components of the Einstein equations can be non-zero if some kind of matter is present, leading to an effect of rotation on the Kasner axes. The constants  $(\lambda_l, \lambda_m, \lambda_n)$  correspond to the structure constants  $C_{11}, C_{22}, C_{33}$  respectively, introduced earlier in Sec. 7.1.2. In particular, we will study in details the cases of the models II, VII, VIII and IX of the Bianchi classification, which correspond to the triplets  $(1, 0, 0)$ ,  $(1, 1, 0)$ ,  $(1, 1, -1)$  and  $(1, 1, 1)$  respectively.

Through the notation

$$\alpha = \ln a, \quad \beta = \ln b, \quad \gamma = \ln c \quad (7.69)$$

and the new temporal variable  $\tau$  defined as

$$dt = abc \, d\tau, \quad (7.70)$$

Eqs. (7.67) and (7.68) simplify to

$$2\alpha_{\tau\tau} = (\lambda_m b^2 - \lambda_n c^2)^2 - \lambda_l^2 a^4 \quad (7.71a)$$

$$2\beta_{\tau\tau} = (\lambda_l a^2 - \lambda_n c^2)^2 - \lambda_m^2 b^4 \quad (7.71b)$$

$$2\gamma_{\tau\tau} = (\lambda_l a^2 - \lambda_m b^2)^2 - \lambda_n^2 c^4, \quad (7.71c)$$

$$\frac{1}{2}(\alpha + \beta + \gamma)_{\tau\tau} = \alpha_{\tau}\beta_{\tau} + \alpha_{\tau}\gamma_{\tau} + \beta_{\tau}\gamma_{\tau}, \quad (7.72)$$

where subscript  $\tau$  denotes the derivative with respect to  $\tau$ . After some algebra on the system (7.71) and using (7.72), one obtains the first integral

$$\begin{aligned} & \alpha_{\tau}\beta_{\tau} + \alpha_{\tau}\gamma_{\tau} + \beta_{\tau}\gamma_{\tau} \\ &= \frac{1}{4} \left( \lambda_l^2 a^4 + \lambda_m^2 b^4 + \lambda_n^2 c^4 - 2\lambda_l\lambda_m a^2 b^2 - 2\lambda_l\lambda_n a^2 c^2 - 2\lambda_m\lambda_n b^2 c^2 \right) \end{aligned} \quad (7.73)$$

involving first derivatives only. The Kasner regime (7.50) discussed before is the solution corresponding to neglecting all terms on the right-hand side of Eqs. (7.71). However, such behavior cannot persist indefinitely as  $t \rightarrow 0$  since there are always some terms on the right-hand side of Eq. (7.71) which are increasing and not negligible up to the singularity.

### 7.3.1 *Bianchi type II: The concept of Kasner epoch*

Introducing the structure constants for the type II model (see Table 7.1), the system (7.67) and Eq. (7.68) reduce to

$$\frac{(\dot{a}bc)_{\cdot}}{abc} = -\frac{a^2}{2b^2c^2}, \quad (7.74a)$$

$$\frac{(\dot{a}\dot{b}c)_{\cdot}}{abc} = \frac{a^2}{2b^2c^2}, \quad (7.74b)$$

$$\frac{(ab\dot{c})_{\cdot}}{abc} = \frac{a^2}{2b^2c^2}, \quad (7.74c)$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0. \quad (7.75)$$

In Eqs. (7.74), the right-hand sides play the role of a perturbation to the Kasner regime; if at a certain instant of time  $t$  they could be neglected, then a Kasner dynamics would take place. This kind of evolution can be stable or not depending on the initial conditions. As shown earlier in Sec. 7.2, the Kasner dynamics has a time evolution which differs along the three directions, growing along two of them and decreasing along the other. For example, for a perturbation growing as  $a^4 \sim t^{4p_l}$  toward the singularity, two scenarios are possible: if the perturbation is associated with one of the two positive indices, it will continue decreasing till the singularity and the Kasner epoch is stable; on the other hand, if  $p_l < 0$ , the perturbation grows

and cannot be indefinitely neglected. In this case, the analysis of the full dynamical system is required, and with the use of the logarithmic variables (7.69)-(7.70) the system (7.71) becomes

$$\alpha_{\tau\tau} = -\frac{1}{2}e^{4\alpha} \quad (7.76a)$$

$$\beta_{\tau\tau} = \gamma_{\tau\tau} = \frac{1}{2}e^{4\alpha}. \quad (7.76b)$$

Equation (7.76a) can be viewed as the motion of a one-dimensional point-particle moving within an exponential potential well: if the initial “velocity”  $d\alpha/d\tau$  is equal to  $p_l$ , then the effect of the potential will result in a slowing down behavior, stopping and accelerating again the point up to a new “velocity”  $-p_l$ . From there on, the potential will remain negligible forever. Furthermore, the second set of equations (7.76b) implies that the conditions

$$\alpha_{\tau\tau} + \beta_{\tau\tau} = \alpha_{\tau\tau} + \gamma_{\tau\tau} = 0 \quad (7.77)$$

hold. Let us consider the explicit solutions of Eq. (7.77)

$$\alpha(\tau) = \frac{1}{2} \ln(c_1 \operatorname{sech}(\tau c_1 + c_2)) \quad (7.78a)$$

$$\beta(\tau) = c_3 + \tau c_4 - \frac{1}{2} \ln(c_1 \operatorname{sech}(\tau c_1 + c_2)) \quad (7.78b)$$

$$\gamma(\tau) = c_5 + \tau c_6 - \frac{1}{2} \ln(c_1 \operatorname{sech}(\tau c_1 + c_2)), \quad (7.78c)$$

where  $c_1 \dots, c_6$  are integration constants. Let us analyze how the solution (7.78) behaves when the time variable  $\tau$  approaches  $+\infty$  and  $-\infty$ ; remembering that

$$dt = abc d\tau = \Lambda d\tau = \exp(\alpha + \beta + \gamma) d\tau$$

we have that

$$\lim_{\tau \rightarrow +\infty} \begin{cases} \alpha_\tau = -c_1/2 \\ \beta_\tau = c_4 + c_1/2 \\ \gamma_\tau = c_6 + c_1/2 \\ t = \frac{1}{\Lambda} \exp(\Lambda\tau) \\ \Lambda = c_4 + c_6 + c_1/2 \end{cases} \quad \lim_{\tau \rightarrow -\infty} \begin{cases} \alpha_\tau = c_1/2 \\ \beta_\tau = c_4 - c_1/2 \\ \gamma_\tau = c_6 - c_1/2 \\ t = \frac{1}{\Lambda'} \exp(\Lambda'\tau) \\ \Lambda' = c_4 + c_6 - c_1/2 \end{cases} \quad (7.79)$$

This means that



$$\lim_{t \rightarrow +\infty} \begin{cases} a(t) = t^{p_l} \\ b(t) = t^{p_m} \\ c(t) = t^{p_n} \end{cases} \quad \lim_{t \rightarrow 0} \begin{cases} a(t) = t^{p'_l} \\ b(t) = t^{p'_m} \\ c(t) = t^{p'_n} \end{cases} \quad (7.80)$$

where we have identified

$$p_l = \frac{-c_1/2}{c_4 + c_6 + c_1/2}, \quad p_m = \frac{c_4 + c_1/2}{c_4 + c_6 + c_1/2}, \quad p_n = \frac{c_6 + c_1/2}{c_4 + c_6 + c_1/2} \quad (7.81)$$

$$p'_l = \frac{c_1/2}{c_4 + c_6 - c_1/2}, \quad p'_m = \frac{c_4 - c_1/2}{c_4 + c_6 - c_1/2}, \quad p'_n = \frac{c_6 - c_1/2}{c_4 + c_6 - c_1/2}. \quad (7.82)$$

These coefficients satisfy the two Kasner relations (7.51): the first follows directly by (7.81) and (7.82), while the latter is obtained when the asymptotic behaviors (7.79) are substituted in (7.72).

We see how this dynamical scheme describes two connected *Kasner epochs* (a Kasner epoch is defined as the period of time during which the solution is well approximated by a Kasner metric and the potential terms are negligible), where the perturbation has the role of changing the values of the Kasner indices. Let us assume that the Universe is initially described by a Kasner epoch for  $\tau \rightarrow +\infty$ , with indices ordered as  $p_l < p_m < p_n$ . The perturbation (which is the term on the r.h.s. of Eq. (7.74a)) starts growing and the Universe undergoes a transition due to the potential term. Then a new Kasner epoch begins, where the old and the new indices (the primed ones) are related among them by the so-called *BKL map*

$$p'_l = \frac{|p_l|}{1 - 2|p_l|}, \quad p'_m = -\frac{2|p_l| - p_m}{1 - 2|p_l|}, \quad p'_n = \frac{p_n - 2|p_l|}{1 - 2|p_l|}, \quad (7.83)$$

$$\Lambda' = (1 - 2|p_l|)\Lambda.$$

The main feature of such a map is the exchange of the negative index between two different directions. This way, in the new epoch, the negative power is no longer related to the  $l$ -direction and the perturbation is damped and vanishes toward the singularity, accordingly to a stable Kasner regime.

### 7.3.2 Bianchi type VII: The concept of Kasner era

The analysis of Bianchi type VII can be performed analogously, considering the Einstein equations (7.67) with  $(\lambda_l, \lambda_m, \lambda_n) = (1, 1, 0)$

$$\frac{(\dot{a}bc)}{abc} = \frac{-a^4 + b^4}{2a^2b^2c^2}, \quad (7.84a)$$

$$\frac{(\dot{a}\dot{b}c)}{abc} = \frac{a^4 - b^4}{2a^2b^2c^2}, \quad (7.84b)$$

$$\frac{(ab\dot{c})}{abc} = \frac{(a^2 - b^2)^2}{2b^2c^2}, \quad (7.84c)$$

together with the constraint (7.68) holding unchanged. Comparison of Eq. (7.84) with Eq. (7.74) allows a similar qualitative analysis: if the right-hand sides of Eq. (7.84) are negligible at a certain instant of time, then the solution is Kasner-like and can be stable or unstable toward the singularity  $t \rightarrow 0$  depending on the initial conditions. If the negative index is associated with the  $n$  direction, then the perturbative terms  $a^4$  and  $b^4$ , evolving as  $t^{4p_l}$  and  $t^{4p_m}$ , decrease up to the singularity and the Kasner solution turns out to be stable; in all other cases, one and only one of the perturbation terms starts growing, blasting the initial Kasner evolution and ending as before in a new Kasner epoch. The main difference between the types II and VII is that many other transitions can occur after the first one, and this can happen, for example, if the new negative Kasner index is associated with the  $m$  direction, i.e. with  $b$ . In this case, the  $b^4$  term would start growing and a new transition would occur with the same law (7.83). The problem of understanding if, when and how this mechanism can break down is unraveled considering the BKL map written in terms of the parameter  $u$ , i.e.

$$\left. \begin{array}{l} p_l = p_1(u) \\ p_m = p_2(u) \\ p_n = p_3(u) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} p'_l = p_2(u-1) \\ p'_m = p_1(u-1) \\ p'_n = p_3(u-1) \end{array} \right. \quad (7.85)$$

Let us represent the initial values of  $u$  describing the dynamics as

$$u_0 = k_0 + x_0, \quad (7.86)$$

where  $k_0$  represents its integer part while  $x_0$  the fractional one (rational or irrational). In this representation and from the properties (7.55) and (7.57) we see how the exact number of exchanges between the  $l$ - and  $m$ -directions equates  $k_0$ . For the first  $k_0$  times, the negative index is exchanged between  $l$  and  $m$  and only afterwards it passes to the  $n$  direction. Now, a new and

final Kasner epoch begins and no more oscillations take place toward the singularity. In fact, when  $u < 2$ , the next value prescribed by the BKL map corresponds to the fractional part  $x_0$  of  $u_0$ . It is easy to restate the parameter  $u$  in its natural interval  $[1, \infty)$  by the replacement  $u_{\text{new}} = 1/x_0$  and making use of Eq. (7.57). The collection of the total  $k_0$  epochs is called a *Kasner era* during which one of the three cosmic scale factors (say, for example,  $c$ ) decreases monotonically toward the singularity: in this sense we can say that, in the general case, the type VII dynamics is composed by one era plus a final epoch.

## 7.4 Bianchi Types VIII and IX Models

### 7.4.1 The oscillatory regime

At this point we are going to address the solution of the system (7.67) for the cases of Bianchi types VIII and IX cosmological models, following the standard approach of Belinskii, Khalatnikov and Lifshitz (BKL). Although the detailed discussion is devoted to the Bianchi IX model, it can be straightforwardly extended to the type VIII.

Explicitly, the Einstein equations (7.67) reduce to

$$2\alpha_{\tau\tau} = (b^2 - c^2)^2 - a^4 \quad (7.87a)$$

$$2\beta_{\tau\tau} = (a^2 - c^2)^2 - b^4 \quad (7.87b)$$

$$2\gamma_{\tau\tau} = (a^2 - b^2)^2 - c^4, \quad (7.87c)$$

together with the constraint (7.72) unchanged, leading to the consequent first integral

$$\alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau = \frac{1}{4} (a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2) . \quad (7.88)$$

Let us therefore consider again the case in which, for instance, the negative power of the  $p_i$  exponents corresponds to the function  $a(t)$ : the perturbation of the Kasner regime results from the terms as  $\lambda_l^2 a^4$  (remember that  $\lambda_l = 1$  for both models) while the other terms decrease with decreasing  $t$ . Preserving only the increasing terms on the right-hand side of Eqs. (7.87) we obtain a system identical to Eqs. (7.76a), whose solution describes the evolution of the metric from its initial state (7.50). We have that if

$$a \sim t^{p_1}, \quad b \sim t^{p_2}, \quad c \sim t^{p_3}, \quad (7.89)$$

then

$$\begin{aligned} abc &= \Lambda t \\ \tau &= \frac{1}{\Lambda} \ln t + \text{const.} \end{aligned} \quad (7.90)$$

where  $\Lambda$  is a constant, so that the initial conditions for Eq. (7.76a) can be formulated as

$$\begin{aligned} \alpha_\tau &= \Lambda p_1, \quad \beta_\tau = \Lambda p_2, \quad \gamma_\tau = \Lambda p_3, \\ \text{as } \tau &\rightarrow \infty. \end{aligned} \quad (7.91)$$

The system (7.76) with Eq. (7.91) is integrated to

$$a^2 = \frac{2 |p_1| \Lambda}{\cosh(2 |p_1| \Lambda \tau)} \quad (7.92a)$$

$$b^2 = b_0^2 \exp[2\Lambda(p_2 - |p_1|)\tau] \cosh(2 |p_1| \Lambda \tau) \quad (7.92b)$$

$$c^2 = c_0^2 \exp[2\Lambda(p_3 - |p_1|)\tau] \cosh(2 |p_1| \Lambda \tau) \quad (7.92c)$$

where  $b_0$  and  $c_0$  are integration constants.

Let us consider the solutions (7.92) in the limit  $\tau \rightarrow \infty$ : towards the singularity they simplify to

$$a \sim \exp[-\Lambda p_1 \tau] \quad (7.93a)$$

$$b \sim \exp[\Lambda(p_2 + 2p_1)\tau] \quad (7.93b)$$

$$c \sim \exp[\Lambda(p_3 + 2p_1)\tau] \quad (7.93c)$$

$$t \sim \exp[\Lambda(1 + 2p_1)\tau] \quad (7.93d)$$

that is to say, in terms of  $t$ ,

$$a \sim t^{p'_l}, \quad b \sim t^{p'_m}, \quad c \sim t^{p'_n}, \quad abc = \Lambda' t, \quad (7.94)$$

where the primed exponents are related to the un-primed ones by

$$p'_l = \frac{|p_1|}{1 - 2 |p_1|}, \quad p'_m = -\frac{2 |p_1| - p_2}{1 - 2 |p_1|}, \quad (7.95a)$$

$$p'_n = \frac{p_3 - 2 |p_1|}{1 - 2 |p_1|}, \quad \Lambda' = (1 - 2 |p_1|) \Lambda, \quad (7.95b)$$

which, clearly, are the same as in the type II case (7.83). Summarizing these results, we see the effect of the perturbation over the Kasner regime: a Kasner epoch is replaced by another one so that the negative power of  $t$  is transferred from the **l** to the **m** direction, i.e. if in the original solution  $p_l$  is negative, in the new solution  $p'_m < 0$ . The previously increasing perturbation  $\lambda_l^2 a^4$  in Eq. (7.67) is damped and eventually vanishes. The

other terms involving  $\lambda_m^2$  instead of  $\lambda_l^2$  will grow, therefore permitting the replacement of a Kasner epoch by another. Such rules of rotation in the perturbing scheme can be summarized with the rules (7.85) of the BKL map, with the greatest of the two positive powers still remaining positive.

The following interchanges are characterized by a sequence of bounces, with an exchange of the negative power between the directions **l** and **m** continuing as long as the integer part of the initial value of  $u$  is not exhausted, i.e. until  $u$  becomes less than unity. Hence, according to Eq. (7.57), the value  $u < 1$  is turned into  $u > 1$ , thus either the exponent  $p_l$  or  $p_m$  is negative and  $p_n$  becomes the smallest of the two positive numbers, say  $p_n = p_2$ . The next sequence of changes will switch the negative power between the directions **n** and **l** or **n** and **m**. In terms of the parameter  $u$ , the map (7.83) takes the form

$$u' = \begin{cases} u - 1 & \text{for } u > 2, \\ \frac{1}{u - 1} & \text{for } u \leq 2. \end{cases} \quad (7.96)$$

The phenomenon of increasing and decreasing of the various terms with transition from a Kasner era to another is repeated infinitely many times up to the singularity.

Let us analyze the implications of the BKL map (7.96) and of the property (7.57).

If we write  $u^0 = k^0 + x^0$  as the initial value of the parameter  $u$ , being  $k^0$  and  $x^0$  its integral and fractional part, the continuous exchange of shrinking and enlarging directions (7.85) proceeds until  $u < 1$ , i.e. it lasts for  $k^0$  epochs, thus leading a *Kasner era* to an end. The new value of  $u$  is  $u' = 1/x^0 > 1$ , with the Kasner indices transforming as in Eq. (7.57), and the subsequent set of exchanges will be  $l \rightarrow n$  or  $m \rightarrow n$ . For an arbitrary, irrational initial value of  $u$  the changes in Eq. (7.85) repeat indefinitely. In the case of an exact solution, the exponents  $p_l$ ,  $p_m$  and  $p_n$  lose their literal meaning thus, in general, it has no sense to consider any exactly defined value of  $u$ , such as for example a rational one.

The evolution of the model towards the singularity consists of successive eras, in which distances along two axes oscillate and along the third axis monotonically decrease while the volume always decreases (approximately) linearly with the synchronous time  $t$ . The order in which the pairs of axes are interchanged and the order in which eras of different lengths (number of Kasner epochs contained in it) follow each other acquire a stochastic character. Successive eras ‘condense’ towards the singularity. Such general

qualitative properties are not changed in the case of space filled with matter according to the analysis of Sec. 7.2.1.

#### 7.4.2 Stochastic properties and the Gaussian distribution

A decreasing sequence of values of the parameter  $u$  corresponds to every  $s$ -th era there. This sequence, from the starting era has the form  $u_{\max}^{(s)}, u_{\max}^{(s)} - 1, u_{\max}^{(s)} - 2, \dots, u_{\min}^{(s)}$ . We can introduce the notation

$$u^{(s)} = k^{(s)} + x^{(s)} \quad (7.97)$$

then

$$u_{\min}^{(s)} = x^{(s)} < 1, \quad u_{\max}^{(s)} = k^{(s)} + x^{(s)}, \quad (7.98)$$

where  $u_{\max}^{(s)}$  is the greatest value of  $u$  for an assigned era and  $k^{(s)} = \left[ u_{\max}^{(s)} \right]$  (square brackets denote the greatest integer less than or equal to  $u_{\max}^{(s)}$ ). The number  $k^{(s)}$  denotes the length of the  $s$ -th era, i.e. the number of Kasner epochs contained in it. For the next era we obtain

$$u_{\max}^{(s+1)} = \frac{1}{x^{(s)}}, \quad k^{(s+1)} = \left[ \frac{1}{x^{(s)}} \right]. \quad (7.99)$$

If the sequence begins as  $k^{(0)} + x^{(0)}$ , the lengths  $k^{(1)}, k^{(2)}, \dots$  are the numbers appearing in the expansion for  $x^{(0)}$  in terms of the continuous fraction

$$x^{(0)} = \frac{1}{k^{(1)} + \frac{1}{k^{(2)} + \frac{1}{k^{(3)} + \dots}}}, \quad (7.100)$$

which is finite if related to a rational number, but in general it is an infinite one.

For the infinite sequence of positive numbers  $u$  ordered as in Eq. (7.99) and admitting the expansion (7.100) it is possible to note that

- (i) a rational number would have a finite expansion;
- (ii) a periodic expansion represents quadratic irrational numbers (i.e. numbers which are roots of quadratic equations with integral coefficients);
- (iii) irrational numbers have an infinite expansion.

The first two cases correspond to sets of zero measure in the space of possible initial conditions.

Since it can be easily checked that the sequence of  $k$ -values in the continuous fraction expansion  $x^0$  is extremely unstable with respect to the initial conditions, and taking into account the approximate nature of the piecewise Kasner representation of this oscillatory regime, we are led to address a statistical description.

An appropriate framework arises from studying the statistical distribution of the eras' sequence and from the analysis of the random properties of the numbers  $x^{(0)}$  over the interval  $(0, 1)$ . For the series  $x^{(s)}$  with increasing  $s$  there exists a limiting, stationary distribution  $w(x)$ , independent of  $s$ , in which the initial conditions are completely forgotten. In fact, instead of a well-defined initial value as in Eq. (7.97) with  $s = 0$ , let us consider a probability distribution for  $x^{(0)}$  over the interval  $(0, 1)$ ,  $W_0(x)$  for  $x^{(0)} = x$ . Then also the numbers  $x^{(s)}$  are distributed with some probability law. Let  $w_s(x)dx$  be the probability that the last term in the  $s$ -th series  $x^{(s)} = x$  lies in the interval  $dx$ . The last term of the  $s - 1$  series must lie in the interval between  $1/(k + 1)$  and  $1/k$ , in order for the length of the  $s$ -th series to be  $k$ . The probability for the series to have length  $k$  is given by

$$W_s(k) = \int_{\frac{1}{1+k}}^{\frac{1}{k}} w_{s-1}(x)dx. \quad (7.101)$$

For each pair of subsequent series, we get the recurrence formula relating the distribution  $w_{s+1}(x)$  to  $w_s(x)$

$$w_{s+1}(x)dx = \sum_{k=1}^{\infty} w_s\left(\frac{1}{k+x}\right) \left|d\frac{1}{(k+x)}\right|, \quad (7.102)$$

or equivalently

$$w_{s+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} w_s\left(\frac{1}{k+x}\right). \quad (7.103)$$

If, for increasing  $n$ , the  $w_{s+n}$  distribution (7.103) tends to a stationary one independent of  $s$ , then  $w(x)$  has to satisfy

$$w(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} w\left(\frac{1}{k+x}\right). \quad (7.104)$$

A normalized solution to Eq. (7.104) is given by

$$w(x) = \frac{1}{(1+x)\ln 2}. \quad (7.105)$$

This can be easily verified by a direct substitution of Eq. (7.105) in Eq. (7.104), giving the identity

$$\begin{aligned}
 \frac{1}{1+x} &= \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} \frac{1}{1+\frac{1}{k+x}} = \sum_{k=1}^{\infty} \frac{1}{(k+x)(1+k+x)} \\
 &= \sum_{k=1}^{\infty} \left( \frac{1}{k+x} - \frac{1}{k+x+1} \right) \\
 &= \frac{1}{1+x} - \frac{1}{2+x} + \frac{1}{2+x} - \frac{1}{3+x} + \frac{1}{3+x} + \dots
 \end{aligned} \tag{7.106}$$

Substituting Eq. (7.105) into Eq. (7.101), we get the corresponding stationary distribution of the lengths of the series  $k$

$$W(k) = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}. \tag{7.107}$$

Finally, since in the stationary limit  $k$  and  $x$  are not independent (i.e.  $x \leftrightarrow 1/(k+x)$ ), they must admit a stationary joint probability distribution

$$w(k, x) = \frac{1}{(k+x)(k+x+1) \ln 2} \tag{7.108}$$

which, for  $u = k + x$ , rewrites as

$$w(u) = \frac{1}{u(u+1) \ln 2}, \tag{7.109}$$

i.e. a stationary distribution for the parameter  $u$ .

The analysis of these chaotic properties of the map (7.96) was firstly pursued in the work of Belinskii, Khalatnikov and Lifshitz at the end of the '60s. Let us finally summarize the fundamental properties exhibited by the Poincaré return map associated to the fractional part  $x$  of the parameter  $u$ , i.e.

$$x^{s+1} = \frac{1}{x^s} - \left\lceil \frac{1}{x^s} \right\rceil, \tag{7.110}$$

which can be easily derived by Eq. (7.96):

- it has positive metric- and topologic-entropy;
- it has the *weak Bernoulli properties* (i.e. the map cannot be finitely approximated);
- it is *ergodic* and *strongly mixing*.



It remains to be discussed separately the case  $u \gg 1$ , the so-called *small oscillations* regime, whose details will be discussed in Sec. 7.4.3.

In this phase, the Kasner exponents approach the values  $(0, 0, 1)$  with the limiting form

$$p_1 \approx -\frac{1}{u}, \quad p_2 \approx \frac{1}{u}, \quad p_3 \approx 1 - \frac{1}{u^2}. \quad (7.111)$$

The transition to the next era is governed by the fact that not all terms on the right hand side of Eq. (7.87) are negligible and two terms have to be simultaneously retained; in such case, the transition is accompanied by a long regime of small oscillations regarding two directions lasting until the next era. Only after this period of evolution a new series of Kasner epochs begins.

The probability  $\pi$  associated to the set of all possible values of  $x^{(0)}$  which lead to a dynamical evolution towards this specific case can be easily recognized to converge to a number  $\pi \ll 1$ . If the initial value of  $x^{(0)}$  is outside such set, the specific case cannot occur; if  $x^{(0)}$  lies within this interval, a characteristic evolution as small oscillations takes place, but after this period the model begins to regularly evolve with a new initial value  $x^{(0)}$ , which can only accidentally fall again in such an interval (with probability  $\pi$ ). The repetition of this situation can lead to these cases only with probabilities  $\pi, \pi^2, \dots$ , which asymptotically approach zero. Anyway, the dynamics associated to this particular behavior will be discussed in the next Subsection.

### 7.4.3 *Small oscillations*

Let us investigate in more detail a particular case of the solution constructed above. We analyze an era during which two of the three functions  $a, b, c$  (for example  $a$  and  $b$ ) oscillate so that their absolute values remain close to each other and the third function (in such case  $c$ ) monotonically decreases, being negligible with respect to  $a$  and  $b$ . As before, we will discuss only the Bianchi IX model, since for the Bianchi VIII case the arguments and results are qualitatively the same.

Let us consider the equations obtained from Eq. (7.71) and Eq. (7.73) imposing  $\exp(\gamma) \ll \exp(\alpha), \exp(\beta)$

$$\alpha_{\tau\tau} + \beta_{\tau\tau} = 0, \quad (7.112a)$$

$$\alpha_{\tau\tau} - \beta_{\tau\tau} = e^{4\beta} - e^{4\alpha}, \quad (7.112b)$$

$$\gamma_{\tau}(\alpha_{\tau} + \beta_{\tau}) = -\alpha_{\tau}\beta_{\tau} + \frac{1}{4}(e^{2\alpha} - e^{2\beta})^2. \quad (7.112c)$$

The solution of Eq. (7.112a) is

$$\alpha + \beta = \frac{2a_0^2}{\xi_0}(\tau - \tau_0) + 2 \ln(a_0), \quad (7.113)$$

where  $a_0$  and  $\xi_0$  are positive constants. In what follows we conveniently replace the time coordinate  $\tau$  with the new one  $\xi$  defined as

$$\xi = \xi_0 \exp \left( \frac{2a_0^2}{\xi_0}(\tau - \tau_0) \right) \quad (7.114)$$

in terms of which Eqs. (7.112b) and (7.112c) rewrite as

$$\chi_{\xi\xi} + \frac{1}{\xi}\chi_{\xi} + \frac{1}{2}\sinh(2\chi) = 0, \quad (7.115a)$$

$$\gamma_{\xi} = -\frac{1}{4\xi} + \frac{\xi}{8}(2\chi_{\xi}^2 + \cosh(2\chi) - 1), \quad (7.115b)$$

where we have introduced the notation  $\chi = \alpha - \beta$  and  $(\ )_{\xi} \equiv d(\ )/d\xi$ . Since  $\tau$  is defined in the interval  $(-\infty, \tau_0]$ , from Eq. (7.114) we have  $\xi \in (0, \xi_0]$ . Provided that a general analytic solution for the system (7.115) is not available, we shall consider the two limiting cases  $\xi \gg 1$  and  $\xi \ll 1$  only.

Let us start with the  $\xi \gg 1$  region. In this approximation, the solution of Eq. (7.115a) reads as

$$\chi = \frac{2A}{\sqrt{\xi}} \sin(\xi - \xi_0), \quad (7.116)$$

$A$  being a constant and therefore leading to  $\gamma \sim A^2(\xi - \xi_0)$ . As we can see, the name “small oscillations” arises from the behavior of the function  $\chi$ . The functions  $a$  and  $b$ , i.e. the expressions of the scale factors, are straightforwardly obtained as

$$a, b = a_0 \sqrt{\frac{\xi}{\xi_0}} \left( 1 \pm \frac{A}{\sqrt{\xi}} \sin(\xi - \xi_0) \right), \quad (7.117a)$$

$$c = c_0 \exp [-A^2(\xi_0 - \xi)]. \quad (7.117b)$$

The synchronous time coordinate  $t$  can be obtained from the relation  $dt = abc d\tau$  as

$$t = t_0 \exp [-A^2(\xi_0 - \xi)]. \quad (7.118)$$

Of course, these solutions only apply when the condition  $c_0 \ll a_0$  is satisfied.

Let us discuss the region where  $\xi \ll 1$ . In such a limit, the function  $\chi$  reads as

$$\chi = K \ln \xi + \theta, \quad \theta = \text{const.}, \quad (7.119)$$

where  $K$  is a constant which, for consistency, is constrained in the interval  $K \in (-1, 1)$ . We can therefore derive all the other related quantities, and in particular

$$a \sim \xi^{(1+K)/2}, \quad b \sim \xi^{(1-K)/2}, \quad (7.120a)$$

$$c \sim \xi^{-(1-K^2)/4}, \quad t \sim \xi^{(3+K^2)/2}. \quad (7.120b)$$

This is again a Kasner solution, with the negative power of  $t$  corresponding to the function  $c$  and the evolution is the same as the general one. Moreover, we can easily note how, for such a Kasner epoch, the parameter  $u$  becomes

$$\text{for } K > 0: \quad u = \frac{1+K}{1-K}, \quad \text{for } K < 0: \quad u = \frac{1-K}{1+K}. \quad (7.121)$$

Summarizing, the system initially crosses a long time interval during which the functions  $a$  and  $b$  satisfy  $(a-b)/a < 1/\xi$  and performs small oscillations of constant period  $\Delta\xi = 2\pi$ , while the function  $c$  decreases with  $t$  as  $c = c_0 t/t_0$ . When  $\xi \sim \mathcal{O}(1)$ , Eqs. (7.117a) and (7.117b) cease to be valid, thus after this period the function  $c$  starts increasing. At the end, when the condition  $c^2/(ab)^2 \sim t^{-2}$  is realized, a new period of oscillations starts (Kasner epochs) and the natural evolution of the system is restored.

Let us now derive a correlation between the two sets of constants  $(K, \theta)$  and  $(A, \xi_0)$  allowing also to relate the initial state of the system to the final one. Let us firstly relate  $A$  and  $\xi_0$  to the initial conditions  $(u_0, a_0)$  that refer to the value of the parameter  $u$  in correspondence to the Kasner epoch just before the small oscillations, and to the value that the functions  $a$  and  $b$  have at the end of that epoch. Imposing the continuity of the time derivatives of the functions  $(\alpha + \beta)$  and  $(\alpha - \beta)$ , for  $\tau = \tau_0$  ( $t = t_0$ ), and because of the relation  $\xi_\tau = 2a_0^2\xi/\xi_0$ , one can get the following conditions:

$$(\alpha + \beta)|_{\tau=\tau_0} = p_2 + p_1 = \frac{1}{1 + u_0 + u_0^2} = \frac{2a_0^2}{\xi_0} \quad (7.122a)$$

$$(\alpha - \beta)|_{\tau=\tau_0} = p_2 - p_1 = \frac{1 + 2u_0}{1 + u_0 + u_0^2} = 4A \frac{a_0^2}{\sqrt{\xi_0}}. \quad (7.122b)$$

Equations (7.122a)-(7.122b) provide the relations we are looking for

$$\xi_0 = 2a_0^2(1 + u_0 + u_0^2) \quad (7.123)$$

$$A = \frac{1 + 2u_0}{(1 + u_0 + u_0^2)^{1/2}} \frac{1}{2\sqrt{2}a_0} \sim \frac{1}{\sqrt{2}a_0}, \quad (u_0 \gg 1). \quad (7.124)$$

(we note here that in such a scheme the interchange between the indices  $p_1$  and  $p_2$  with respect to the functions  $\alpha$  and  $\beta$  has the only effect of changing  $A \rightarrow -A$ ).

The correlation between the set of constants  $(K, \theta)$  and  $(A, \xi_0)$  can be obtained noting that, for  $\chi \ll 1$ , Eq. (7.115a) becomes

$$\chi_{\xi\xi} + \frac{1}{\xi}\chi_{\xi} + \chi = 0, \quad (7.125)$$

by replacing the hyperbolic sinus term with its argument. The general solution to this equation is

$$\chi = c_1 J_0(\xi) + c_2 N_0(\xi) \quad (7.126)$$

where  $J_0$  and  $N_0$  denote the Bessel and the Neumann functions to zeroth order, respectively. A solution to Eq. (7.125) admits the two asymptotic expressions

$$\text{for } \xi \gg 1: \quad \chi = \frac{\sqrt{2}}{\pi\xi} c_1 \cos(\xi - \pi/4) + c_2 \sin(\xi - \pi/4) + O(1/\xi) \quad (7.127)$$

$$\text{for } \xi \ll 1: \quad \chi = \frac{2}{\pi} c_2 \ln \xi + \frac{2}{\pi} c_2 \ln(1/2 + c) + c_1. \quad (7.128)$$

The comparison of Eqs. (7.127) and (7.128) with Eqs. (7.116) and (7.119), provides the following identifications

$$c_2 = \sqrt{\pi} A (\cos \xi_0 + \sin \xi_0) \quad (7.129a)$$

$$c_1 = \sqrt{\pi} A (\cos \xi_0 - \sin \xi_0) \quad (7.129b)$$

$$K = \frac{2}{\pi} c_2 = \frac{2A}{\sqrt{\pi}} (\cos \xi_0 + \sin \xi_0) \quad (7.129c)$$

$$\theta = \frac{2}{\pi} c_2 (\ln 1/2 + c) + c_1. \quad (7.129d)$$

Finally, by means of Eq. (7.123) and Eq. (7.124),  $K$  can be obtained as a function of  $u_0$  and  $a_0$  as

$$K = \frac{1}{\sqrt{2\pi a_0}} \frac{1 + 2u_0}{(1 + u_0 + u_0^2)^{1/2}} \times \{ \cos [2a_0^2 (1 + u_0 + u_0^2)] + \sin [2a_0^2 (1 + u_0 + u_0^2)] \}. \quad (7.130)$$

The substitution of this expression in Eq. (7.121) yields the new value of  $u_1$  as a function of the initial conditions:  $u_1 = u_1(u_0, a_0)$ .

Rigorously speaking, Eq. (7.130) is valid only for small values of the function  $\chi$ , i.e. its validity is limited to the region  $K \ll 1$ . The interest in that relation, however, relies on the peculiar initial conditions required to realize such a situation ( $K \ll 1$ ). Since  $K$ , as provided by Eq. (7.130), is in general not close to zero for generic  $(u_0, a_0)$ , we can conclude that the existence of a long era  $u_0 \gg 1$  does not imply that the successive evolution

has a similar behavior, i.e.  $u_1 \gg 1$ . In other words, the system is expected to escape small oscillations to recover its natural evolution associated to a finite value of  $u$ .

Although in the Bianchi type VIII case the derivation presented here is valid in its guidelines, in such model we have to distinguish between the two cases for the monotonically decreasing function corresponding or not to the negative constant  $\nu = -1$  during the small oscillations phase. If it does, the description is exactly the same while, if it does not, the system (7.115b) slightly changes, but with marginal effects on the whole evolution.

## 7.5 Dynamical Systems Approach

We have discussed the properties and the dynamics of the homogeneous models by means of the Einstein field equations following the path of the Landau school. We will now discuss the framework known as *Dynamical Systems Approach*, which is based on the fact that the Einstein equations for a spatially homogeneous model can be written as an autonomous system of first order differential equations in the time variable only. The evolution curves of an autonomous system partition  $\mathbb{R}^n$  into orbits, so to obtain a dynamical system on  $\mathbb{R}^n$ . Such reduction allows one to adopt the standard techniques of this field so that, for example, the asymptotic behavior as  $t \rightarrow \pm\infty$  can be described in terms of asymptotically stable equilibrium points (sinks), asymptotically stable periodic orbits, or more general attractors of higher dimensions. A relevant feature of this approach is that it allows a description in terms of dimensionless variables. This can be done by means of a conformal transformation of the metric whose conformal factor brings the unique dimensional unit (which can be chosen to be a length by setting  $\kappa = 1$ ), as detailed at the end of Sec. 7.5.1.

We have seen in Sec. 7.1 how homogeneous spaces can be classified accordingly to a scheme firstly given by Bianchi. This formulation is based on the introduction of a group invariant *time-independent* frame, where the time dependence is all encoded in the matrix  $\eta_{ab}(t)$  as in Eq. (7.17). At the same time, it is possible to give a different but equivalent classification by introducing a group invariant *orthonormal* frame such that

$$h_{\alpha\beta} = \delta_{ab} e^a_{\alpha}(t, x) e^b_{\beta}(t, x), \quad \delta_{ab} = \text{diag}(1, 1, 1). \quad (7.131)$$

This classification studies all the inequivalent forms that the Ricci coefficients  $\gamma_{IJK}$  (see Eq. (2.111a)) and their linear combinations  $\lambda_{IJK}$  (as in Eq. (2.111b)) can take. Let us briefly sketch this picture.

We take an orthonormal tetrad  $e_I \equiv e_I^i \partial_i$ , with the time-like vector  $e_0^i$  coincident with the normalized four-velocity  $u^i$

$$e_0^i = u^i. \quad (7.132)$$

Given any function  $f$ , we have that the following expressions hold for the commutators

$$[e_I, e_J] f = \lambda^K_{IJ} e_K f. \quad (7.133)$$

Because of Eq. (7.132), we have that both the vorticity  $\omega_{ij}$  (see Sec. 2.7.2 for definitions and notation; notice that we adopt the signature  $(-, +, +, +)$ ) and the acceleration vector  $u^j \nabla_j u^i$  vanish. This implies that  $\lambda^0_{0a} = \lambda^0_{ab} = 0$ , and the remaining non-zero components can be decomposed as

$$\lambda^a_{0b} = -\theta_{ab} + \epsilon_{abc} \Omega^c, \quad \Omega^I = \frac{1}{2} \epsilon^{IJKL} u_J e_K \dot{e}_L \quad (7.134)$$

where  $u_I$  is the tetradic projection of the four-velocity  $u_i$ . The functions  $\lambda_{abc}$  can be decomposed by virtue of a symmetric matrix  $n^{ab}$  and a vector  $a^a$  (in analogy with Eq. (7.33))

$$\lambda^a_{bc} = \epsilon_{bcd} n^{ad} + \delta^a_c a_b - \delta^a_b a_c. \quad (7.135)$$

The quantities  $\theta_{ab}, n^{ab}, a^c, \Omega^a$  completely determine the coefficients  $\lambda_{IJK}$ .

Now, if one repeats the same calculations developed in Sec. 7.1, then it is led to a similar classification not dealing anymore with constants, but with functions of the time variable; the results are summarized in Table 7.2 in analogy to Table 7.1.

Table 7.2 Inequivalent structure functions corresponding to the Bianchi classification. The signs stand for the positive (+) or negative (-) character of the functions  $n_a$  and  $a$ .

Type	a	n <sub>1</sub>	n <sub>2</sub>	n <sub>3</sub>
I	0	0	0	0
II	0	+	0	0
VII	0	+	+	0
VI	0	+	-	0
<b>IX</b>	<b>0</b>	<b>+</b>	<b>+</b>	<b>+</b>
<b>VIII</b>	<b>0</b>	<b>+</b>	<b>+</b>	<b>-</b>
V	+	0	0	0
IV	+	0	0	+
VII <sub>a</sub>	+	0	+	+
VI <sub>a</sub>	+	0	+	-

### 7.5.1 Equations for orthogonal Bianchi class A models

We will consider only *orthogonal models*, i.e. models in which the vector  $u^i$  is parallel to the vector  $n^i$  normal to the spatial hypersurfaces, and class A models, i.e. those in which  $a_c = 0, 0, 0$ .

The quantities  $n_{ab}(t)$  determine the Bianchi type of the isometry group, and the curvature of the group orbits  $t = \text{const}$ . This curvature can be described by the trace-free Ricci tensor  ${}^3S_{ab}$  and the Ricci scalar  ${}^3R$  of the metric induced on the group orbits. It can be shown that if

$$b_{ab} = 2n_a^c n_{cb} - (n_d^d) n_{ab}, \quad (7.136)$$

then

$${}^3S_{ab} = b_{ab} - \frac{1}{3}(b_c^c)\delta_{ab} \quad (7.137)$$

$${}^3R = -\frac{1}{2}b_a^a. \quad (7.138)$$

Using Eq. (2.112b) together with the decomposition (7.134) and (7.135), the 00- and the  $ab$ -components of the Einstein equations in the presence of a perfect fluid tensor yield the Raychaudhuri equation for the expansion  $\theta$  and an equation for the shear as

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{2}(\rho + 3P), \quad (7.139a)$$

$$\dot{\sigma}_{ab} = -\theta\sigma_{ab} - {}^3S_{ab}, \quad (7.139b)$$

while the  $0a$  components yield an algebraic constraint for the shear components

$$\epsilon_{abc} n^{cd} \sigma_d^b = 0. \quad (7.139c)$$

Thus, we obtain the first integral from the trace of the Einstein equations expressed as

$$\rho = \frac{1}{3}\theta^2 - \sigma^2 + \frac{1}{2}{}^3R. \quad (7.139d)$$

From the Jacobi identities applied to the vierbein vectors  $e_I$ , in analogy with Eq. (7.29), we get the evolution equation for  $n_{ab}$  as

$$\dot{n}_{ab} = 2\sigma^c_{(a} n_{b)c} - \frac{1}{3}\theta n_{ab}. \quad (7.140)$$

It is possible to show that  $\sigma_{ab}$  and  $n_{ab}$  can be diagonalized simultaneously with a rotation, so that one can study only the eigenfunction components  $\sigma_a$  and  $n_a$ .

Equations (7.139) form a six-dimensional autonomous system of differential equations for the variables  $y_i = (\theta, \sigma_a, n_a)$  (from the definition of vorticity one has  $\text{Tr}(\sigma_{ab}) = 0$ ), of the form

$$\dot{y}_i = F_i(y_j), \quad i, j = 1, \dots, 6. \quad (7.141)$$

The fundamental observation which is at the ground of such a formulation is that, since  ${}^3S_{ab}$  and  $\rho$  are quadratic expressions of the variables  $y_i$ , the functions  $F_i$  are homogeneous of degree 2, which implies that the system is invariant under a scale transformation

$$Y_i = L y_i, \quad \frac{d\tau}{dt} = L, \quad (7.142)$$

where  $L$  is a length of reference and  $\tau$  a new time variable. This allows us to introduce dimensionless variables, thereby reducing the dimension of the system by one unit. Another reason for introducing such dimensionless variables is the fact that the variables  $y_i$  do not enable one to distinguish different asymptotic states, since at the singularity these variables typically diverge, while at late times in ever-expanding models they tend to zero. A relevant physical scale is given by the Hubble function  $H$ , defined as

$$H = \theta/3, \quad (7.143)$$

so that it is natural to introduce the dimensionless shear  $\Sigma_a$ , the dimensionless spatial curvature variables  $N_a$  and the dimensionless density<sup>5</sup>  $\Omega$

$$(\Sigma_a, N_a) = (\sigma_a, n_a) / H, \quad \Omega = \rho / (3H^2) \quad (7.144)$$

together with the new time variable  $\tau$

$$\frac{d\tau}{dt} = H. \quad (7.145)$$

It is worth noting how the parametrization (7.144) correctly describes the dynamics for all the type A models. In the case of the type IX, this is true only for the expanding phase, i.e. when  $H > 0$ . Finally we introduce  $\mathcal{R}$ , proportional to the Ricci scalar

$$\mathcal{R} = -\frac{1}{6} {}^3R. \quad (7.146)$$

We will now show how the Einstein field equations can be recast in a set of dimensionless system of ordinary equations coupled to a single equation that brings and describes the evolution of the typical length of the system.

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<sup>5</sup>In the first works, the standard length adopted was the expansion  $\theta$  instead of the Hubble length  $H$ .



Let us assume that the equation of state of matter is given by Eq. (2.15) with  $\gamma > 1$ , and write down the full system of equations as

$$\frac{dH}{d\tau} = -(1+q)H. \quad (7.147a)$$

$$\frac{d\Sigma_a}{d\tau} = -(2-q)\Sigma_a - S_a, \quad \text{no sum over } a \quad (7.147b)$$

$$\frac{dN_a}{d\tau} = (q+2\Sigma_a)N_a, \quad \text{no sum over } a \quad (7.147c)$$

$$q = \frac{1}{3}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) + \frac{1}{2}(\rho + 3P) \equiv 2\Sigma^2 + \frac{1}{2}(\rho + 3P). \quad (7.147d)$$

From the definition of the shear (2.146), we have the constraint

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0. \quad (7.147e)$$

The first integral (7.139d), because of Eq. (7.146), becomes

$$\Omega = 1 - \Sigma^2 - \mathcal{R}, \quad (7.147f)$$

also addressed as the *Gauss constraint*. The last equation to be added to this set is the evolution equation for the density  $\Omega$ , obtained from the four-divergence of the energy-momentum tensor of the perfect fluid which reads as

$$\frac{d\Omega}{d\tau} = (2q-1)\Omega - 3P = [2q - (3\gamma-5)]\Omega. \quad (7.147g)$$

From this last equation, together with Eqs. (7.144) and (7.147a), we can conclude that for all orthogonal perfect fluid models with a linear equation of state, the energy density diverges toward the singularity (which is at  $\tau \rightarrow -\infty$ ). By integrating the equations for the Hubble parameter and for the energy density, we obtain that

$$\left. \begin{aligned} \Omega &\propto \exp \left[ \int (2q - 3\gamma + 5) d\tau \right] \\ H &\propto \exp \left[ - \int (1 + q) d\tau \right] \end{aligned} \right\} \Rightarrow \rho = 3H^2\Omega \propto \exp(-3(\gamma-1)\tau). \quad (7.148)$$

### 7.5.2 The Bianchi I model and the Kasner circle

Let us firstly consider the vacuum case. For the Bianchi I model, we have that  $N_1 = N_2 = N_3 = 0$ , implying that  $\mathcal{R} = S_a = 0$ . For the vacuum case,

$\Omega = 0$  and the system (7.147) reduces to

$$\frac{dH}{d\tau} = -(1+q)H \quad (7.149a)$$

$$\frac{d\Sigma_a}{d\tau} = -(2-q)\Sigma_a \quad \text{no sum over } a \quad (7.149b)$$

$$q = 2\Sigma^2 \quad (7.149c)$$

$$\Sigma^2 = 1. \quad (7.149d)$$

This set of equations can be easily integrated as

$$q = 2, \quad \Sigma_a = \text{const}, \quad H = H_0 e^{-3\tau} \quad (7.150)$$

while the time variable reads explicitly from Eq. (7.145)

$$t + t_0 = \frac{1}{3H_0} \exp(\tau), \quad (7.151)$$

where  $t_0$  is an integration constant.

Since Eq. (7.149d) holds, we can define the so-called *Kasner circle*  $K^0$  that well represents the Bianchi I *vacuum subset*  $\mathcal{B}_I^{\text{vacuum}}$ , and is sketched in Fig. 7.2.

Furthermore, from Eqs. (7.147e) and (7.149d), we can recover the standard Kasner relation (7.51) by setting

$$(\Sigma_1, \Sigma_2, \Sigma_3) = (3p_1 - 1, 3p_2 - 1, 3p_3 - 1), \quad (7.152)$$

thus implying that each point on  $K^0$  represents a Kasner solution. Because of the permutational symmetries of such a representation,  $K^0$  can be divided into six sectors, labeled by a triplet  $(abc)$  that represents a permutation of the fundamental triplet (123). This way, for each point in a particular set, we have that  $p_a < p_b < p_c$ .

If we consider the  $u$  parametrization of the Kasner exponents (7.55), each point on a sector is represented by a unique value of  $u$ . The boundaries of the sectors are six points associated with solutions that are locally rotationally symmetric:

- $Q_a$  are characterized by  $(\Sigma_a, \Sigma_b, \Sigma_c) = (2, -1, -1)$  or, equivalently,  $(p_a, p_b, p_c) = (-1/3, 2/3, 2/3)$ . They all correspond to the value  $u = 1$ .
- $T_a$  are the *Taub points* given by  $(\Sigma_a, \Sigma_b, \Sigma_c) = (-2, 1, 1)$  or, equivalently,  $(p_a, p_b, p_c) = (1, 0, 0)$ ; they provide the Taub representation of Minkowski spacetime, given by  $u = \infty$ .

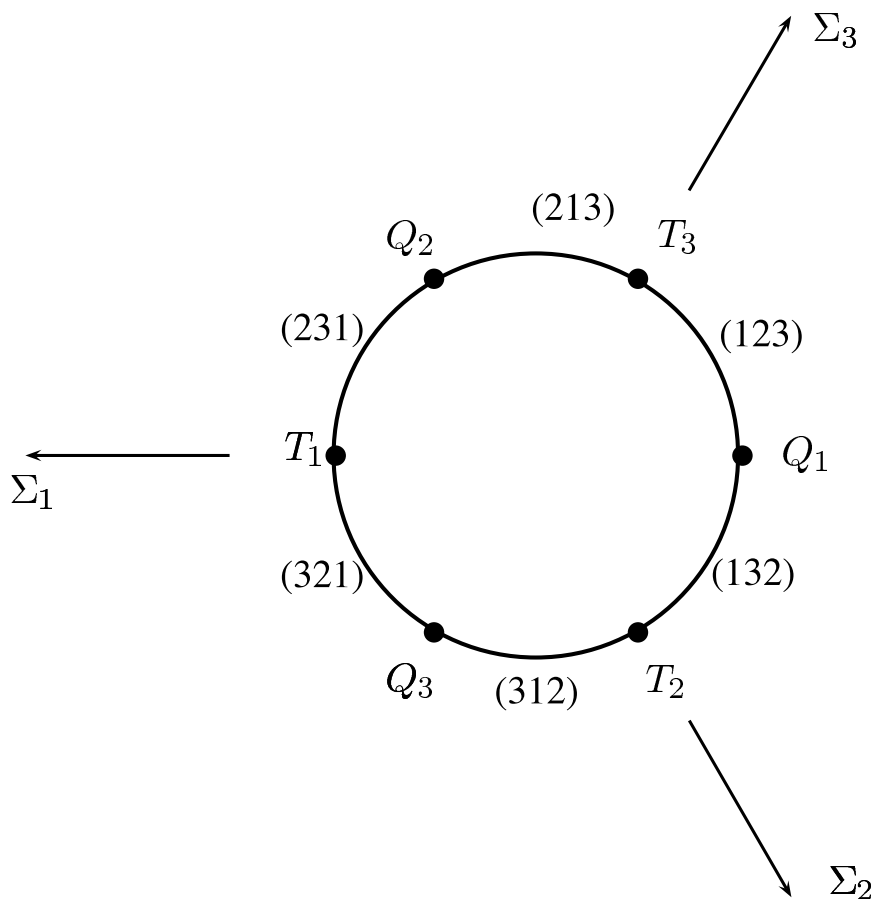


Figure 7.2 Representation of the Kasner circle  $K^0$ . The circle can be divided in six equivalent sectors that correspond to different choices in the ordering the Kasner exponents  $p_a$ ; each point on the circle represents a different Kasner solution.

In the case of a perfect fluid with a linear equation of state, we have the

following set of equations

$$\frac{dH}{d\tau} = -(1+q)H, \quad (7.153a)$$

$$\frac{d\Sigma_a}{d\tau} = -(2-q)\Sigma_a, \quad (7.153b)$$

$$q = 2\Sigma^2 + \frac{1}{2}(3\gamma - 5)\Omega, \quad (7.153c)$$

$$1 - \Omega = \Sigma^2 < 1, \quad (7.153d)$$

$$\frac{d\Omega}{d\tau} = (2q + 5 - 3\gamma)\Omega. \quad (7.153e)$$

This set can be integrated for a generic linear equation of state. If we take  $\{\Omega(\tau_0) = \Omega_0, \Sigma_a(\tau_0) = \Sigma_{a0}, \tau_0 = 0\}$  as initial conditions, we have that

$$\Omega(\tau) = \frac{\Omega_0}{\Omega_0 + (1 - \Omega_0) \exp[3(w - 2)\tau]}, \quad (7.154a)$$

$$\Sigma_a(\tau) = \frac{\Sigma_{a0} \sqrt{6(1 - \Omega_0)}}{\sqrt{(\Omega_0 - 1) + \Omega_0 \exp[3\tau(2 - \omega)]}}. \quad (7.154b)$$

From Eqs. (7.154) one has that

$$\Sigma_{2(3)}(\tau) = \frac{\Sigma_{2(3)0}}{\Sigma_{10}} \Sigma_1(\tau), \quad (7.155)$$

that, together with Eq. (7.153d), implies that the solution can be represented in the  $\Sigma_a$  plane as a straight line from  $K^0$  ending, for  $\tau \rightarrow +\infty$ , in the center of the circle (the so-called Friedmann fixed point).

### 7.5.3 The Bianchi II model in vacuum

In the case of the type II model, only one of the three  $N_a$  is different from zero; we can take  $N_1 = N_2 = 0, N_3 > 0$ . Furthermore, the Gauss constraint (7.147f) can be used to eliminate  $N_3$ . This way, the system (7.147) reduces to

$$\frac{d\Sigma_{a(b)}}{d\tau} = (q - 2)\Sigma_{a(b)} + 4(1 - \Sigma^2) \quad (7.156a)$$

$$\frac{d\Sigma_c}{d\tau} = (q - 2)\Sigma_c - 8(1 - \Sigma^2). \quad (7.156b)$$

In the vacuum case  $\Omega = 0$  there are no fixed points, while the boundary of this vacuum subset coincides with the Kasner circle  $K^0$ . The solutions can be represented as straight lines that connect one point on  $K^0$  to a

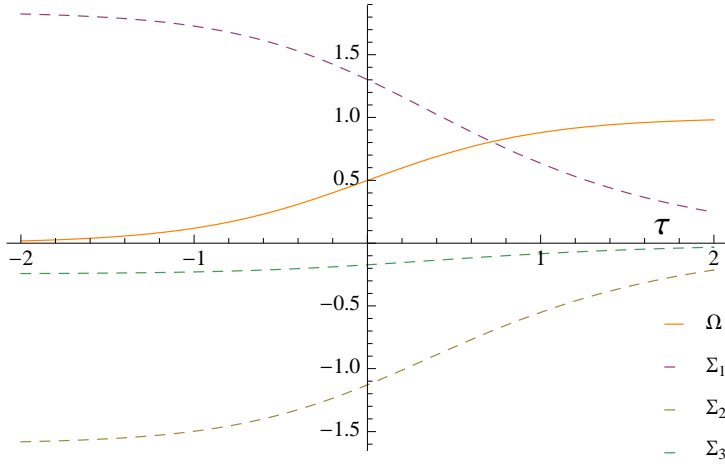


Figure 7.3 Plot of the solution (7.154a) for a given set of initial conditions ( $\tau_0 = 0$ ,  $\Omega_0 = 0.5$ ,  $\Sigma_{10} = 0.751783$ ,  $\Sigma_{20} = -0.651783$ ,  $\Sigma_{30} = -0.1$ ). It can be seen how the dimensionless energy density  $\Omega$  evolves from 0 at the singularity to 1 as  $\tau$  grows.

different one, as sketched in Fig. 7.4. This is the standard result we have already discussed in the BKL approach in the previous sections. An easy check is the analysis of the evolution of the “reduced” variables  $\Sigma_{\pm}$  defined as

$$\Sigma_1 = \Sigma_+ + \sqrt{3}\Sigma_- \quad (7.157a)$$

$$\Sigma_2 = \Sigma_+ - \sqrt{3}\Sigma_- \quad (7.157b)$$

$$\Sigma_3 = -2\Sigma_+ . \quad (7.157c)$$

From these definitions, it follows that the constraint (7.147e) is automatically satisfied and  $\Sigma^2 = \Sigma_+^2 + \Sigma_-^2$ . Then, equations (7.156a)-(7.156b) read as

$$\Sigma'_+ = 2(\Sigma^2 - 1)(\Sigma_+ - 2) , \quad (7.158a)$$

$$\Sigma'_- = 2(\Sigma^2 - 1)\Sigma_- , \quad (7.158b)$$

which can be implicitly solved and represented in the  $\Sigma$ -plane as

$$\Sigma_+ = A(\Sigma_- + 2) , \quad (7.159)$$

where  $A$  is a real parameter that characterizes the single orbit. The solution (7.159) draws a star of straight lines originating from  $(\Sigma_+, \Sigma_-) = (2, 0)$  or,

equivalently,  $(\Sigma_1, \Sigma_2, \Sigma_3) = (2, 2, -4)$ . This is the point  $M_1$  in Fig. 7.4. Then the orbit of Bianchi II vacuum model is just a chord starting from a point on  $K^0$  and ending in another point on  $K^0$ . A direct calculation yields the standard BKL map for the  $p_a$  indices.

This solution is also addressed as the Bianchi type II vacuum subset  $\mathcal{B}_{II}^{vacuum}$ .

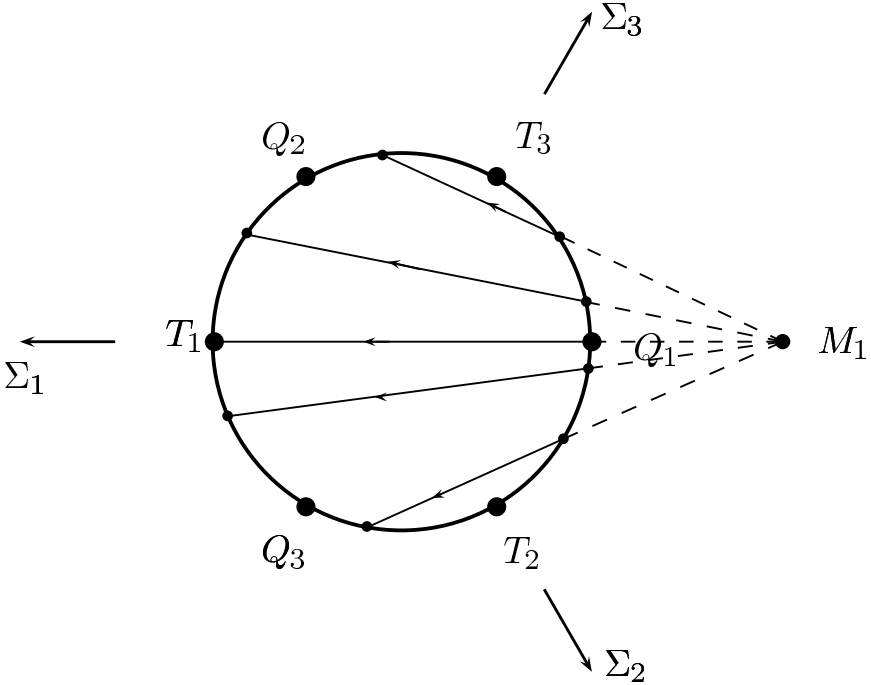


Figure 7.4 In the figure above, the type II transitions are sketched in the  $\Sigma_1, \Sigma_2, \Sigma_3$  plane. A generic transition connects a point on the Kasner circle  $K^0$  with a different point on the same circle, and the arrow denote the evolution toward the singularity. The common focal point  $M_1$  of these straight lines is at  $(\Sigma_1, \Sigma_2, \Sigma_3) = (-4, 2, 2)$ .

#### 7.5.4 The Bianchi IX model and the Mixmaster attractor theorem

The type IX case has  $N_1 > 0, N_2 > 0, N_3 > 0$  and the corresponding system of equations results to be given by the full set (7.147). From the analysis of the type II model we can obtain an equivalent description as

the one given in Sec. 7.4 where we gave a piecewise representation for the dynamics of the Bianchi IX model. In the dynamical systems framework, the same approximation corresponds to maintaining in Eqs. (7.147) only one of the functions  $N_a$ , as we did there with the scale factors  $a, b, c$ . This way, the piecewise solution can be represented as in Fig. 7.5. We stress that, in the  $\Sigma$  plane, a Kasner epoch corresponds to a point on  $K^0$ , while a transition to a chord connecting two points on  $K^0$ . An era is identified with the oscillations among two of the six subsets in which the Kasner circle is partitioned (in Fig. 7.5, an example of era is given by the sequence of transitions between the points belonging to the circle sectors  $Q_1 - T_3$  and  $T_3 - Q_2$ ).

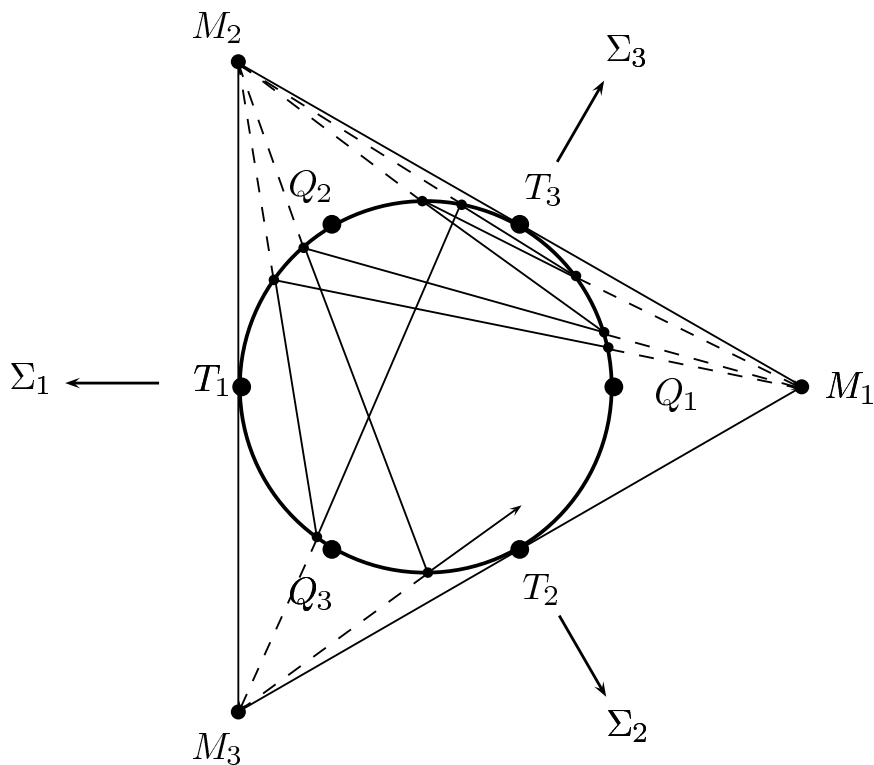


Figure 7.5 In this figure we can see the piecewise solution obtained in Sec. 7.4 as it is described in the  $(\Sigma_1, \Sigma_2, \Sigma_3)$  plane in the dynamical systems approach.

It is worth noting that, when dealing with the Bianchi IX model, it is easy to reconstruct the metric starting by the only functions  $n_a$ . In the

orthonormal frame, the relation between the diagonal terms of  $\eta_{ab}(t)$  and  $n_a$  is given by

$$n_a(t) = \frac{\eta_{aa}}{\eta}, \quad (\text{no summation over } a). \quad (7.160)$$

For a generic homogeneous case, a specific algebraic technique to reconstruct the metric exists. As soon as one of the  $n_a$  vanishes, this technique envisages the use of other frame variables like  $H$  or  $\sigma_a$ .

There is an important theorem whose proof was given by Ringström (2001), and it can be stated as follows<sup>6</sup>

**Theorem 7.1.** *Let  $(\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3)$  be a generic solution of the type IX model. Then*

$$\lim_{\tau \rightarrow -\infty} N_1 N_2 + N_2 N_3 + N_3 N_1 = 0 \quad (7.161a)$$

$$\lim_{\tau \rightarrow -\infty} \Omega = 0. \quad (7.161b)$$

This theorem is the only exact mathematical result about the dynamics of the Bianchi type IX model; however, this result does not completely solve the main question, whether the exact Mixmaster dynamics is chaotic or not. Indeed, chaos is associated with the Kasner map that is valid only in the piecewise approximation, and it is commonly believed that this map reliably represents the exact dynamics.

This theorem can be restated in a different way if we define the *Mixmaster attractor*  $\mathcal{A}_{IX}$  as follows

**Definition 7.3.** The Mixmaster attractor  $\mathcal{A}_{IX}$  is the set given by the union of the Bianchi type I and type II subsets. Since the Bianchi type II consists of three equivalent representations it can be written as

$$\mathcal{A}_{IX} = \mathcal{B}_I \text{vacuum} \cup \mathcal{B}_{II} \text{vacuum} = K^0 \cup \mathcal{B} \text{vacuum}_{N_1} \cup \mathcal{B} \text{vacuum}_{N_2} \cup \mathcal{B} \text{vacuum}_{N_3}. \quad (7.162)$$

Then, an equivalent formulation can be the following

**Theorem 7.2.** *Let  $Y(\tau) = (\Sigma_a(\tau), N_a(\tau))$  be a generic solution of the Bianchi type IX model. Then*

$$\lim_{\tau \rightarrow -\infty} |Y(\tau) - \mathcal{A}_{IX}| = \lim_{\tau \rightarrow -\infty} \min_{Z \in \mathcal{A}_{IX}} |Y(\tau) - Z| = 0. \quad (7.163)$$

---

<sup>6</sup>Indeed, different proofs of this theorem exist, approaching the statement from different points of view.



This theorem definitely states that the attractor of the type IX model belongs to  $\mathcal{A}_{IX}$ , but it does not tell if they coincide or it is only a subset, being still an open issue. Furthermore, the Mixmaster attractor theorem does not provide any information about the detailed asymptotic evolution.

For a complete discussion of the several implications we recommend the interested reader to analyze the wide literature on the topic.

## 7.6 Multidimensional Homogeneous Universes

The question of chaos in higher dimensional cosmologies has been widely investigated over the last 20 years. In the case of diagonal models (in the canonical basis), many authors showed that none of higher-dimensional extensions of the Bianchi IX type possesses proper chaotic features: the crucial difference is given by the finite number of oscillations characterizing the dynamics near the singularity. Chaos, however, is restored (up to 9 spatial dimensions) as soon as different symmetry groups are considered, such as the non-diagonal models. In this Section, we will follow the analysis proposed by Halpern (1985) of the diagonal, homogeneous models with four spatial dimensions, and conclude with some remarks on the non-diagonal case.

Indeed, the work of Fee in 1979 classifies the four-dimensional homogeneous spaces into 15 types, named  $G_0 - G_{14}$ , and is based on the analysis of the corresponding Lie groups.

The line element can be written using the Cartan basis of left-invariant forms and explicitly reads as ( $N = 1$ )

$$ds^2 = dt^2 - {}^4\eta_{rs}(t)\omega^r \otimes \omega^s. \quad (7.164)$$

The 1-forms  $\omega^r$  obey the relation  $d\omega^r = \frac{1}{2}C^r_{pq}\omega^p \wedge \omega^q$ , where the  $C^r_{pq}$  are the four-dimensional structure constants. We limit our attention to the case of a diagonal matrix  ${}^4\eta_{rs}$

$${}^4\eta_{rs} = \text{diag}(a^2, b^2, c^2, d^2). \quad (7.165)$$

The Einstein equations are obtained from the metric (7.165) by the stan-

dard procedure outlined in Sec. 7.3 as

$$R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\ddot{d}}{d} = 0, \quad (7.166a)$$

$$R_1^1 = \frac{(\dot{a}bcd)}{abcd} + {}^4R_1^1 = 0, \quad (7.166b)$$

$$R_2^2 = \frac{(\dot{a}bcd)}{abcd} + {}^4R_2^2 = 0, \quad (7.166c)$$

$$R_3^3 = \frac{(\dot{a}bcd)}{abcd} + {}^4R_3^3 = 0, \quad (7.166d)$$

$$R_4^4 = \frac{(\dot{a}bcd)}{abcd} + {}^4R_4^4 = 0, \quad (7.166e)$$

$$R_n^0 = \left( \frac{\dot{x}_n}{x_n} - \frac{\dot{x}_m}{x_m} \right) C_{mn}^m = 0, \quad (7.166f)$$

where  $x_n$  ( $n = 1, 2, 3, 4$ ) denote the scale factors  $a, b, c, d$ , respectively, and the  ${}^4R_a^b$  are the tetradic components of the spatial four-dimensional Ricci tensor  ${}^4R_\alpha^\beta$  defined as in Eq. (7.40).

Equations (7.166) can be restated by the use of the logarithmic variables

$$\alpha = \ln a, \quad \beta = \ln b, \quad \gamma = \ln c, \quad \delta = \ln d, \quad (7.167a)$$

and the logarithmic time  $\tau$ , i.e.

$$dt = abcd \, d\tau. \quad (7.167b)$$

We thus obtain the system

$$\begin{aligned} \alpha_{\tau\tau} &= -(abcd)^2 {}^4R_1^1, & \beta_{\tau\tau} &= -(abcd)^2 {}^4R_2^2, \\ \gamma_{\tau\tau} &= -(abcd)^2 {}^4R_3^3, & \delta_{\tau\tau} &= -(abcd)^2 {}^4R_4^4, \end{aligned} \quad (7.168a)$$

$$\begin{aligned} \alpha_{\tau\tau} + \beta_{\tau\tau} + \gamma_{\tau\tau} + \delta_{\tau\tau} \\ = 2\alpha_\tau\beta_\tau + 2\alpha_\tau\gamma_\tau + 2\alpha_\tau\delta_\tau + 2\beta_\tau\gamma_\tau + 2\beta_\tau\delta_\tau + 2\gamma_\tau\delta_\tau, \end{aligned} \quad (7.168b)$$

and

$$R_n^0 = 0. \quad (7.168c)$$

The dynamical scheme (7.168) is valid for any of the 15 models using the corresponding  ${}^4R_a^b$ .

The simplest group to be considered is  $G_0$ . For such model the functions  ${}^4R_a^b$  are all equal to zero and the solution simply generalizes the Kasner dynamics to four spatial dimensions, which corresponds to the line element discussed below in Eq. (7.170) when considering a Kasner epoch as a phase

of evolution in the  $G13$  model.

Among the five-dimensional homogeneous space-times,  $G13$  is the analogous of the Bianchi type IX, having the same set of structure constants. The Einstein equations can be written as

$$2\alpha_{\tau\tau} = [(b^2 - c^2)^2 - a^4] d^2, \quad (7.169a)$$

$$2\beta_{\tau\tau} = [(a^2 - c^2)^2 - b^4] d^2, \quad (7.169b)$$

$$2\gamma_{\tau\tau} = [(b^2 - a^2)^2 - c^4] d^2, \quad (7.169c)$$

$$\delta_{\tau\tau} = 0, \quad (7.169d)$$

together with Eq. (7.168b). If we assume that the BKL approximation is valid, i.e. that the right-hand sides of equations (7.169) are negligible, then the asymptotic solution for  $\tau \rightarrow -\infty$  is the five-dimensional and Kasner-like line element

$$ds^2 = dt^2 - \sum_{r=1}^4 t^{2p_r} (dx^r)^2, \quad (7.170)$$

with the Kasner exponents  $p_r$  satisfying the generalized Kasner relations

$$\sum_{r=1}^4 p_r = \sum_{r=1}^4 p_r^2 = 1. \quad (7.171)$$

This regime can only hold until the BKL approximation is satisfied. However, as soon as  $\tau$  approaches the singularity, one or more of the terms may increase. Let us assume  $p_1$  as the smallest index; then  $a = \exp(\alpha)$  is the largest contribution and we can neglect all other terms, thus obtaining

$$\begin{aligned} \alpha_{\tau\tau} &= -\frac{1}{2} \exp(4\alpha + 2\delta), \\ \beta_{\tau\tau} &= \gamma_{\tau\tau} = \frac{1}{2} \exp(4\alpha + 2\delta), \\ \delta_{\tau\tau} &= 0. \end{aligned} \quad (7.172)$$

When considering the asymptotic limits for  $\tau \rightarrow \pm\infty$ , from the solution of (7.172) we obtain the map

$$\begin{aligned} p'_1 &= -\frac{p_1 + p_4}{1 + 2p_1 + p_4}, & p'_2 &= \frac{p_2 + 2p_1 + p_4}{1 + 2p_1 + p_4}, \\ p'_3 &= \frac{p_3 + 2p_1 + p_4}{1 + 2p_1 + p_4}, & p'_4 &= \frac{p_4}{1 + 2p_1 + p_4}; \end{aligned} \quad (7.173)$$

$$abcd = \Lambda' t, \quad \Lambda' = (1 + 2p_1 + p_4)\Lambda. \quad (7.174)$$

In the new Kasner epoch, the leading terms on the right-hand sides of (7.169) evolve according to

$$a^4 d^2 \sim t^{2(2p'_1+p'_4)} \quad (7.175a)$$

$$b^4 d^2 \sim t^{2(2p'_2+p'_4)} \quad (7.175b)$$

$$c^4 d^2 \sim t^{2(2p'_3+p'_4)} . \quad (7.175c)$$

From Eq. (7.175a), it follows that a new transition can occur only if one of the three exponents is negative. Nevertheless this is generally not true, because there exists a region where the other exponents are greater than zero. After eliminating  $p'_4$  and  $p'_3$  from Eq. (7.175a) with Eq. (7.171), this region satisfies the following inequalities

$$3p_1^2 + p_2^2 + p_1 - p_2 - p_1 p_2 \geq 0 , \quad (7.176a)$$

$$3p_2^2 + p_1^2 + p_2 - p_1 - p_1 p_2 \geq 0 , \quad (7.176b)$$

$$3p_1^2 + p_2^2 - 5p_1 - 5p_2 + 5p_1 p_2 + 2 \geq 0 , \quad (7.176c)$$

plus a reality condition for  $p_3$

$$1 - 3p_1^2 - 3p_2^2 - 2p_1 p_2 + 2p_1 + 2p_2 \geq 0 . \quad (7.177)$$

The region defined by the validity of (7.176) and (7.177) is plotted in Fig. 7.6. Thus the Universe undergoes a certain number of transitions of Kasner epochs and eras; as soon as the Kasner indices  $p_1, p_2$  assume values in the shaded region, then no more transitions can take place and the evolution remains Kasner-like until the singular point is reached.

The type *G*14 case is quite similar to *G*13: for this model, the structure constants are the same as Bianchi type VIII and, under the same hypotheses, only a finite sequence of epochs occurs.

This analysis can be repeated for any of the diagonal homogeneous 4+1-dimensional models manifesting the same behavior: the absence of chaos in the asymptotic regime toward the singularity. Furthermore, these results hold even in higher dimensions.

### 7.6.1 On the non-diagonal cases

The full BKL-like dynamics can be recovered up to 10-dimensional space-times as soon as the assumption of diagonal metric (in the canonical basis) is relaxed. The main difference with respect to the diagonal case is that now the “Kasner axes”  $\omega_K^a$  do not coincide with the time-independent 1-forms  $\omega^b$  of the space, but are linear and time dependent combinations of them as

$$\omega_K^a = A_b^a(t) \omega^b . \quad (7.178)$$

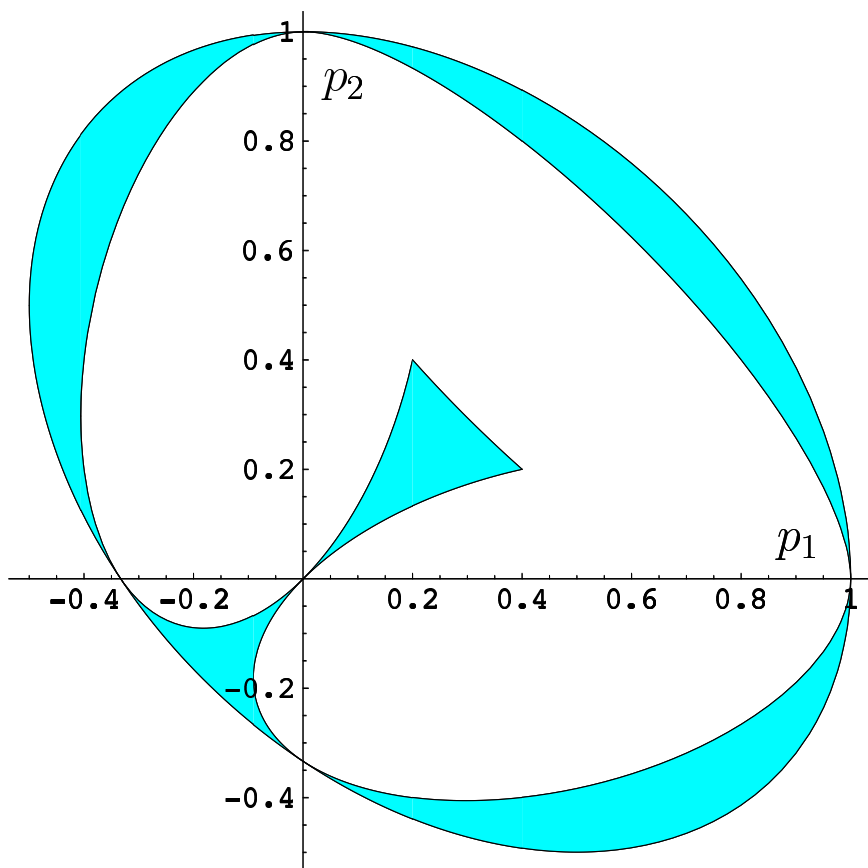


Figure 7.6 The shaded region corresponds to all of the couples  $(p_1, p_2)$  not satisfying Eq. (7.176b): as soon as a set  $(p_1, p_2)$  takes values in that portion, the BKL mechanism breaks down and the Universe experiences the last Kasner epoch till the singular point.

In the basis of the Kasner vectors, although the time dependence of the spatial-geometry is non-diagonal, the  $d$ -dimensional metric  ${}^d\eta_{ab}$  can still be kept as diagonal

$${}^d\eta_{ab} = \text{diag} (a_1^2(t), a_2^2(t), \dots, a_d^2(t)) \quad (7.179)$$

and the same analysis developed in Secs. 7.2 and 7.3 for the Bianchi models I, II and IX can be generalized to obtain similar results. In particular, when we neglect the Ricci scalar in the Einstein equations, so recovering a generalized Kasner-like solution, we obtain also that the functions  $A_b^a(t)$  are constant during each epoch. This means that we need to specify  $d(d +$

1) – 2 integration constants, though remaining with still enough arbitrary constants to fit assigned initial data. The key difference with the previous analysis is just in the introduction of the functions  $A_b^a$  and their constant behavior during the epochs: in the diagonal case, these coincide with the Kronecker delta  $\delta_b^a$  and the system loses  $d^2 - d$  arbitrary functions. This restriction does not allow for  $C_{jk}^i \neq 0$  for general  $i, j, k$  the reason why diagonal models are not chaotic. The introduction of the  $A_b^a$  functions is equivalent to a rotation of the triad vectors, and when they behave as constants, their net effect is to generate new non-zero structure constants of the group as linear combination of the original ones.

It has been explicitly shown that some homogeneous  $d$ -dimensional models (up to  $d = 9$ ) possess  $C_{jk}^i \neq 0$  in a generic non-canonical basis. This way chaos is still present in higher dimensional homogenous models

We refer to Sec. 9.7 for a more general and detailed analysis of the perturbation terms to a Kasner regime in the inhomogeneous multidimensional case.

## 7.7 Guidelines to the Literature

For the analysis of the homogeneous spaces presented in Sec. 7.1 we refer to the textbooks by Ryan & Shepley [406] and by Stephani et al. [427]. For formal aspects concerning the geometrical objects introduced here, see the books of Wald [456], while a good text on group theory is for example that by Zhong-Qi Ma [330]. The original derivation of the Bianchi classification appeared in [87].

In particular, for the application to Cosmology, for the Bianchi classification and the corresponding properties we refer to Landau & Lifshitz [301]. The demonstration of the re-collapsing behavior of the type IX model, both in vacuum and in presence of a perfect fluid with  $\gamma > 1$ , is given in [318,319].

The derivation of the Kasner solution given in Sec. 7.2 can be found in the original paper [267]. However, a satisfactory discussion is also provided by the standard book [301].

For the dynamics of the Bianchi models considered in Sec. 7.3 we refer to the review article [354] and references therein.

A valuable introduction to the Einstein equations under the homogeneity hypothesis is offered by the textbooks [301] and by Misner, Thorne & Wheeler [347].

The study of the oscillatory regime pursued in Sec. 7.4 is properly ad-

dressed by the original work by Belinskii, Khalatnikov and Lifshitz [64] and later reviewed in [65]. A more detailed presentation of the stochastic properties associated to the BKL map is given in [316] and in [314].

The standard textbook on the topics of Dynamical Systems Approach, discussed in Sec. 7.5 is the one edited by Wainwright & Ellis [454]. A clear derivation of the Einstein field equations for a homogeneous model in a group invariant, orthonormal frame can be found in [168, 332] while its application to cosmological settings were firstly reviewed in [125]. A discussion on the dynamical properties of orthogonal Bianchi model of class A can be found in [455]. The original demonstration of the Mixmaster Attractor Theorem is in [393, 394], while a different derivation is for example in [234]. A comprehensive review on the implications of such a result on the dynamics, as discussed in Sec. 7.5.4, can be found in [235, 236].

We did not face the topic of Consistent Potential Method, firstly developed by Grubišić & Moncrief in [207] with the related topic of [261]. We refer the reader interested in some applications to [80, 208] and for a wide review to [77]. The research line that deals with the Painlevé analysis of the Mixmaster equations was firstly proposed in [133] and later developed for example in [13, 126].

The first works on the homogeneous Mixmaster dynamics in higher dimensional cases, as discussed in Sec. 7.6, are [43, 182, 224] which underline that chaos is suppressed, at least in diagonal model. The extension to the chaotic, non-diagonal case, presented in Sec. 7.6.1, is discussed in [145].

## Chapter 8

# Hamiltonian Formulation of the Mixmaster

In this Chapter we provide the Hamiltonian formulation of the Mixmaster dynamics, describing in detail how the infinite sequence of Kasner epochs takes the suggestive form of a two-dimensional point particle performing an infinite series of bounces within a potential well. After specializing the Einstein-Hilbert action to the case of homogeneous models, we deal with a three-dimensional system, whose generalized coordinates correspond to the three independent logarithmic scale factors. Indeed all the dynamical content is summarized in the time behavior of the three spatial directions, while the spatial dependence of the three-geometries enters through the structure constants only (any other space dependence is integrated out).

As far as we perform a Legendre transformation, we are naturally led to introduce the so-called Misner variables, which allow to diagonalize the kinetic term in the Hamiltonian function. Once recognized how the isotropic component of the metric (summarized by the Misner variable  $\alpha$ ) plays the natural role of time for the configuration space of the anisotropies degrees of freedom (namely the Misner variables  $\beta_{\pm}$ ), we reduced the Mixmaster model to the very intuitive picture of a bouncing particle within an equilateral-shaped receding potential in the evolution toward the singularity ( $\alpha \rightarrow -\infty$ ). After introducing the Misner–Chitré like variables, we are able to get a dynamical scheme in which the potential walls are fixed in time and, asymptotically to the singularity, are modeled by an infinite potential well. This representation of the Mixmaster model shows how it is isomorphic in a generic time gauge to a well-known chaotic system, i.e. to a billiard-ball in a two-dimensional Lobačevskij space.

We also address the Mixmaster model in the Misner–Chitré like variables as viewed in the framework of statistical mechanics. The existence of an energy-like constant of motion characterizes the corresponding chaos in



terms of a microcanonical ensemble. The stochastic properties of the system are then summarized by the associated Liouville invariant measure.

The covariance of the Mixmaster chaos with respect to the time choice is then discussed, comparing and contrasting different results, with particular attention to the so-called fractal boundaries method.

Finally we characterize the Mixmaster dynamics when it is influenced by a scalar field, an Abelian vector potential and a cosmological constant. These studies allow to determine the cosmological implementation of this model, especially in view of the inflationary paradigm (of which the massless scalar field and the cosmological constant are a schematic representation), as presented in Chap. 5.

## 8.1 Hamiltonian Formulation of the Dynamics

In order to face the Lagrangian analysis of the Mixmaster model, we re-state for convenience the geometrical scheme associated to the homogeneity constraint. In what follows, we restrict our attention to the type VIII and IX models only, but this analysis is indeed valid for any class A model of the Bianchi classification. The only difference at the kinematical level is in the value of the structure constants.

Let us start by considering the line element for a generic homogeneous space-time in the standard ADM form

$$ds^2 = N(t)^2 dt^2 - h_{\alpha\beta} dx^\alpha dx^\beta, \quad (8.1)$$

where

$$h_{\alpha\beta} = e^{q_l} l_\alpha(x^\gamma) l_\beta(x^\gamma) + e^{q_m} m_\alpha(x^\gamma) m_\beta(x^\gamma) + e^{q_n} n_\alpha(x^\gamma) n_\beta(x^\gamma), \quad (8.2)$$

with  $q_a$  ( $a = l, m, n$ ) being functions of time only. The three linear independent vectors  $\mathbf{l}$ ,  $\mathbf{m}$  and  $\mathbf{n}$ , due to the homogeneity constraint satisfy the conditions

$$-\frac{1}{v} \mathbf{l} \cdot \text{curl} \mathbf{l} = \lambda_l \quad (8.3a)$$

$$-\frac{1}{v} \mathbf{m} \cdot \text{curl} \mathbf{m} = \lambda_m \quad (8.3b)$$

$$-\frac{1}{v} \mathbf{n} \cdot \text{curl} \mathbf{n} = \lambda_n, \quad (8.3c)$$

$$(8.3d)$$

where  $v = \mathbf{l} \cdot \mathbf{m} \wedge \mathbf{n}$  (as in Eq. (7.20)) and the three constants  $\lambda_a = (\lambda_l, \lambda_m, \lambda_n)$  correspond to the structure constants of the Bianchi classification  $(n_1, n_2, n_3)$  given in Table 7.1. For the Mixmaster model they

explicitly read as

$$\begin{aligned} \text{Type VIII : } \lambda_l &= 1, \lambda_m = 1, \lambda_n = -1, \\ \text{Type IX : } \lambda_l &= 1, \lambda_m = 1, \lambda_n = 1. \end{aligned} \quad (8.4)$$

The line element for the Bianchi space can also be expressed in terms of the 1-forms by setting

$$h_{\alpha\beta} dx^\alpha dx^\beta = \eta_{ab} \omega^a \omega^b = e^{q_a} \delta_{ab} \omega^a \omega^b, \quad (8.5)$$

so that the correspondence to **l**, **m** and **n** is obtained from

$$\begin{aligned} \omega^1 &= -\sinh \psi \sinh \theta d\phi + \cosh \psi d\theta \\ \text{(VIII)} \quad \omega^2 &= -\cosh \psi \sinh \theta d\phi + \sinh \psi d\theta \\ \omega^3 &= \cosh \theta d\phi + d\psi \end{aligned} \quad (8.6a)$$

$$\begin{aligned} \omega^1 &= \sin \psi \sin \theta d\phi + \cos \psi d\theta \\ \text{(IX)} \quad \omega^2 &= -\cos \psi \sin \theta d\phi + \sin \psi d\theta \\ \omega^3 &= \cos \theta d\phi + d\psi, \end{aligned} \quad (8.6b)$$

where  $\theta \in [0, \pi)$ ,  $\phi \in [0, 2\pi)$  and  $\psi \in [0, 4\pi)$  are the Euler angles. The Einstein-Hilbert action in vacuum (2.11) can be integrated over the spatial coordinates (involved through the 1-forms) which factorize out providing the term <sup>1</sup>

$$\int \omega^1 \wedge \omega^2 \wedge \omega^3 = \int \sin \theta d\phi \wedge d\theta \wedge d\psi = (4\pi)^2. \quad (8.7)$$

It is worth noting that this is just the surface of a three-sphere of radius 2: in fact, the closed RW model is a particular case of Bianchi IX, for  $q_a = q_b = q_c$  (see Sec. 3.2.4). Thus, the vacuum dynamical evolution of the Bianchi types VIII and IX models is summarized in terms of the variational principle

$$\delta S_B = \delta \int_{t_1}^{t_2} \mathcal{L}_B(q_a, \dot{q}_b) dt = 0. \quad (8.8)$$

Here  $t_1$  and  $t_2$  denote two fixed instants of time ( $t_1 < t_2$ ), while the Lagrangian  $\mathcal{L}_B$  reads as

$$\mathcal{L}_B = -\frac{8\pi^2 \sqrt{\eta}}{\kappa} \left[ \frac{1}{2N} (\dot{q}_l \dot{q}_m + \dot{q}_l \dot{q}_n + \dot{q}_m \dot{q}_n) - N^3 R \right]. \quad (8.9)$$

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<sup>1</sup>The integration for the Bianchi type VIII is considered over a spatial volume  $(4\pi)^2$  in order to have the same integration constant used for the type IX and to keep a uniform formalism.

A direct computation yields ( $a, b = l, m, n$ )

$$\eta^3 R = -\frac{1}{2} \left( \sum_a \lambda_a^2 e^{2q_a} - \sum_{a \neq b} \lambda_a \lambda_b e^{q_a + q_b} \right), \quad (8.10)$$

$$\eta = \det(\eta_{ab}) = \exp \left( \sum_a q_a \right). \quad (8.11)$$

From the Lagrangian formulation, the Hamiltonian for the Mixmaster dynamics is obtained by performing a Legendre transformation, i.e. by calculating the momenta  $p_a$  conjugate to the generalized coordinates  $q_a$  as

$$p_l \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_l} = -\frac{4\pi^2}{\kappa} \frac{\sqrt{\eta}}{N} (\dot{q}_m + \dot{q}_n) \quad (8.12a)$$

$$p_m \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_m} = -\frac{4\pi^2}{\kappa} \frac{\sqrt{\eta}}{N} (\dot{q}_n + \dot{q}_l) \quad (8.12b)$$

$$p_n \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_n} = -\frac{4\pi^2}{\kappa} \frac{\sqrt{\eta}}{N} (\dot{q}_l + \dot{q}_m) \quad (8.12c)$$

and then taking the standard transformation

$$N\mathcal{H}_B = \sum_{a=l,m,n} p_a \dot{q}_a - \mathcal{L}_B \quad (8.13)$$

where the  $\dot{q}_a$  are obtained from Eqs. (8.12). This way, we get the action

$$S_B = \int dt (p_a \dot{q}^a - N\mathcal{H}_B), \quad (8.14)$$

$$\mathcal{H}_B = \frac{\kappa}{8\pi^2 \sqrt{\eta}} \left[ \sum_a (p_a)^2 - \frac{1}{2} \left( \sum_b p_b \right)^2 - \frac{64\pi^4}{\kappa^2} \eta^3 R \right], \quad (8.15)$$

where  $\mathcal{H}_B = 0$  is the scalar constraints for these models.

Let us introduce the “anisotropy parameters”, defined as

$$Q_a \equiv \frac{q^a}{\sum_b q^b}, \quad \sum_a Q_a = 1. \quad (8.16)$$

The functions in Eq. (8.16) allow one to interpret the last term on the right-hand side of Eq. (8.15) as a potential for the dynamics. It can be rewritten in the form

$$\eta^3 R = -\frac{1}{2} \left( \sum_a \lambda_a^2 \eta^{2Q_a} - \sum_{b \neq c} \lambda_b \lambda_c \eta^{Q_b + Q_c} \right). \quad (8.17)$$

The main advantage of writing the potential as in Eq. (8.17), arises when investigating its proprieties in the asymptotic behavior toward the cosmological singularity ( $\eta \rightarrow 0$ ). In fact, the second term in Eq. (8.17) becomes negligible, while the value of the first one results to be strongly sensitive to the sign of the  $Q_a$ . Thus, the potential can be modeled by an infinite well as

$$-\eta^3 R = \sum_a \Theta_\infty(Q_a) \quad (8.18)$$

where

$$\Theta_\infty(x) = \begin{cases} +\infty, & \text{if } x < 0 \\ 0, & \text{if } x > 0. \end{cases} \quad (8.19)$$

By Eq. (8.18) we see how the dynamics of the Universe resembles that of a particle moving in the domain  $\Pi_Q$ , defined by the simultaneous positivity of all the anisotropy parameters  $Q_a$ .

The same Hamiltonian formulation can also be obtained from the standard ADM description of GR, as in Chap. 2. In that case, the line element (8.5) has to be inserted in Eq. (2.72a), where one defines the momenta conjugate to the three-metric as

$$\Pi^{\alpha\beta} = p_a e^{-q_a} \delta^{ab} e_a^\alpha e_b^\beta, \quad (8.20)$$

and a direct calculation yields Eq. (8.15). Furthermore, the condition of space homogeneity implies that the super-momentum constraint  $\mathcal{H}_\alpha = 0$ , as in Eq. (2.72b), is identically satisfied.

## 8.2 The Mixmaster Model in the Misner Variables

In this Section, we state the Hamiltonian dynamics of the Mixmaster model in terms of the so-called Misner-Chitré like variables, which diagonalize the kinetic part of the Hamiltonian and provide the simple scheme for the dynamics as corresponding to a two-dimensional point particle moving within a closed potential domain. When expressed in terms of the Misner-Chitré like variables the potential well becomes a stationary domain and we can determine the chaotic properties of the system, which are approached by means of a Jacobi metric representation of the geodesic flow. Indeed, in the next Section we deal with a billiard-ball on a Lobačevskij plane, i.e. a dynamically closed domain on a constantly negative curved surface.

### 8.2.1 Metric reparametrization

The main advantage arising from the variables  $q_a$  when treating the homogeneous dynamics is an explicit description of the evolution in terms of expanding and contracting axes. For the Kasner solution such variables can be easily related to the Kasner exponents  $p_l, p_m, p_n$  as

$$q_a(t) = 2p_a \ln t. \quad (8.21)$$

Let us consider a set of variables that diagonalize the kinetic term

$$KT = \sum_a (p_a)^2 - \frac{1}{2} \left( \sum_b p_b \right)^2 \quad (8.22)$$

in order to have a description resembling that of a point-particle. The quadratic form (8.22) has a negative determinant, i.e. it cannot be reduced to the typical kinetic energy of Hamiltonian systems  $\sum p_a^2$ , since it is not positive defined due to the presence of one term with a negative sign. Given the matrix  $B$

$$B = \begin{pmatrix} \frac{2}{c_1} & \frac{2}{c_1} & \frac{2}{c_1} \\ \frac{2}{c_2} & \frac{2}{c_2} & -\frac{4}{c_2} \\ \frac{2\sqrt{3}}{c_3} & -\frac{2\sqrt{3}}{c_3} & 0 \end{pmatrix}, \quad (8.23)$$

the canonical transformation

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = B^{-1} \begin{pmatrix} p_\alpha \\ p_+ \\ p_- \end{pmatrix} \quad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = B^T \begin{pmatrix} \alpha \\ \beta_+ \\ \beta_- \end{pmatrix} \quad (8.24)$$

diagonalizes  $KT$  to the form

$$KT = \frac{1}{24} (-c_1^2 p_\alpha^2 + c_2^2 p_+^2 + c_3^2 p_-^2). \quad (8.25)$$

Among all possible choices for  $c_1, c_2, c_3$ , the set  $(1, 1, 1)$  corresponds to the standard form of the so-called *Misner coordinates*  $\alpha, \beta_\pm$  defined as

$$\begin{cases} q_1 = 2(\alpha + \beta_+ + \sqrt{3}\beta_-) \\ q_2 = 2(\alpha + \beta_+ - \sqrt{3}\beta_-) \\ q_3 = 2(\alpha - 2\beta_+) \end{cases} \quad (8.26)$$

The peculiarity of the Misner variables can be outlined through the complementary definition of  $\eta_{ab}$ , expressed as the factorization

$$\eta_{ab} = e^{2\alpha} (e^{2\beta})_{ab} \leftrightarrow (\ln \eta)_{ab} = 2\alpha \delta_{ab} + 2\beta_{ab}. \quad (8.27)$$

In Eq. (8.27), the exponential term containing  $\alpha$  is related to the Universe volume while  $\beta_{ab}$  is a three-dimensional symmetric matrix with null trace representing the Universe anisotropies which can accordingly be chosen as

$$\beta_{11} = \beta_+ + \sqrt{3}\beta_- \quad (8.28a)$$

$$\beta_{22} = \beta_+ - \sqrt{3}\beta_- \quad (8.28b)$$

$$\beta_{33} = -2\beta_+ . \quad (8.28c)$$

The exponential matrix is defined as a power series of matrices, so that

$$\det(e^{2\beta}) = e^{2 \operatorname{tr} \beta} = 1 , \quad (8.29)$$

$$\eta = e^{6\alpha} . \quad (8.30)$$

### 8.2.2 Kasner solution

In terms of the variables  $(\alpha, \beta_+, \beta_-)$  we get the relations for the Kasner solution as

$$\alpha = \frac{1}{3} \ln t \quad (8.31a)$$

$$\beta_+ = \frac{1-3p_3}{6} \ln t = \frac{1-3p_3}{2} \alpha = \frac{1}{2} \left( \frac{1-3u-3u^2}{1+u+u^2} \right) \alpha \quad (8.31b)$$

$$\beta_- = \frac{p_1-p_2}{2\sqrt{3}} \ln t = \sqrt{3}(p_1-p_2)\alpha = -\frac{\sqrt{3}}{2} \frac{1+2u}{1+u+u^2} \alpha . \quad (8.31c)$$

From Eq. (8.31a) it clearly arises that  $\alpha$  is a variable expressing the isotropic component of the Universe, proportional to the logarithm of its volume, while  $\beta_{\pm}$  are linked to the anisotropy of the space. Furthermore, the first of the Kasner relations (7.51) is automatically satisfied.

As soon as we change the time variable by means of Eq. (8.31a), we can define the so-called *anisotropy velocity*  $\beta'$

$$\beta' \equiv \left( \frac{d\beta_+}{d\alpha}, \frac{d\beta_-}{d\alpha} \right) , \quad (8.32)$$

which measures the variation of the anisotropy amount with respect to the expansion, parametrized by  $\alpha$ . The expression (8.32) resembles the second Kasner relation in Eq. (7.51). The volume of the Universe behaves as  $e^{3\alpha}$  and tends to zero towards the singularity ( $\alpha \rightarrow -\infty$ ), being directly related to the temporal parameter. Thus, in the Misner variables the Kasner conditions take the simple form corresponding to the unitarity of the velocity vector  $\beta'$ , i.e.

$$|\beta'|^2 = 1 . \quad (8.33)$$

### 8.2.3 Lagrangian formulation

The variational principle (8.15) rewritten under the coordinate transformation (8.26) stands as

$$\delta S_B = \delta \int \left( p_\alpha \dot{\alpha} + p_+ \dot{\beta}_+ + p_- \dot{\beta}_- - N \mathcal{H}_B \right) dt = 0 \quad (8.34)$$

in which  $\mathcal{H}_B$  is given by

$$\mathcal{H}_B = \frac{\kappa}{3(8\pi)^2} e^{-3\alpha} \left( -p_\alpha^2 + p_+^2 + p_-^2 + \mathcal{V} \right) \quad (8.35)$$

and the potential  $\mathcal{V}$  takes the form

$$\mathcal{V} \equiv -\frac{6(4\pi)^4}{\kappa^2} \eta^3 R = \frac{3(4\pi)^4}{\kappa^2} e^{4\alpha} U_B(\beta_+, \beta_-) \quad (8.36)$$

where  $U_B$  is specified for the two Bianchi models under consideration as

$$U_{\text{VIII}} = e^{-8\beta_+} + 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+} \left( \cosh(4\sqrt{3}\beta_-) - 1 \right) \quad (8.37a)$$

$$U_{\text{IX}} = e^{-8\beta_+} - 4e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+} \left( \cosh(4\sqrt{3}\beta_-) - 1 \right). \quad (8.37b)$$

We can reconstruct the expression of the conjugate momenta by varying the action (8.34) with respect to  $p_\alpha, p_\pm$  and then inverting the relation, on the basis of the system

$$p_\alpha = -\frac{6(4\pi)^2}{N\kappa} e^{3\alpha} \dot{\alpha} \quad (8.38a)$$

$$p_\pm = \frac{6(4\pi)^2}{N\kappa} e^{3\alpha} \dot{\beta}_\pm. \quad (8.38b)$$

The variation with respect to the lapse function  $N$  generates the super-Hamiltonian constraint  $\mathcal{H} = 0$ , in agreement with the analysis performed in Sec. 2.3.3.

### 8.2.4 Reduced ADM Hamiltonian

In order to obtain the Einstein equations, the variational principle requires  $\delta S$  to vanish for arbitrary and independent variations of  $(p_\pm, p_\alpha, \beta_\pm, \alpha, N)$ . As we have seen in Sec. 2.3.3, the ADM reduction procedure prescribes the choice of one of the field variables as the temporal coordinate and to solve the constraint (8.35) with respect to the corresponding conjugate variable. It is convenient to set  $t = \alpha$  and solve  $\mathcal{H}_B = 0$  with respect to  $p_\alpha$  as

$$\mathcal{H}_{\text{ADM}} \equiv -p_\alpha = \sqrt{p_+^2 + p_-^2 + \mathcal{V}}. \quad (8.39)$$

Through Eq. (8.39) we express  $p_\alpha$  in the action integral, so that the reduced variational principle in the canonical form reads as

$$\delta S_{\text{ADM}} = \delta \int (p_+ d\beta_+ + p_- d\beta_- - \mathcal{H}_{\text{ADM}} d\alpha) = 0. \quad (8.40)$$

The dynamical picture is completed by taking into account the choice  $\dot{\alpha} = 1$  which fixes the temporal gauge according to (8.38a), i.e.

$$N_{\text{ADM}} = \frac{6(4\pi)^2}{\mathcal{H}_{\text{ADM}} \kappa} e^{3\alpha}. \quad (8.41)$$

The dynamical evolution of the Bianchi type VIII and IX cosmological models using the isotropic variable  $\alpha$ , characterizing the Universe volume as the appropriate time variable, has been established. Correspondingly, the pure gravitational degrees of freedom are identified to the variables describing the Universe anisotropy ( $\beta_\pm$ ). Finally, we introduce the anisotropy parameters  $Q_\alpha$ , that, in terms of the Misner variables, read as

$$Q_1 = \frac{1}{3} + \frac{\beta_+ + \sqrt{3}\beta_-}{3\alpha} \quad (8.42a)$$

$$Q_2 = \frac{1}{3} + \frac{\beta_+ - \sqrt{3}\beta_-}{3\alpha} \quad (8.42b)$$

$$Q_3 = \frac{1}{3} - \frac{2\beta_+}{3\alpha}. \quad (8.42c)$$

### 8.2.5 Mixmaster dynamics

In this Subsection, we present the approach to the Mixmaster dynamics as developed by Misner in 1969.

The Hamiltonian introduced so far differs from the typical expression of classical mechanics for the non-positive definiteness of the kinetic term, i.e. the sign in front of  $p_\alpha^2$ . The potential term is a function of  $\alpha$  (i.e. of *time*) and of the Universe anisotropy parametrized by  $\beta_\pm$ .

The Hamiltonian approach as in Eq. (8.38) provides the equations of motion as

$$\dot{\alpha} = N \frac{\partial \mathcal{H}_{\text{IX}}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -N \frac{\partial \mathcal{H}_{\text{IX}}}{\partial \alpha}, \quad (8.43a)$$

$$\dot{\beta}_\pm = N \frac{\partial \mathcal{H}_{\text{IX}}}{\partial p_\pm}, \quad \dot{p}_\pm = -N \frac{\partial \mathcal{H}_{\text{IX}}}{\partial \beta_\pm}. \quad (8.43b)$$

This set, considered together with the explicit form of the potential (see Figs. 8.1 and 8.2), can be interpreted as the motion of a “point-particle” in a potential well, where the term  $\mathcal{V}$  is proportional to the curvature scalar.



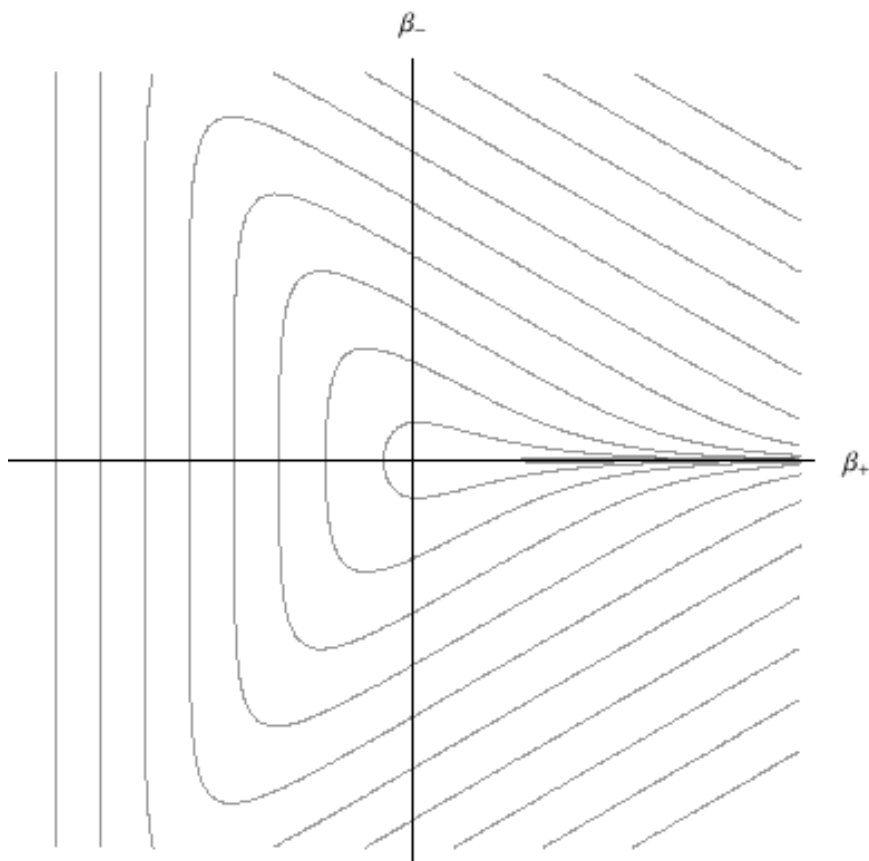


Figure 8.1 Equipotential lines of the Bianchi type VIII model in the  $\beta_+, \beta_-$  plane.

In the regions of the configuration space where  $\mathcal{V}$  can be neglected, the dynamics resembles the pure Kasner behavior, corresponding to  $|\beta'| = 1$ .

Asymptotically close to the origin  $\beta_{\pm} = 0$ , the equipotential lines for the Bianchi type VIII are ellipses

$$U_{\text{VIII}}(\beta_+, \beta_-) = 5 - 16\beta_+ + 40\beta_+^2 + 72\beta_-^2 + \mathcal{O}(\beta_{\pm}^3), \quad (8.44)$$

while for the Bianchi type IX are approximated by circles

$$U_{\text{IX}}(\beta_+, \beta_-) = -3 + 24(\beta_+^2 + \beta_-^2) + \mathcal{O}(\beta_{\pm}^3). \quad (8.45)$$

The expressions of the equipotential lines for large values of  $|\beta_+|$  and small

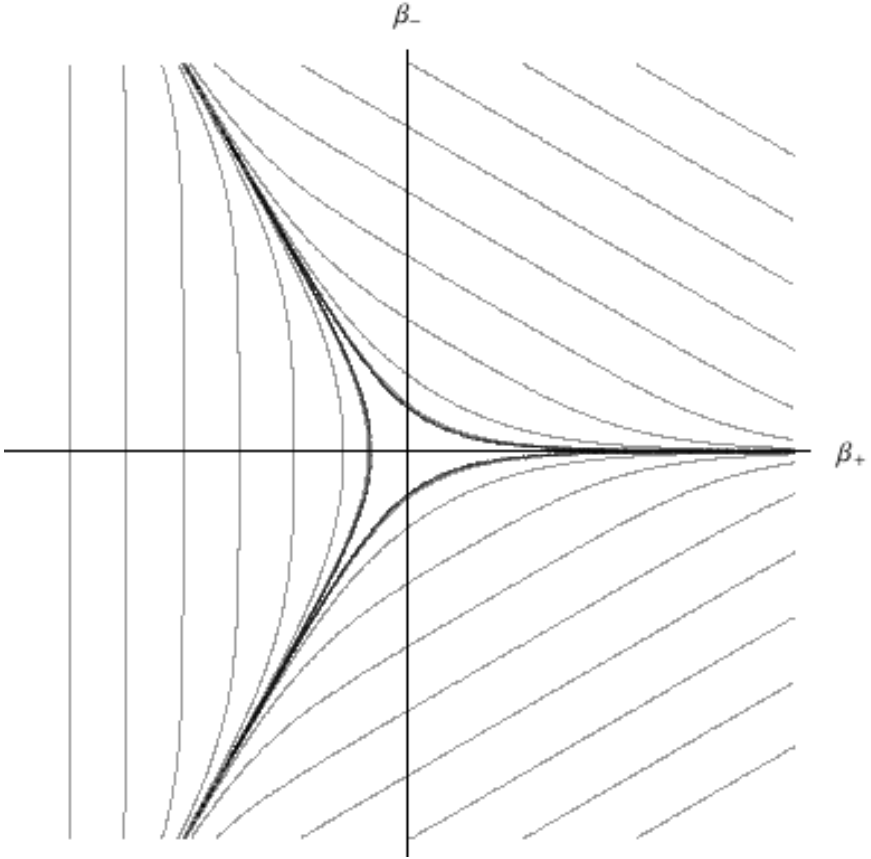


Figure 8.2 Equipotential lines of the Bianchi type IX model in the  $\beta_+, \beta_-$  plane.

$|\beta_-|$  are the same for both types

$$U(\beta) \simeq \begin{cases} e^{-8\beta_+}, & \beta_+ \rightarrow -\infty, |\beta_-| \ll 1 \\ 48\beta_-^2 e^{4\beta_+}, & \beta_+ \rightarrow +\infty, |\beta_-| \ll 1. \end{cases} \quad (8.46)$$

Figures 8.1 and 8.2 represent some of the equipotential lines  $U(\beta) = \text{const.}$ , for which the potential values increment of a factor  $e^8 \sim 3 \times 10^3$  for  $\Delta\beta \sim 1$ .

The Universe evolution is described as the motion of a point-like particle governed by such potential terms, and is characterized by a sequence of bounces against the potential walls when the system evolves towards the singularity. Analogously to the BKL approach of Sec. 7.4, the evolution consists of a series of Kasner epochs when  $|\beta'| = 1$ , i.e. the point-Universe

moves far from the walls; then a new epoch with different Kasner parameters takes place after a bounce according to the BKL map.

Let us describe in more detail the bounces performed by the billiard ball representing the Universe. From the asymptotic form (8.46) for the Bianchi IX potential, we get the equipotential line  $\beta_{\text{wall}}$  cutting the region where the potential terms are significant. The condition for the potential to be relevant near the cosmological singularity is given by  $e^{4(\alpha-2\beta_+)} \simeq \mathcal{H}_{\text{ADM}}^2$  or, in terms of  $\beta_{\text{wall}}$ , by

$$\beta_+ \simeq \beta_{\text{wall}} = \frac{\alpha}{2} - \frac{1}{8} \ln(\mathcal{H}_{\text{ADM}}^2). \quad (8.47)$$

As described before, inside the allowed potential domain, the dynamics is governed by the Kasner evolution, i.e.  $\mathcal{H}_{\text{ADM}}$  is constant as in the Bianchi I model. From Eq. (8.47) we get  $|\beta'_{\text{wall}}| = 1/2$ , i.e. the point in the  $(\beta_+, \beta_-)$  plane moves twice as fast as the receding potential wall. The point-Universe will thus collide against the wall and will be reflected from one straight-line motion (Kasner dynamics) to another one.

A reflection-like relation lays for the bounces. This interval of evolution for Bianchi IX, i.e. a two-dimensional particle bouncing against a single wall, is equivalent to the dynamics of the Bianchi II model and it is analytically integrable (see Sec. 7.3.1). Its ADM Hamiltonian is given by

$$\mathcal{H}_{\text{ADM}}^{\text{II}} = \left( p_+^2 + p_-^2 + \frac{3(4\pi)^4}{\kappa^2} e^{4(\alpha-2\beta_+)} \right)^{1/2}, \quad (8.48)$$

which is independent of  $\beta_-$ , i.e.  $p_-$  is a constant of motion. Let us search for another first integral of motion: such quantity can be recovered by a linear combination of  $p_+$  and  $\mathcal{H}_{\text{ADM}}^{\text{II}}$ , in particular as

$$\Omega = \mathcal{H}_{\text{ADM}}^{\text{II}} - p_+/2. \quad (8.49)$$

The reflection law for the incoming and outgoing particle nearby the wall can be obtained as follows. Let us denote the angles of incidence and of reflection of the particle off the potential wall as  $\theta_i$  and  $\theta_f$ , respectively.

The velocity  $\beta'$  is parametrized before the bounce as  $(\beta'_+)_i = -\cos\theta_i$ ,  $(\beta'_-)_i = \sin\theta_i$ , and as  $(\beta'_+)_f = \cos\theta_f$  and  $(\beta'_-)_f = \sin\theta_f$  after the bounce. Therefore, considering that  $p_-$  and  $\Omega$  are constants of motion, as well as remembering that  $\beta'_\pm = p_\pm/\mathcal{H}$ , the relation

$$\sin\theta_f - \sin\theta_i = \frac{1}{2} \sin(\theta_i + \theta_f) \quad (8.50)$$

holds. This represents the reflection map for the bounce for which a limit angle for the collisions appears. The maximum angle such that a bounce

against the wall occurs is given by

$$|\theta_i| < |\theta_{\max}| = \arccos\left(\frac{\beta'_{\text{wall}}}{\beta'}\right), \quad (8.51)$$

and hence, since  $\beta'_{\text{wall}}/\beta' = 1/2$ , the maximum incidence angle is given by  $|\theta_{\max}| = \pi/3$ . In the foregoing bounces the  $\beta$ -particle will collide on a different wall and, because of the wall motion, the angle  $|\theta_f| > \pi/2$  is allowed. Let us observe that, in terms of the parameter  $u$  introduced in Sec. 7.3.2, the relation (8.50) reads as  $u_f = u_i - 1$ .

### 8.3 Misner-Chitré Like Variables

A valuable framework of analysis of the Mixmaster evolution, able to join together the two points of view of the map approach and of the continuous dynamics evolution, relies on a Hamiltonian treatment of the equations in terms of the Misner-Chitré variables, firstly introduced by Chitré in his PhD thesis (1972). Such formulation allows one to fix the existence of an asymptotic (energy-like) constant of motion once an ADM reduction is performed. By this result, the stochasticity of the Mixmaster can be treated either in terms of statistical mechanics (by the microcanonical ensemble), either by its characterization as isomorphic to a billiard on a two-dimensional Lobačevskij space. Such scheme can be constructed independently of the choice of a time variable, simply providing very general Misner-Chitré like (MCl) coordinates. The standard Misner-Chitré variables  $(\tau, \zeta, \theta)$  are the following

$$\alpha = -e^\tau \cosh \zeta \quad (8.52a)$$

$$\beta_+ = e^\tau \sinh \zeta \cos \theta \quad (8.52b)$$

$$\beta_- = e^\tau \sinh \zeta \sin \theta \quad (8.52c)$$

where  $0 \leq \zeta < \infty$ ,  $0 \leq \theta < 2\pi$ , and  $-\infty < \tau < \infty$ . In order to discuss the results concerning chaoticity and dynamical properties, it is useful to deal with a slight modification to the set (8.52) via the MCl coordinates  $(\Gamma(\tau), \xi, \theta)$  through the transformations

$$\alpha = -e^{\Gamma(\tau)} \xi \quad (8.53a)$$

$$\beta_+ = e^{\Gamma(\tau)} \sqrt{\xi^2 - 1} \cos \theta \quad (8.53b)$$

$$\beta_- = e^{\Gamma(\tau)} \sqrt{\xi^2 - 1} \sin \theta \quad (8.53c)$$

where  $1 \leq \xi < \infty$ , and  $\Gamma(\tau)$  stands for a *generic* function of  $\tau$ : the variables in Eq. (8.52) correspond to setting  $\Gamma(\tau) = \tau$  and  $\xi = \cosh \zeta$ . Such modified

set of variables permits to write the anisotropy parameters  $Q_a$  defined in Eq. (8.42) as *independent* of the variable  $\Gamma$  in the form

$$Q_1 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi} \left( \cos \theta + \sqrt{3} \sin \theta \right) \quad (8.54a)$$

$$Q_2 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi} \left( \cos \theta - \sqrt{3} \sin \theta \right) \quad (8.54b)$$

$$Q_3 = \frac{1}{3} + 2 \frac{\sqrt{\xi^2 - 1}}{3\xi} \cos \theta. \quad (8.54c)$$

The dynamical quantities, if expressed in terms of the relations (8.54), will be independent of  $\Gamma(\tau)$  too.

The variational principle and the Hamiltonian (8.35) in these new variables read as

$$\delta \int \left( p_\xi \dot{\xi} + p_\theta \dot{\theta} + p_\tau \dot{\tau} - N\mathcal{H} \right) dt = 0, \quad (8.55)$$

$$\mathcal{H} = \frac{\kappa}{3(8\pi)^2} \frac{e^{-2\Gamma}}{\sqrt{\eta}} \left[ -\frac{p_\tau^2}{\left( \frac{d\Gamma}{d\tau} \right)^2} + p_\xi^2 (\xi^2 - 1) + \frac{p_\theta^2}{\xi^2 - 1} + \mathcal{V} e^{2\Gamma} \right], \quad (8.56)$$

$$\sqrt{\eta} = \exp \left\{ -3\xi e^{\Gamma(\tau)} \right\}. \quad (8.57)$$

The solution to the super-Hamiltonian constraint leads to the expression involving  $\mathcal{H}_{\text{ADM}}$  as

$$-p_\tau \equiv \frac{d\Gamma}{d\tau} \mathcal{H}_{\text{ADM}} = \frac{d\Gamma}{d\tau} \sqrt{\varepsilon^2 + \mathcal{V} e^{2\Gamma}}, \quad (8.58)$$

where

$$\varepsilon^2 \equiv (\xi^2 - 1) p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1}. \quad (8.59)$$

In terms of this constraint, the principle (8.55) reduces to the form

$$\delta \int \left( p_\xi \dot{\xi} + p_\theta \dot{\theta} - \dot{\Gamma} \mathcal{H}_{\text{ADM}} \right) dt = 0, \quad (8.60)$$

whose variation provides the Hamiltonian equations for  $\xi$  and  $\theta$  as

$$\dot{\xi} = \frac{\dot{\Gamma}}{\mathcal{H}_{\text{ADM}}} (\xi^2 - 1) p_\xi \quad (8.61a)$$

$$\dot{\theta} = \frac{\dot{\Gamma}}{\mathcal{H}_{\text{ADM}}} \frac{p_\theta}{(\xi^2 - 1)} \quad (8.61b)$$

$$\dot{p}_\xi = -\dot{\Gamma} \left[ \frac{\xi}{\mathcal{H}_{\text{ADM}}} \left( p_\xi^2 - \frac{p_\theta^2}{(\xi^2 - 1)^2} \right) + \frac{e^{2\Gamma}}{2\mathcal{H}_{\text{ADM}}} \frac{\partial \mathcal{V}}{\partial \xi} \right], \quad (8.61c)$$

$$\dot{p}_\theta = -\dot{\Gamma} \frac{e^{2\Gamma}}{2\mathcal{H}_{\text{ADM}}} \frac{\partial \mathcal{V}}{\partial \theta}. \quad (8.61d)$$

Analogously to the derivation of Eq. (8.41), the time-gauge relation is expressed as

$$N_{\text{ADM}}(t) = \frac{3(8\pi)^2}{2\kappa} \frac{\sqrt{\eta} e^{2\Gamma}}{\mathcal{H}_{\text{ADM}}} \frac{d\Gamma}{d\tau} \dot{\tau}, \quad (8.62)$$

thus our analysis remains fully independent of the choice of the time variable until the form of  $\Gamma$  and  $\dot{\tau}$  is fixed.

### 8.3.1 The Jacobi metric and the billiard representation

In this Section, we construct the Jacobi metric associated to the dynamics of the *billiard ball* discussed above, relying on a formulation independent of the choice of a specific gauge. In fact, all the analyses can be restated in terms of the time variable  $\Gamma$  without specifying the form of the lapse function but, for the sake of convenience, we will take the restriction  $\dot{\tau} = 1$ . The variational principle (8.60) can be rewritten as

$$\delta \int \left( p_\xi \frac{d\xi}{d\Gamma} + p_\theta \frac{d\theta}{d\Gamma} - \mathcal{H}_{\text{ADM}} \right) d\Gamma = 0. \quad (8.63)$$

Nevertheless, for any choice of the time variable  $\tau$  (for example  $\tau = t$ ), there exists a corresponding function  $\Gamma(\tau)$  (i.e. a set of MCl variables leading to the scheme (8.63)) defined by the invertible relation

$$\frac{d\Gamma}{dt} = \frac{2\kappa}{3(8\pi)^2} \frac{N_{\text{ADM}} \mathcal{H}_{\text{ADM}}}{\sqrt{\eta}} e^{-2\Gamma}. \quad (8.64)$$

The asymptotically vanishing of  $\sqrt{\eta}$  near the initial singularity is ensured by the Landau-Raichadhury theorem (see Sec. 2.4), which stands in this general scheme too, as far as  $\Gamma(t)$  is an increasing and unbounded function of  $t$

$$\sqrt{\eta} \rightarrow 0 \quad \Rightarrow \quad \Gamma(t) \rightarrow \infty. \quad (8.65)$$

Approaching the initial singularity, the limit  $\sqrt{\eta} \rightarrow 0$  for the Mixmaster potential (8.17) implies an infinite potential well behavior, as discussed in Sec. 8.1. In this reduced Hamiltonian formulation, the term  $\Gamma(t)$  plays the role of a parametric function of time and the anisotropy parameters  $Q_a$  are functions of the variables  $\xi, \theta$  only (see Eq. (8.54)). Therefore in the dynamically allowed domain  $\Pi_Q$  (see Fig. 8.3) the ADM Hamiltonian becomes asymptotically an integral of motion as

$$\forall \{\xi, \theta\} \in \Pi_Q \quad \begin{cases} \mathcal{H}_{\text{ADM}} = \sqrt{\varepsilon^2 + e^{2\Gamma} \mathcal{V}} \cong \varepsilon \\ \frac{\partial \mathcal{H}_{\text{ADM}}}{\partial \Gamma} = 0 \Rightarrow \varepsilon = E = \text{const.} \end{cases} \quad (8.66)$$

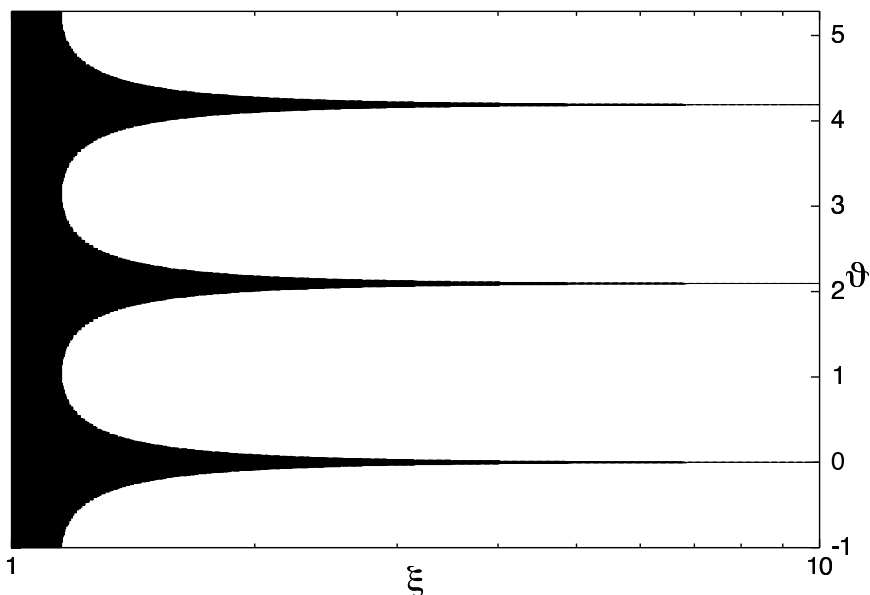


Figure 8.3 The region  $\Pi_Q(\xi, \theta)$  of the configuration space where the conditions  $Q_a \geq 0$  are fulfilled. The dynamics of the point Universe is restricted by means of the curvature term which corresponds to an infinite potential well.

The variational principle (8.63) reduces to

$$\delta \int (p_\xi d\xi + p_\theta d\theta - \varepsilon d\Gamma) = \delta \int (p_\xi d\xi + p_\theta d\theta) = 0, \quad (8.67)$$

which holds since the third term of the integral on the left-hand side behaves as an exact differential ( $\varepsilon = E$ ).

By following the standard Jacobi procedure to reduce the variational principle to a geodesic one in terms of the configuration variables  $x^a$ , we set  $x^{a'} = dx^a/d\Gamma \equiv g^{ab}p_b$  and, by the Hamiltonian equations (8.61) expressed in terms of  $\Gamma$ , we obtain the metric

$$g^{\xi\xi} = \frac{1}{E} (\xi^2 - 1), \quad g^{\theta\theta} = \frac{1}{E} \frac{1}{\xi^2 - 1}. \quad (8.68)$$

By Eq. (8.68) and using the fundamental constraint relation obtained rewriting Eq. (8.59) as

$$(\xi^2 - 1) p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1} = E^2, \quad (8.69)$$

it can be shown that

$$g_{ab} x^{a'} x^{b'} = \frac{1}{E} \left[ (\xi^2 - 1) p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1} \right] = E. \quad (8.70)$$

Using the definition

$$x^{a'} = \frac{dx^a}{ds} \frac{ds}{d\Gamma} = u^a \frac{ds}{d\Gamma}, \quad (8.71)$$

Eq. (8.70) is rewritten as

$$g_{ab} u^a u^b \left( \frac{ds}{d\Gamma} \right)^2 = E, \quad (8.72)$$

leading to the relation

$$d\Gamma = \sqrt{\frac{g_{ab} u^a u^b}{E}} ds. \quad (8.73)$$

Indeed Eq. (8.73) together with  $p_\xi \xi' + p_\theta \theta' = E$  allows us to put the variational principle (8.67) in the geodesic form

$$\delta \int E d\Gamma = \delta \int \sqrt{g_{ab} u^a u^b E} ds = \delta \int \sqrt{G_{ab} u^a u^b} ds = 0, \quad (8.74)$$

where the metric  $G_{ab} \equiv E g_{ab}$  satisfies the normalization condition  $G_{ab} u^a u^b = 1$  and therefore

$$\frac{ds}{d\Gamma} = E. \quad (8.75)$$

In Eq. (8.73) we adopted the positive root, according to the requirement that the curvilinear coordinate  $s$  increases monotonically with increasing values of  $\Gamma$ , i.e. approaching the initial cosmological singularity.

Summarizing, the dynamical problem in the region  $\Pi_Q$  reduces to a geodesic flow on a two-dimensional Riemannian manifold described by the line element

$$ds^2 = E^2 \left[ \frac{d\xi^2}{\xi^2 - 1} + (\xi^2 - 1) d\theta^2 \right]. \quad (8.76)$$

The above metric has negative curvature, since the associated curvature scalar is  $R = -2/E^2$ . Therefore the point-Universe moves over a negatively curved bidimensional space on which the potential wall (8.18) cuts the region  $\Pi_Q$ , depicted in Fig. 8.3. By a way completely independent of the time gauge, a full representation of the system as isomorphic to a billiard ball on a Lobačevskij plane has been provided. Indeed, the freedom of the gauge choice relies on the possibility to express  $\Gamma(t)$  via a generic lapse function (8.62) which, for  $\dot{\Gamma} = 1$ , reads as

$$N_{\text{ADM}}(t) = \frac{3(8\pi)^2 \sqrt{\eta} e^{2\Gamma}}{2\kappa \mathcal{H}_{\text{ADM}}}. \quad (8.77)$$



### 8.3.2 Some remarks on the billiard representation

From a geometrical point of view, the domain defined by the potential walls is not strictly closed, since there are three directions corresponding to the three corners in the traditional Misner picture from which the point Universe could in principle escape (see Fig. 8.3). However, as discussed in Sec. 7.4 for the Bianchi models in the BKL framework, the only case in which an asymptotic solution of the field equations shows this behavior corresponds to having two scale factors equal to each other (i.e.  $\theta = 0$ ). Nevertheless, these cases, corresponding to the Taub Universe (see Sec. 10.10.1) are dynamically unstable and correspond to sets of zero measure in the space of the initial conditions. Thus, in this sense we can neglect the probability to reach such configurations and the domain is *de facto* dynamically closed.

The bounces (in the billiard configuration) against the potential walls together with the geodesic flow instability on a closed domain of the Lobačevskij plane imply the Mixmaster point-Universe to have stochastic features. Indeed, types VIII and IX are the only Bianchi models having a compact configuration space, hence the claimed compactness of the domain guarantees that the geodesic instability is upgraded to a real stochastic behavior. On the other hand, the possibility to deal with a stochastic scattering is justified by the constant negative curvature of the Lobačevskij plane and therefore these two notions (compactness and curvature) are necessary for these considerations.

## 8.4 The Invariant Liouville Measure

In this Section the derivation of the invariant measure of the Mixmaster model is provided in a generic time gauge. Indeed, the ADM reduction of the variational problem asymptotically close to the cosmological singularity permits to model the Mixmaster dynamics by a two-dimensional point-Universe randomizing in a closed domain with fixed “energy” (just the ADM kinetic energy), as in Eq. (8.66).

From the statistical mechanics point of view, such stochastic motion within the closed domain  $\Pi_Q$  induces in the phase-space a suitable micro-canonical ensemble representation in view of the existence of the “energy-like” constant of motion. The stochasticity of this system can then be

described in terms of the Liouville invariant measure

$$d\rho = \text{const} \times \delta(E - \varepsilon) d\xi d\theta dp_\xi dp_\theta \quad (8.78)$$

characterizing the microcanonical ensemble. The particular value taken by the variable  $\varepsilon$  ( $\varepsilon = E$ ) does not influence the stochastic properties of the system and must be fixed by the initial conditions. This redundant information for the statistical dynamics is removable by integrating over all admissible values of  $\varepsilon$ . Introducing the natural variables  $(\varepsilon, \phi)$  in place of  $(p_\xi, p_\theta)$  by the transformation

$$p_\xi = \frac{\varepsilon}{\sqrt{\xi^2 - 1}} \cos \phi, \quad p_\theta = \varepsilon \sqrt{\xi^2 - 1} \sin \phi, \quad 0 \leq \phi < 2\pi \quad (8.79)$$

the Dirac distribution is integrated out, leading to the uniform and normalized invariant measure

$$d\mu = d\xi d\theta d\phi \frac{1}{8\pi^2}. \quad (8.80)$$

The approximation on which this analysis is based (i.e. the potential wall model) is reliable since it is dynamically induced, no matter what time variable  $\Gamma$  is adopted. Furthermore, such invariant measure turns out to be independent of the choice of the temporal gauge, as shown by Eq. (8.77).

It can be shown that, by virtue of the system (8.61), the asymptotic functions  $\xi(\Gamma), \theta(\Gamma), \phi(\Gamma)$  during the free geodesic motion are governed by the equations

$$\frac{d\xi}{d\Gamma} = \sqrt{\xi^2 - 1} \cos \phi, \quad (8.81a)$$

$$\frac{d\theta}{d\Gamma} = \frac{\sin \phi}{\sqrt{\xi^2 - 1}}, \quad (8.81b)$$

$$\frac{d\phi}{d\Gamma} = -\frac{\xi \sin \phi}{\sqrt{\xi^2 - 1}}, \quad (8.81c)$$

which admit a parametric solution. However, the global behavior of  $\xi$  along the whole geodesic flow is described by the invariant measure (8.80) and therefore the temporal behavior of  $\Gamma(t)$  acquires a stochastic character. If we assign one of the two functions  $\Gamma(t)$  or  $N(t)$  with an *arbitrary* analytic functional form, then the other one will exhibit a stochastic behavior by virtue of the  $\xi$ -dependence for the quantity  $\sqrt{\eta}$ . Finally, by retaining the same dynamical scheme adopted in the construction of the invariant measure, the one-to-one correspondence between any lapse function and the associated set of MCl variables (8.53) guarantees the covariance with respect to the time gauge.

## 8.5 Invariant Lyapunov Exponent

The application of standard methods to characterize the chaotic behavior of the Mixmaster model has taken a large amount of the efforts made over the last two decades on this cosmological model. Despite this, a definitive assertion fully based on an exact dynamics is still lacking because the standard chaos indicators, typically adopted for classical systems, are not straightforwardly extendible to relativistic systems. Indeed two main features make the Mixmaster model challenging to be treated as a dynamical system:

- the vanishing of its Hamiltonian
- the kinetic term is not positive definite.

A detailed discussion of these aspects, as well as a brief review of successes and failures concerning various attempts made over the years, is given in Sec. 8.6. Here we show that the possibility to reduce the Mixmaster dynamics to a two-dimensional one, endowed with an energy-like constant of motion  $\varepsilon$ , allows us to apply the standard notion of Lyapunov exponents. Furthermore, the gauge-free nature of this representation implies a covariant characterization of the chaotic feature associated to a positive Lyapunov exponent for a compact configuration space.

This approach relies on a billiard configuration, resulting from the dynamical evolution of the real system when the singularity is approached. In fact, the infinite walls schematization of the potential picture comes out from the asymptotic vanishing behavior of the metric determinant. Despite its viability, the treatment we are going to describe replaces a precise dynamical system (the *exact* Mixmaster) with an approximated scheme.

The dynamical instability of the billiard in terms of an invariant treatment (with respect to the choice of the coordinates  $(\xi, \theta)$ ) emerges introducing the orthonormal tetradic basis

$$v^i = \left( \frac{1}{E} \sqrt{\xi^2 - 1} \cos \phi, \frac{1}{E} \frac{\sin \phi}{\sqrt{\xi^2 - 1}} \right) \quad (8.82a)$$

$$w^i = \left( -\frac{1}{E} \sqrt{\xi^2 - 1} \sin \phi, \frac{1}{E} \frac{\cos \phi}{\sqrt{\xi^2 - 1}} \right). \quad (8.82b)$$

Indeed, the vector  $v^i$  is nothing but the geodesic field, i.e. it satisfies

$$\frac{dv^i}{ds} + \Gamma_{kl}^i v^k v^l = 0, \quad (8.83)$$

while the vector  $w^i$  is parallel transported along the geodesics, according to the equation

$$\frac{dw^i}{ds} + \Gamma_{kl}^i v^k w^l = 0, \quad (8.84)$$

where  $\Gamma_{kl}^i$  are the Christoffel symbols constructed by the reduced metric (8.76). Projecting the geodesic deviation equation along the vector<sup>2</sup>  $w^i$  the corresponding connecting vector (tetradic) component  $Z$  satisfies the equivalent equation

$$\frac{d^2 Z}{ds^2} = \frac{Z}{E^2}. \quad (8.85)$$

This expression, as a projection on the tetradic basis, is a scalar one and therefore completely independent of the choice of the variables. Its general solution reads as

$$Z(s) = c_1 e^{s/E} + c_2 e^{-s/E}, \quad c_{1,2} = \text{const.}, \quad (8.86)$$

and the corresponding invariant Lyapunov exponent is defined as

$$\lambda_v = \sup \lim_{s \rightarrow \infty} \frac{\ln \left( Z^2 + \left( \frac{dZ}{ds} \right)^2 \right)}{2s}, \quad (8.87)$$

which, in terms of Eq. (8.86), takes the value

$$\lambda_v = \frac{1}{E} > 0. \quad (8.88)$$

The limit (8.87) is well defined as soon as the curvilinear coordinate  $s$  approaches infinity. In fact, from Eq. (8.75) the singularity corresponds to the limit  $\Gamma \rightarrow \infty$ , and this implies  $s \rightarrow \infty$ .

When the point-Universe bounces against the potential walls, it is reflected from a geodesic to another one, thus making each of them unstable. Though with the limit of the potential wall approximation, this result shows that, independently of the choice of the temporal gauge, the Mixmaster dynamics is isomorphic to a well-known chaotic system. Equivalently, in terms of the BKL representation, the free geodesic motion corresponds to the evolution during a Kasner epoch and the bounces against the potential walls to the transition between two of them. The positivity of the Lyapunov exponent (8.88) is not enough to ensure the system chaoticity, since its derivation remains valid for any Bianchi type model. The crucial point is that for the Mixmaster (types VIII and IX) the potential walls reduce

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<sup>2</sup>Its component along the geodesic field  $v^i$  does not provide any physical information about the system instability.

the configuration space to a compact region ( $\Pi_Q$ ), i.e. the geodesic motion fills the entire configuration space.

Furthermore, it can be shown that the Mixmaster asymptotic dynamics and the structure of the potential walls fulfill the hypotheses at the basis of the Wojtkowsky theorem, thus ensuring that the largest Lyapunov exponent has a positive sign almost<sup>3</sup> everywhere.

Generalizing, for any choice of the time variable, one gets a stochastic representation of the Mixmaster model, provided the factorized coordinate transformation in the configuration space

$$\alpha = -e^{\Gamma(\tau)} a(\theta, \xi) \quad (8.89a)$$

$$\beta_+ = e^{\Gamma(\tau)} b_+(\theta, \xi) \quad (8.89b)$$

$$\beta_- = e^{\Gamma(\tau)} b_-(\theta, \xi) , \quad (8.89c)$$

where  $\Gamma, a, b_{\pm}$  denote generic functional forms of the variables  $\tau, \theta, \xi$ .

The present analysis relies on the use of a standard ADM reduction of the variational principle (which reduces the system by one degree of freedom) and overall on adopting MCl variables, ensuring that the asymptotic potential walls are fixed in time.

## 8.6 Chaos Covariance

We have discussed the oscillatory regime in the Hamiltonian framework characterizing the behavior of the Bianchi types VIII and IX cosmological models as discussed in Chap. 7 within the BKL formalism near a physical singularity, outlining their chaotic properties: firstly, the dynamical evolution of the Kasner exponents characterized the sequence of the Kasner epochs, each one described by its own line element, with the epoch sequence nested in multiple eras. Secondly, the use of the parameter  $u$  and its relation to dynamical functions offered the statistical treatment connected to each Kasner era, finding an appropriate expression for the distribution over its domain of variation: the entire evolution has been decomposed in a discrete mapping in terms of the rational/irrational initial values attributed to BKL.

---

<sup>3</sup>It can be shown that, given a dynamical system of the form  $d\mathbf{x}/dt = \mathbf{F}(x)$ , the positivity of the associated Lyapunov exponents are invariant under the diffeomorphism:  $\mathbf{y} = \phi(\mathbf{x}, t)$ ,  $d\tau = \lambda(\mathbf{x}, t)dt$ , as soon as several requirements hold; these requirements are fulfilled by the present approximation.

### 8.6.1 *Shortcomings of Lyapunov exponents*

We can outline two conceptual limits for the said approaches:

- the BKL formalism corresponds to a non-continuous evolution toward the initial singularity
- the Hamiltonian approach lacks a proper definition of chaos according to the indicators commonly used in the theory of dynamical systems, i.e. of Lyapunov exponents.

A wide literature faced over the years this subject in order to provide the best possible understanding of the resulting chaotic dynamics.

The research activity developed overall in two different, but related, directions:

- (i) on one hand, the dynamical analysis was devoted to remove the limits of the BKL approach related to its discrete nature (by analytical treatments and by numerical simulations)
- (ii) on the other hand, to get a better characterization of the Mixmaster chaos (especially in view of its properties of covariance).

The first line of investigation provided satisfactory representations of the Mixmaster dynamics in terms of continuous variables, mainly studying the properties of the BKL map and its reformulation in the Poincaré plane.

In parallel to these studies, detailed numerical descriptions have been performed aiming to test the validity of the analytical results.

The efforts to develop a precise characterization of chaos relies on the ambiguity to apply the standard indicators to relativistic systems. In fact, the chaotic properties summarized so far were questioned when numerical evolution of the Mixmaster equations yielded zero Lyapunov exponents. Nevertheless, other numerical studies found an exponential divergence of initially nearby trajectories with positive Lyapunov numbers. This discrepancy was solved when considering that, both numerically and analytically, such calculations depended on the choice of the time variable and, in parallel, on the failure of the conservation of the Hamiltonian constraint in the numerical simulations.

In particular, the first clear distinction between the direct numerical study of the dynamics and the map approximation, stating the appearance of chaos and its relation with the increase of entropy, has been introduced by Burd, Buric and Tavakol (1991). The puzzle consisted of simulations providing zero Lyapunov numbers, claiming that the Mixmaster Universe

is non-chaotic with respect to the intrinsic time (associated with the function  $\alpha$  introduced for the Hamiltonian formalism) but chaotic with respect to the synchronous time (i.e., the temporal parameter  $t$ ). The non-zero claims about Lyapunov exponents, using different time variables, have been obtained reducing the Universe dynamics to a geodesic flow on a pseudo-Riemannian manifold. Moreover, a geometrized model of dynamics defining an average rate of separation of nearby trajectories in terms of a geodesic deviation equation in a Fermi basis has been interpreted for detection of chaotic behavior. A non-definitive result was given: the principal Lyapunov exponent results always positive in the BKL approximation but, if the period of oscillations in the long phase is infinite (corresponding to the long oscillations when the particle enters the corners of the potential), the principal Lyapunov exponent tends to zero.

For example, Berger in 1990 reports the dependence of the Lyapunov exponent on the choice of the time variable. Through numerical simulations, the Lyapunov exponents were evaluated along some trajectories in the  $(\beta_+, \beta_-)$  plane for different choices of the time variable, more precisely  $\tau$  (BKL),  $\Omega$  (Misner) and  $\lambda$ , the “mini-superspace” one, i.e.  $d\lambda = |-p_\Omega^2 + p_+^2 + p_-^2|^{1/2} d\tau$ . The same trajectory giving zero Lyapunov exponent for  $\tau$  or  $\Omega$ -time, fails for  $\lambda$ .

Such contrasting results are explained by the non-covariant nature of the indicators adopted due to their inapplicability to hyperbolic manifolds. This feature prevented, up to now, to say a definitive word about the general picture concerning the covariance of the Mixmaster chaos, with particular reference to the possibility of removing the observed chaotic features by a suitable choice of the time variable, apart from the indications provided in the next subsection.

The ambiguity which arises when changing the time variable depends on the vanishing of the Mixmaster Hamiltonian and its non-positive definite kinetic term (typical of a gravitational system). These features prevent the direct application of the most used criteria for characterizing the chaotic behavior of a dynamical system.

Although a whole line of research opened up, the first widely accepted indications in favor of covariance were derived with a fractal formalism by Cornish and Levin (1997). Indeed, a complete covariant description of the Mixmaster chaos, in terms of continuous dynamical variables, is lacking due to the discrete nature of the fractal approach.

### 8.6.2 On the occurrence of fractal basin

In order to give an invariant characterization of the dynamics chaoticity, many methods along the years have been proposed, but not all approaches have reached an undoubtable consensus. An interesting one, as introduced above, relying on the fractal basin of initial conditions evolution has been proposed in 1997 by Cornish and Levin and opened a debate. The conflict among the different approaches has been tackled by using an observer-independent fractal method, though leaving some questions open about the conjectures lying at its basis.

The asymptotic behavior towards the initial singularity of a Bianchi type IX trajectory depends on whether or not one has a rational or irrational initial condition for the parameter  $u$  in the BKL map. The numerical treatment on which the fractal basin boundary method is based necessarily deals with rational values for the initial conditions. In such a scheme, the effect of the Gauss map has been considered together with the evolution of the equations of motion, in order to “uncover” dynamical properties about the possible configurations varying with the initial conditions. Nevertheless, such approach led to some doubts regarding the reliability of the method itself.

In fact, let us observe that initial conditions with rational numbers are dense but yet constitute a set of zero measure and correspond to *fictitious* singularities. The nature of this initial set needs to be compared with the *complete* set of initial conditions given by the whole real set, with finite measure over a finite interval: the conclusions arising from the dynamical evolution are not complementary between the two domains of initial conditions. In this sense, predictions coming out from the set of rational initial conditions only cannot be extrapolated to the general case.

Cornish and Levin used a coordinate-independent fractal method to show that the Mixmaster Universe is indeed chaotic. By exploiting techniques originally developed for the chaotic scattering, they found a fractal structure, namely *the strange repellor* (see Fig. 8.4) for the Mixmaster cosmology that indeed well describes chaos. A strange repellor is the collection of all Universes periodic in the space of the model parameters while an aperiodic one experiences a transient age of chaos if it brushes against the repellor. The fractal pattern was exposed in the numerical integration of the Einstein equations and in the discrete map used to approximate the solution. The fractal approach would be independent of the time coordinate and the chaos reflected in the fractal weave of Mixmaster Universes



would be unambiguous.

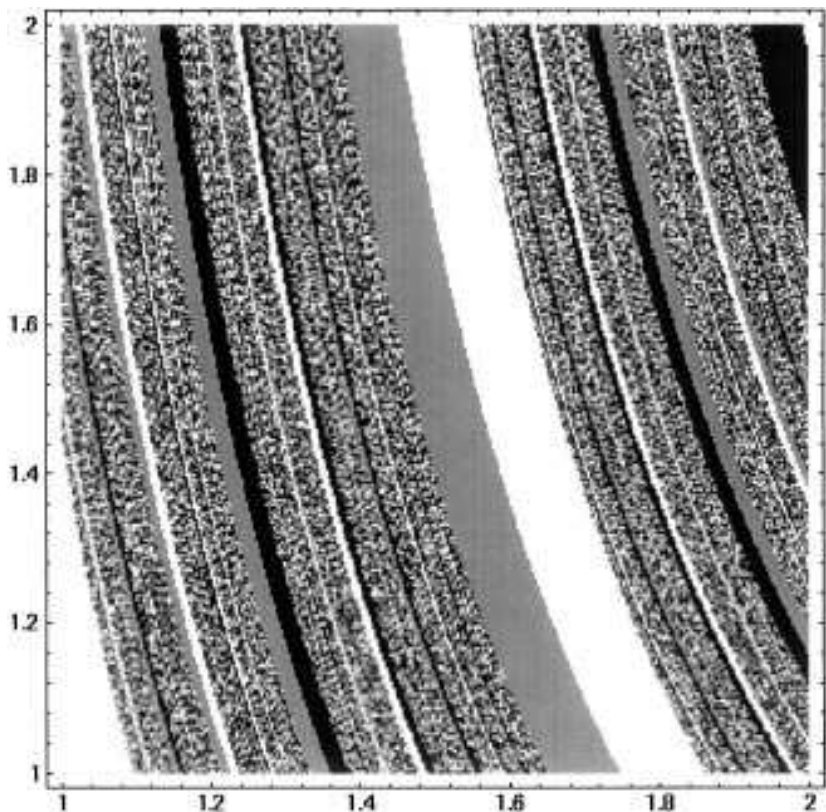


Figure 8.4 The numerically generated basin boundaries in the  $(u, v)$  plane are built of Universes which ride the repeller for many orbits before being thrown off. Similar fractal basins can be found by viewing alternative slices through the phase space, such as the  $(\beta, \dot{\beta})$  plane. The overall morphology of the basin is altered little by demanding more strongly anisotropic outcomes (Reprinted figure with permission from N.J. Cornish and J.J. Levin, *Phys. Rev. Lett.*, 78, 998 (1997). Copyright (1997) by the American Physical Society. <http://link.aps.org/doi/10.1103/PhysRevLett.78.998>).

This work is widely accepted in the literature but it is worth noting the following points:

- (1) the chosen points representing this framework are the ones whose dynamics never reaches the singularity due to the intrinsic numerical limit;

- (2) in the exact Mixmaster dynamics, the natural dynamical evolution predicts that the point particle representing the Universe evolution enters the corner with the velocity *not parallelly* oriented towards the corner's bisecting line and, after some oscillations, it is sent back to the middle of the potential. This effect is altered when opening the potential corners, as requested by the numerics.
- (3) The artificial opening up of the potential corners adopted in the basin boundary approach could induce itself the fractal nature.

## 8.7 Cosmological Chaos as a Dimensional and Matter Dependent Phenomenon

### 8.7.1 The role of a scalar field

Here we face the influence of a scalar field when approaching the cosmological singularity showing how it can suppress the Mixmaster oscillations.

Let us consider the Mixmaster Universe in the presence of a self-interacting scalar field  $\phi$ . The Einstein equations are obtained from the variation of the action in the Hamiltonian form associated to the constraint  $\mathcal{H} = 0$ . In particular, the scalar field is rescaled in order that the relative factor between  $p_\phi^2$  and  $p_\alpha^2$  equals the unity and we choose the gauge  $N \propto e^{3\alpha}$  in order to simplify the form of the super-Hamiltonian (8.35) which reads as

$$\mathcal{H} = KT + PT, \quad (8.90)$$

$KT$  and  $PT$  being the kinematic and potential parts of the Hamiltonian, respectively, i.e.

$$KT = -p_\alpha^2 + p_+^2 + p_-^2 + p_\phi^2, \quad (8.91)$$

$$PT = \frac{3(4\pi)^4}{\kappa^2} e^{4\alpha} U_{\text{IX}} + e^{6\alpha} V(\phi), \quad (8.92)$$

where  $V(\phi)$  denotes a generic potential of the scalar field. In the Misner variables the cosmological singularity appears as  $\alpha \rightarrow -\infty$ . Therefore, unless  $V(\phi)$  contains terms growing enough with  $|\alpha|$ , the very last term in Eq. (8.92) can be neglected at early times, i.e.  $e^{6\alpha} V(\phi) \rightarrow 0$  as  $\alpha \rightarrow -\infty$ .

Assuming that the spatial curvature can be neglected, i.e. dealing only with  $KT$ , then it is easy to verify that the equations of motion admit the following solution expressed in terms of  $\alpha$

$$\begin{cases} \beta_\pm = \beta_\pm^0 + \pi_\pm |\alpha| \\ \phi = \phi^0 + \pi_\phi |\alpha|, \end{cases} \quad (8.93)$$

where  $\pi_{\pm} = p_{\pm}/|p_{\alpha}|$  and  $\pi_{\phi} = p_{\phi}/|p_{\alpha}|$ . The constraint  $KT = 0$  then becomes

$$\pi_+^2 + \pi_-^2 + \pi_{\phi}^2 = 1. \quad (8.94)$$

Let us study the behavior of the potential (8.92) when the scalar field is not present. From Eq. (8.94) evaluated for  $\pi_{\phi} = 0$ , we can parametrize

$$\pi_+ = \cos \theta \quad (8.95a)$$

$$\pi_- = \sin \theta. \quad (8.95b)$$

Through Eq. (8.93), the potential (8.92) rewrites as

$$PT \sim e^{-4|\alpha|(1+2\cos\theta)} + e^{-4|\alpha|(1-\cos\theta-\sqrt{3}\sin\theta)} + e^{-4|\alpha|(1-\cos\theta+\sqrt{3}\sin\theta)}, \quad (8.96)$$

where we retained the dominant terms only, i.e. the first three in Eq. (8.92). Except for the set of zero measure  $\theta = (0, 2\pi/3, 4\pi/3)$ , any generic value of  $\theta$  will cause the growth of one of the terms on the r.h.s. of Eq. (8.96) as  $\alpha \rightarrow -\infty$ ; thus the Kasner regime is not stable toward the singularity. Let us consider the case  $\phi \neq 0$  and hence  $\pi_{\phi}^2 > 0$ . Equation (8.94) is restated by

$$\pi_+^2 + \pi_-^2 = 1 - \pi_{\phi}^2 < 1, \quad (8.97)$$

thus none of the terms in Eq. (8.96) grows if the following conditions are satisfied

$$\begin{cases} 1 + 2\pi_+ > 0, \\ 1 - \pi_+ - \sqrt{3}\pi_- > 0, \\ 1 - \pi_+ + \sqrt{3}\pi_- > 0, \end{cases} \quad (8.98)$$

situation which realizes for  $\pi_+^2 < 1/2$  and  $\pi_-^2 < 1/12$ , that is  $2/3 < \pi_{\phi}^2 < 1$ . It can be shown that  $p_{\alpha}$  decreases at each bounce and therefore, for any initial value of  $p_{\phi}$ , the condition above will be satisfied.

As we have seen, the approach to the singularity of the vacuum Bianchi IX model is described by a particle moving in a potential with exponentially closed walls bounding a triangular domain. During the evolution, the particle bounces against the walls providing an infinite number of oscillations toward the singularity. The scalar field influences such dynamics so that for values of  $\pi_{\pm}$  satisfying the conditions (8.98), there are not further bounces and the solution approaches a final stable Kasner regime. In other words, there will be an instant of time after which the point-Universe will never reach the potential walls again and no more oscillations will appear. In this sense the scalar field can suppress the chaotic Mixmaster dynamics toward the classical cosmological singularity.

### 8.7.2 The role of a vector field

In this Section, the effects of an Abelian vector field on the dynamics of a generic  $(n+1)$ -dimensional homogeneous model in the BKL scheme are investigated.

A generic  $(n+1)$ -dimensional space-time coupled to an Abelian vector field  $A_\mu = (\varphi, A_\alpha)$ , with  $\alpha = 1, 2, \dots, n$ , in the ADM framework is described by the action

$$S_{\text{g+EM}} = \int d^n x dt \left( \Pi^{\alpha\beta} \frac{\partial}{\partial t} h_{\alpha\beta} + E^\alpha \frac{\partial}{\partial t} A_\alpha + \varphi \mathcal{D}_\alpha E^\alpha - N\mathcal{H} - N^\alpha \mathcal{H}_\alpha \right), \quad (8.99)$$

$$\mathcal{H} = \frac{1}{\sqrt{h}} \left[ \Pi_\beta^\alpha \Pi_\alpha^\beta - \frac{1}{n-1} (\Pi_\alpha^\alpha)^2 + \frac{1}{2} h_{\alpha\beta} E^\alpha E^\beta + h \left( \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - {}^n R \right) \right], \quad (8.100a)$$

$$\mathcal{H}_\alpha = -2\nabla_\beta \Pi_\alpha^\beta + E^\beta F_{\alpha\beta}, \quad (8.100b)$$

where  $F_{\alpha\beta}$  is the spatial electromagnetic tensor, and the relation  $\mathcal{D}_\alpha \equiv \partial_\alpha + A_\alpha$  holds (see Sec. 2.2.4). Moreover,  $E^\alpha$  and  $\Pi^{\alpha\beta}$  are the conjugate momenta to the electromagnetic field and to the  $n$ -metric tensor, respectively, which are a vector and a tensorial density of weight  $1/2$ . The variation with respect to the lapse function  $N$  yields the super-Hamiltonian constraint  $\mathcal{H} = 0$ , while the one with respect to  $\varphi$  provides the Gauss constraint  $\mathcal{D}_\alpha E^\alpha = 0$ .

We deal with a sourceless Abelian vector field and thus consider the transverse (or Lorentz) components for  $A_\alpha$  and  $E^\alpha$  only. Therefore, we choose the gauge conditions  $\varphi = 0$  and  $\mathcal{D}_\alpha E^\alpha = 0$ , in order to prevent the longitudinal components of the vector field from taking part to the action. In the general case, i.e. either in the presence of the sources or in the case of non-Abelian vector fields, this simplification can no longer take place in such explicit form and the terms  $\varphi(\partial_\alpha + A_\alpha)E^\alpha$  must be considered in the action principle.

A BKL-like analysis can be developed: after introducing a set of Kasner vectors<sup>4</sup>  $\mathbf{l}_a$  and the Kasner-like scale factors  $\exp(q^a/2)$ , the dynamics is

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<sup>4</sup>We recall that this notation implies that the vectors have to be formally treated as having Euclidean components.

dominated by a potential of the form  $\sum e^{2q_a} \tilde{\lambda}_a^2$ , where  $\tilde{\lambda}_a$  are the projections of the momenta of the Abelian field along the Kasner vectors. With the same spirit of the Mixmaster analysis developed in Sec. 7.3.1, an unstable  $n$ -dimensional Kasner-like evolution arises; nevertheless the potential term inhibits the solution to last up to the singularity and induces the BKL-like transition to another epoch. Given the relation  $\exp(q^a) = t^{2p_a}$ , the map that links two consecutive epochs is ( $a = 2, \dots, n$ )

$$p'_1 = \frac{-p_1}{1 + \frac{2}{n-2}p_1}, \quad p'_a = \frac{p_a + \frac{2}{n-2}p_1}{1 + \frac{2}{n-2}p_1}, \quad (8.101a)$$

$$\tilde{\lambda}'_1 = \tilde{\lambda}_1, \quad \tilde{\lambda}'_a = \tilde{\lambda}_a \left( 1 - 2 \frac{(n-1)p_1}{(n-2)p_a + np_1} \right). \quad (8.101b)$$

An interesting new feature, resembling that of the inhomogeneous Mixmaster, is the *rotation of the Kasner vectors*, expressed as

$$\mathbf{l}'_a = \mathbf{l}_a + \sigma_a \mathbf{l}_1, \quad (8.102a)$$

$$\sigma_a = \frac{\tilde{\lambda}'_a - \tilde{\lambda}_a}{\tilde{\lambda}_1} = -2 \frac{(n-1)p_1}{(n-2)p_a + np_1} \frac{\tilde{\lambda}_a}{\tilde{\lambda}_1}, \quad (8.102b)$$

which completes the dynamical scheme.

The homogeneous Universe approaches the initial singularity described by a metric tensor with oscillating scale factors and rotating Kasner vectors. Passing from one Kasner epoch to another, the negative Kasner index  $p_1$  is exchanged between different directions (for instance  $\mathbf{l}_1$  and  $\mathbf{l}_2$ ) and, at the same time, these directions rotate in space according to the rule (8.102b). The presence of a vector field, independently of the considered model, induces a dynamically closed domain on the configuration space. In correspondence to these oscillations of the scale factors, the Kasner vectors  $\mathbf{l}_a$  rotate and the quantities  $\sigma_a$  remain constant during a Kasner epoch to lowest order in  $q^a$ . The vanishing of the determinant  $h$  approaching the singularity does not significantly affect the rotation law (8.102b). The resulting dynamics provides a map exhibiting a *dimensional-dependence*, and it reduces to the standard BKL one for  $n = 3$ .

## 8.8 Isotropization Mechanism

The isotropic FRW model is accurate to describe the backward evolution of the Universe up to the decoupling time, i.e.  $3 \times 10^5$  years after the Big Bang. Moreover there are well-established indications that the isotropic dynamics

is the natural scenario for the primordial nucleosynthesis process, i.e. the validity of the RW geometry up to  $10^{-2} - 10^{-3}$  seconds after the Big Bang. There is no argument against the idea that the isotropic Universe can be extrapolated up to the inflationary age, i.e.  $t \sim 10^{-34}$ s. The comparison of predictions from inflation with the CMB data (see Sec. 4.4) provides a significant evidence that after the inflationary process our Universe retains an isotropic morphology up to a very high degree of precision on a scale depending on the model parameters (see Chap. 5). On the other hand, the description of the very early stages requires more general models, like as the homogeneous ones, as suggested by the instability shown in the backward evolution of the FRW Universe with respect to tensor perturbations. Therefore it is interesting to investigate the mechanisms allowing a transition between these two cosmological epochs. When the anisotropy of the Universe is sufficiently suppressed, one deals with a quasi-isotropization of the model and such configuration can be regarded as a “bridge” between the two stages. In this paragraph we discuss the origin of a background space<sup>5</sup> when a real self-interacting scalar field  $\phi$  is taken into account. We adopt the same rescaling for  $\phi$  as in Sec. 8.7.1.

Let us restate the Misner-like variables to include the scalar field as

$$\beta_+ \equiv \beta^1, \quad \beta_- \equiv \beta^2, \quad \phi \equiv \sqrt{3}\beta^3. \quad (8.103)$$

The action describing the Universe in such scheme reads as

$$S_{g+\phi} = \int dt \left[ p_r \partial_t \beta^r + p_\alpha \partial_t \alpha - \frac{N\kappa}{3(8\pi)^2} e^{-3\alpha} \left( \sum_r p_r^2 - p_\alpha^2 + \mathcal{U} \right) \right], \quad (8.104)$$

where  $r = 1, 2, 3$ ,  $p_r$  and  $p_\alpha$  are the conjugate momenta to  $\beta_r$  and  $\alpha$ , respectively. The potential term is defined as  $\mathcal{U} = e^{6\alpha} W(\phi) + \mathcal{V}$ ,  $\mathcal{V}$  follows from Eq. (8.36) and  $W(\phi) = 1/2[h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi] + V(\phi)$ . From Eq. (8.104) an inflationary solution comes out imposing the constraint

$$e^{-6\alpha}\mathcal{U} \simeq V(\phi) \simeq \text{const.} \gg e^{-2\alpha}U_{\text{IX}} \quad (8.105)$$

which can be realized by an appropriate process of spontaneous symmetry breaking, exhaustively studied in Chap. 5.

Let us consider the situation when  $\mathcal{U} = e^{6\alpha}\rho_\Lambda$ , where  $\rho_\Lambda = \text{const.}$  The Hamilton-Jacobi equation then takes the form

$$\sum_r \left( \frac{\delta S}{\delta \beta^r} \right)^2 - \left( \frac{\delta S}{\delta \alpha} \right)^2 + \exp(6\alpha)\rho_\Lambda = 0, \quad (8.106)$$

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<sup>5</sup>The chaotic nature of the evolution toward the singularity implies that the geometry, and therefore all the geometrical quantities, should be described in an average sense only. With this respect, during the vacuum Mixmaster, the Universe does not possess a stable background near the singularity.

whose solution can be expressed as

$$S(\beta^r, \alpha) \sim \sum_r K_r \beta^r + \frac{1}{3} K_\alpha + \frac{K}{6} \ln \left| \frac{K_\alpha - K}{K_\alpha + K} \right|, \quad (8.107)$$

where  $K_\alpha(K_r, \alpha) = \pm \sqrt{\sum_r K_r^2 + \rho_\Lambda \exp(6\alpha)}$ , with some generic constants  $K = \sqrt{\sum_r K_r^2}$  and  $K_r$ . The equation of motion for  $\alpha$  is readily obtained from Eq. (8.104) as

$$\frac{d\alpha}{dt} = -\frac{2N\kappa}{3(8\pi)^2} p_\alpha e^{-3\alpha}. \quad (8.108)$$

Choosing  $\alpha$  as the time coordinate, i.e.  $d\alpha/dt = 1$ , the time gauge condition becomes  $N = -3(8\pi)^2 \exp(3\alpha)/(2\kappa p_\alpha)$ . Since the lapse function is positive defined we must also have  $p_\alpha < 0$ .

According to the Hamilton-Jacobi method, one has firstly to differentiate with respect to  $K^r$  and then to equate the result to arbitrary constants  $\beta_0^r$ , so getting

$$\frac{\delta S}{\delta K_r} = \beta_0^r \quad \Rightarrow \quad \beta^r(\alpha) = \beta_0^r + \frac{K_r}{6|K|} \ln \left| \frac{K_\alpha - K}{K_\alpha + K} \right|. \quad (8.109)$$

Let us consider the two limits of interest. First of all, for  $\alpha \rightarrow \infty$  ( $K_\alpha \rightarrow \infty$ ) the solution (8.109) transforms into the inflationary one corresponding to the quasi-isotropization of the model as the functions  $\beta^r$  approach the constants  $\beta_0^r$ . Such values are reabsorbed in the Kasner vectors and therefore this limit is equivalent to getting a vanishing Universe anisotropy.

On the opposite limit, i.e. for  $\alpha \rightarrow -\infty$  ( $K_\alpha \rightarrow K$ ), the solution (8.109) provides the generalized Kasner one as expected, simply modified by the presence of the scalar field

$$\beta^r(\alpha) = \beta_0^r - \frac{K_r}{K} \alpha. \quad (8.110)$$

The existence of the solution (8.109) shows how the inflationary scenario can provide the necessary dynamical “bridge” between the fully anisotropic and the quasi-isotropic epochs during the Universe evolution. In fact, during that time, the anisotropies  $\beta_\pm$  are dumped away and the only effective dynamical variable is  $\alpha$ , i.e. the isotropic volume of the Universe. This shows how the dominant term during the inflation is  $\rho_\Lambda e^{6\alpha}$  and any term involving the spatial curvature becomes more and more negligible although increasing like (at most)  $e^{4\alpha}$ .

## 8.9 Guidelines to the Literature

For an introduction to the original formulation of the Hamiltonian dynamics, presented in Sec. 8.1, associated to the Mixmaster model see [345]. An analysis of the Bianchi models in the variables  $Q_a$  can be found in [286].

A reference textbook providing a satisfactory and advanced description of analytical mechanics topics relevant to the presented approach is that of Arnold [15].

The Hamiltonian analysis in the variables, presented in Sec. 8.2 diagonalizing the super-Hamiltonian was firstly introduced by Misner in [345].

For a comprehensive discussion of the reduced ADM-procedure of the Bianchi models dynamics, given in Sec. 8.2.4 in terms of the Misner variables, we refer to the textbook of Misner, Thorne & Wheeler [347], Ch. 30. The general dynamical scheme underlying such reduction was firstly provided in [19].

The restatement of the reduced Hamiltonian dynamics in terms of generalized Misner-Chitré like variables, provided in Sec. 8.2.5 can be recovered in [258]. The first proposal for this type of variables is due to Chitré in [119].

A textbook on the general features of the ergodic theory, as arising for the Mixmaster model in the Misner-Chitré like variables, is the one by Cornfeld, Fomin & Sinai [130].

A valuable textbooks on the Jacobi metric associated to a geodesic flow and on the corresponding ergodic properties (Sec. 8.3), are the ones by Anosov [14] and by Arnold & Avez [16]. For the specific application to the Bianchi models, see [119] and [258].

The formulation of the invariant Liouville measure, presented in Sec. 8.4, had two fundamental steps. For the first characterization of a measure in the configuration space of the Mixmaster, see [118] and for its extension to the phase-space see [286] (for a discussion of the Artin theorem adopted in such approach, see [20]). For a discussion of the covariance of the Liouville measure, see [257]. The analysis of non-stationary corrections to the Mixmaster invariant measure can be recovered in [352]. An additional interesting literature on the stochastic properties of the Mixmaster can be found in [39, 40, 429].

A general analysis of the properties of classical chaotic systems can be found in [369]. A collection of interesting reviews on the problem of chaos in General Relativity is provided by the proceedings volume [211].

The debate of invariant Lyapunov exponents, introduced in Sec. 8.5 was characterized by different conceptual stages. A demonstration of the co-



variance of the Mixmaster chaos in the billiard-ball representation is given in [258]. For a discussion of the notion of Lyapunov exponents in relativistic cosmological systems, see [105] (the original definition of Lyapunov exponents can be found [421]). For a derivation of the Wojtkowski theorem, see [468]. A generic result that links the Lyapunov exponents in different frames is given in [356]. There is a wide literature in this subject in order to provide the best possible understanding of the resulting chaotic dynamics. The research activity developed overall in two different, but related, directions: on the one hand the removal of the limits of the BKL approach due to its discrete nature (by analytical treatments [40, 106, 118] and by numerical simulations [74, 75, 79, 88, 405]); or getting a better characterization of the Mixmaster chaos (especially in view of its properties of covariance [171, 172, 238, 429]). For a review on such a topic, see [77, 354].

For a discussion of the main features concerning the fractal boundary approach and its implementation on the Mixmaster dynamics, in Sec. 8.6.2, see the following literature: [89, 131, 132, 141, 142, 210, 238, 410, 430].

The first derivation of the chaos removal when the Mixmaster has a massless scalar field source, describe in Sec. 8.7.1, was provided in [59, 60]. For a discussion in the Hamiltonian formalism of the same features, see [76].

A discussion of the role that an Abelian vector field plays in the Mixmaster dynamics in a multi-dimensional space-time, as in Sec. 8.7.2, is introduced in [69]. This paper contains the details of the calculation reproduced in this Section.

An analysis of the role that a cosmological constant can have in isotropizing the Mixmaster dynamics, discussed in Section 8.8, can be found in [287] and in the review article [354].

## Chapter 9

# The Generic Cosmological Solution Near the Singularity

In this Chapter we will analyze the extension of the Mixmaster dynamics to the inhomogeneous sector by constructing the generic cosmological solution near the initial singularity. After a discussion on the Bianchi IX model instability towards the singularity, we will outline how the Kasner evolution introduced in Chap. 7 can be upgraded to describe an inhomogeneous regime, approaching a singular point and having three physically arbitrary functions available to specify the Cauchy problem on a non-singular space-like hypersurface. Such generalized Kasner solution can be stable up to the vanishing of the space volume only if a given condition holds for its metric functions, which prevents to deal with four physical degrees of freedom, as required by the generality of the Cauchy problem.

By relaxing such restriction, we are naturally led to represent the generic cosmological solution as a piecewise Kasner approximation by an iterative scheme fully equivalent to that singled out for the homogeneous Mixmaster. Indeed the spatial points dynamically decouple toward the singularity and play only a parametric role in the Einstein equations. The BKL map retains exactly the same form as in Chap. 7, but its point-like nature induces a coupling between the chaotic time dependence of the Mixmaster and the spatial morphology of the three-hypersurfaces. In particular near enough to the singularity the space-time takes the structure of a real foam, with a classical statistical nature.

The Hamiltonian formulation of the inhomogeneous Mixmaster is considered in the Misner variables with stationary Kasner axes, and also in a more general gauge-independent framework, when the ADM reduction of the system is fully developed. Such general scheme allows to extend the co-variant study performed in Chap. 8, of the Mixmaster chaos covariance to the generic inhomogeneous sector. An estimate of the spatial gradient

behavior shows how the BKL conjecture stating the local validity of the Mixmaster (say, within each causal horizon) is statistically well-grounded. Finally, the Mixmaster dynamics is re-analyzed by introducing a set of variables appropriate to deal with a dynamical system approach.

This Chapter is concluded with a multidimensional analysis of the inhomogeneous Mixmaster model which outlines how, up to ten space-time dimensions, the chaotic features are preserved. However, for higher dimensional cases, we outline the existence of an open region of the Kasner sphere where the Kasner regime is stable (known as the *Kasner Stability Region*). The attractive character of this region in the parameter space can be inferred by the properties of the multidimensional BKL map. For space-times with more than ten dimensions, the chaotic features of the inhomogeneous Mixmaster are suppressed in favor of a stable Kasner epoch reaching the initial singularity.

## 9.1 Inhomogeneous Perturbations of Bianchi IX

In this Section we describe the inhomogeneous perturbations to a homogeneous Mixmaster Universe. The interest in such topic is twofold:

- (i) it represents a first step towards introducing more degrees of freedom than those available for the homogeneous sector of GR, thus linking the gravitational (toy) models with the full field theory.
- (ii) The dynamics of the perturbations should probe some insight into the BKL conjecture (see Sec. 9.2) about the generic (namely inhomogeneous) cosmological singularity. This topic has been firstly studied by Regge and Hu in 1972.

Dealing with a homogeneous space allows to simplify the usual construction of perturbations. In such spaces every point is equivalent to any other under the action of an isometric group (see Sec. 7.1.1) and one can perform all computations at one specific point in space. The general form of the equations are then generated by simple group invariant operations on the manifold and the set of tensors are composed by the direct product of the basis invariant forms operating on the representation function of the group. In this case, one does not have to construct basis tensor harmonics as functions of the whole space (like the hyper-spherical tensor harmonics in the FRW metric; see Sec. 3.5), but rather one can evaluate the product at one point and use the invariant operators to generate the complete set.

Any tensor field in a homogeneous space can be expanded in terms of these tensor harmonics. The perturbations can in general be expressed in terms of the (left) invariant 1-forms (7.28) with time-dependent expansion coefficients coupled to the representation functions of the underlying symmetry group of the manifold. For a  $SO(3)$ -homogeneous space-time (namely the Bianchi IX model), the representation functions are the so-called Wigner  $D$ -functions  $D_{m'm}^j(g)$ , labeled by the spin number  $j$  and the magnetic numbers  $(m', m)$ . We remind that, because of the identification of the (topological) three-sphere  $S^3$  with the group manifold  $SU(2) \simeq SO(3)$ , we can use group elements  $g = g(x^\gamma)$  to coordinate the physical space of Bianchi IX (which has the  $S^3$  topology).

Let us firstly recall the construction of the Wigner  $D$ -functions, obtained from their definition in terms of matrix elements of the rotation operator

$$\mathcal{R}(\phi, \theta, \psi) = \exp(-i\phi \mathbf{j}_x) \exp(-i\theta \mathbf{j}_y) \exp(-i\psi \mathbf{j}_z), \quad (9.1)$$

in which  $\mathbf{j}_x, \mathbf{j}_y, \mathbf{j}_z$  are generators of the  $su(2)$  Lie algebra and  $(\phi, \theta, \psi)$  are the Euler angles parametrizing the  $SO(3)$  left-invariant 1-forms (8.6b). The Wigner functions can be explicitly expressed as

$$D_{m'm}^j = \langle j m' | \mathcal{R}(\phi, \theta, \psi) | j m \rangle = e^{-im'\phi} d_{m'm}^j(\theta) e^{-im\psi}, \quad (9.2)$$

where

$$d_{m'm}^j(\theta) = C_{m'm}^j \sum_s \tilde{C}_{m'm}^j(s) \left( \cos \frac{\theta}{2} \right)^{2j+m-m'-2s} \left( \sin \frac{\theta}{2} \right)^{m'-m+2s} \quad (9.3a)$$

$$C_{m'm}^j = \sqrt{(j+m')!(j-m')!(j+m)!(j-m)!} \quad (9.3b)$$

$$\tilde{C}_{m'm}^j(s) = \frac{(-1)^{m'-m+s}}{[(j+m-s)!s!(m'-m+s)!(j-m'-s)!]}, \quad (9.3c)$$

where the sum is over the values of  $s$  for which the factorials are non-negative. For  $j = l \in \mathbb{N}$ , these functions are simply related to the spherical harmonics  $Y_{l,m}(\theta, \phi)$  as

$$D_{m'm}^l(g) = (-1)^{-m'} \sqrt{4\pi/(2l+1)} Y_{l,-m'}(\theta, \phi) e^{im\psi}. \quad (9.4)$$

The Wigner  $D$ -functions can be obtained requiring to satisfy the differential equations (expressed in terms of the Euler angles)

$$\begin{aligned} \hat{L}^2 D_{m'm}^j &= \left[ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} - 2 \cos \theta \frac{\partial^2}{\partial \phi \partial \psi} + \frac{\partial^2}{\partial \psi^2} \right) \right] D_{m'm}^j \\ &= j(j+1) D_{m'm}^j, \end{aligned} \quad (9.5a)$$

$$\hat{L}_3 D_{m'm}^j = -i \frac{\partial}{\partial \psi} D_{m'm}^j = m' D_{m'm}^j, \quad (9.5b)$$

$$\hat{L}_z D_{m'm}^j = -i \frac{\partial}{\partial \phi} D_{m'm}^j = m D_{m'm}^j. \quad (9.5c)$$

In Eqs. (9.5) we have introduced the two bases  $\{\hat{L}_1, \hat{L}_2, \hat{L}_3\}$  and  $\{\hat{L}_x, \hat{L}_y, \hat{L}_z\}$  of generators for the  $su(2)$  algebra and their common Casimir operator

$$\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \quad (9.6)$$

which is an invariant of the group. The angular momentum operators  $\{\hat{L}_1, \hat{L}_2, \hat{L}_3\}$  of the three-dimensional rotation group in quantum mechanics (which are the intrinsic angular momentum operators of a rigid body) are related to the (left-invariant) vector fields  $e_\alpha^a$  (7.19) via the relations

$$\hat{L}_1 = ie_1^\alpha \partial_\alpha, \quad \hat{L}_2 = ie_2^\alpha \partial_\alpha, \quad \hat{L}_3 = ie_3^\alpha \partial_\alpha. \quad (9.7)$$

The angular momentum operators  $\{\hat{L}_x, \hat{L}_y, \hat{L}_z\}$  are in turn related to the (right-invariant) Killing vector fields  $\xi_\alpha^a$  (see Sec. 7.1.1) by the formulae

$$\hat{L}_x = -i\xi_1^\alpha \partial_\alpha, \quad \hat{L}_y = -i\xi_2^\alpha \partial_\alpha, \quad \hat{L}_z = -i\xi_3^\alpha \partial_\alpha. \quad (9.8)$$

Let us describe the inhomogeneous perturbations. Be  $\bar{h}_{\alpha\beta}(x, t)$  the (unperturbed) spatial metric of the Bianchi IX model. A generic perturbation

$$\gamma_{\alpha\beta}(x, t) = h_{\alpha\beta}(x, t) - \bar{h}_{\alpha\beta}(x, t) \quad (9.9)$$

to the unperturbed three-metric can be translated into a matrix of space-scalars  $\gamma_{ab}(x, t)$  by projecting it on the invariant 1-forms, that is

$$\gamma_{\alpha\beta}(x, t) = \gamma_{ab}(x, t) \omega_\alpha^a(x) \omega_\beta^b(x). \quad (9.10)$$

According to the previous discussion, these scalars can be decomposed in terms of definite angular-momentum components of the three-metric  $\gamma_{ab}^{jm}(x, t)$ , labeled by spin and magnetic numbers ( $j, m$ )

$$\gamma_{ab}(x, t) = \sum_{j, m} \gamma_{ab}^{jm}(x, t). \quad (9.11)$$

These, in turns, can be expressed in terms of Wigner  $D$ -functions (9.2) as

$$\gamma_{ab}^{jm}(x, t) = \sum_{m'=-j}^j \gamma_{ab}^{jm'}(t) D_{m'm}^j(g) \quad (9.12)$$

and therefore we are expanding inhomogeneous perturbations (intended as scalar harmonic functions on  $S^3$ ) as a linear combination of Wigner  $D$ -functions. The time dependent amplitudes  $\gamma_{ab}^{jm'}(t)$  represent, at fixed  $j$ ,  $6(2j+1)$  inhomogeneous degrees of freedom (in the case of diagonal matrices  $\gamma_{ab}$  there are  $3(2j+1)$  inhomogeneous components), governed by a set of coupled differential equations (see below).

At a first sight, the dependence of the scalar harmonic functions  $\gamma_{ab}^{jm}(x, t)$  on only the modes  $j, m$  in Eq. (9.11) and Eq. (9.12) seems incompatible with a generic decomposition of  $\gamma_{ab}(x, t)$ . In fact, a scalar function  $\gamma(g)$  (for the moment we drop the internal indices  $a, b$ , irrelevant for the discussion) on  $SU(2)$  can be expanded, by the Peter-Weyl theorem, as

$$\gamma(g) = \sum_{j, m', m} \gamma^{jm'm} D_{m'm}^j(g). \quad (9.13)$$

The components, at fixed  $j$ , will be  $(2j+1)^2$  and no longer  $(2j+1)$  because they are label by both magnetic numbers  $(m', m)$ . However, the choice of not summing over the  $j, m$  labels contracted with the Wigner  $D$  functions relies on the fact that the Einstein equations allow to decouple  $j, m$  states from  $m'$  states, allowing to fix the perturbations to the metric with definite  $j, m$ . Indeed, only the  $m'$  states are mixed by the action of the derivative operators and thus by the linearized Einstein tensor. In fact, from Eqs. (9.5a)-(9.5c), it follows that

$$\begin{aligned} \hat{L}_+ D_{m'm}^j &= \left( \hat{L}_1 + i \hat{L}_2 \right) D_{m'm}^j \\ &= i \sqrt{(j+m')(j-m'+1)} D_{(m'-1)m}^j \end{aligned} \quad (9.14a)$$

$$\begin{aligned} \hat{L}_- D_{m'm}^j &= \left( \hat{L}_1 - i \hat{L}_2 \right) D_{m'm}^j \\ &= i \sqrt{(j+m'+1)(j-m')} D_{(m'+1)m}^j \end{aligned} \quad (9.14b)$$

$$\hat{L}_3 D_{m'm}^j = m' D_{m'm}^j. \quad (9.14c)$$

Any invariant operator can be rewritten in terms of Cartesian coordinates  $x_A = \{x_1, x_2, x_3, x_4\}$  in the Euclidean space  $E^4$  in which the three-sphere  $S^3$  is embedded. Instead of the three co-frames  $\omega^a$  in the Euler angles chart, the invariant basis in the Euclidean space is given by the coordinate differentials  $dx^A$ , related to  $\omega^a$  by the transformation matrices  $S_A^a(x_A)$  as

$$\omega^a = 2 S_A^a(x_A) dx^A. \quad (9.15)$$

Conversely, the coordinate differentials of  $E^4$  can be expressed in terms of  $\omega^a$  as  $dx^A = S_a^A \omega^a / 2$ . The coordinate derivatives, which are the vector fields of  $E^4$ , are given by

$$\frac{\partial}{\partial x_A} = 2 S_A^a(x_A) e_a. \quad (9.16)$$

It turns out to be much easier, using the homogeneity of the  $S^3$  spatial slices, to evaluate the Cartesian derivatives at the pole  $x_P = \{x_4 = 1, x_1 = x_2 = x_3 = 0\}$ , where the transformation matrices reduce to  $S_A^a = -\delta_A^a$ , and yielding the relations

$$\left. \frac{\partial}{\partial x_A} \right|_{x_P} = -2 e_a. \quad (9.17)$$

Equation (9.17) allows us, by means of Eqs. (9.7), to express the derivatives in terms of invariant operators. On the other hand, Eqs. (9.14) specify the actions of the invariant operators on the Wigner  $D$ -functions. It follows that, once the Einstein equations for the Bianchi IX model have been rewritten in terms of Cartesian coordinates  $x^A$ , the perturbations to homogeneity contain states with definite  $j, m$ , thus only perturbations labeled by  $m'$  states are mixed.

The above construction allows us to straightforwardly obtain the perturbation equations to Bianchi IX. In particular, the Christoffel symbols and their derivatives can be computed at the pole  $x_P$ . Also the (generic) Regge-Wheeler perturbation equations on an empty background metric

$$2 \delta R_{ij} = \nabla_k \nabla^k \gamma_{ij} - \nabla_j \nabla^k \gamma_{ik} - \nabla_i \nabla^k \gamma_{jk} + \nabla_i \nabla_j \gamma_k^k = 0 \quad (9.18)$$

can be evaluated at  $x_P$ . This leads to ordinary (in the time coordinate) differential equations for  $\gamma_{ab}^{jm'}(t)$  in which the magnetic number  $m'$  is mixed by the spatial derivatives. As a result of the numeric integration (at the lowest mode  $j = 1/2$ ) the perturbations decrease as the volume of the Universe increases from the singularity and vice-versa. Thus, the Bianchi IX model is stable in the expanding picture but is unstable when the cosmological singularity is approached from a non-singular hypersurface, suggesting to abandon the symmetry requirements when treating the initial singularity.

## 9.2 Formulation of the Generic Cosmological Problem

From the '60s, the Landau school started to investigate the properties and the behavior of the *generic cosmological solution* of the Einstein equations. A generic solution of the field equations corresponds to a metric  $g_{ij}$  that

possesses the correct number of free functions to formulate any Cauchy problem on some non-singular hypersurface. From the study the Cauchy problem for the Einstein equations can be recognized that in vacuum we need four unknown functions to specify the physical degrees of freedom for the gravitational field, and eight functions if a perfect fluid is included into the dynamics.

The first results in this direction were obtained by Khalatnikov and Lifshitz in 1963 who extended the Kasner solution to the case when the homogeneous hypothesis is relaxed. Such solution is stable when reaching a singular point in the past as soon as a particular condition is imposed, so reducing the number of arbitrary functions to three (treated in Sec. 9.2.1). In the following years, by relaxing such condition, this solution was generalized by Belinskii, Lifshitz and Khalatnikov outlining a very complex behavior which resembles that of the homogeneous Mixmaster model studied in Chaps. 7 and 8. This is now called the *BKL conjecture* and, even if a rigorous mathematical proof does not exist yet, it is commonly used to describe the detailed evolution of the Einstein equations in the neighborhood of a cosmological singularity (see Sec. 9.2.2).

The construction can be achieved firstly by considering the inhomogeneous solution for the individual Kasner epochs and then providing a general description for the alternation of two successive epochs. The answer to the first question is given by the so-called generalized Kasner metric, while the solution to the latter is in close analogy to the replacement rule for the homogeneous indices.

### 9.2.1 The Generalized Kasner solution

Lifshitz and Khalatnikov showed that the Kasner solution can be generalized to the inhomogeneous case, near the singularity, as

$$\begin{cases} dl^2 = h_{\alpha\beta} dx^\alpha dx^\beta , \\ h_{\alpha\beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta , \end{cases} \quad (9.19)$$

where

$$a \sim t^{p_l} , \quad b \sim t^{p_m} , \quad c \sim t^{p_n} , \quad (9.20)$$

and  $p_l, p_m, p_n$  are functions of spatial coordinates subjected to the conditions

$$p_l(x^\gamma) + p_m(x^\gamma) + p_n(x^\gamma) = p_l^2(x^\gamma) + p_m^2(x^\gamma) + p_n^2(x^\gamma) = 1 . \quad (9.21)$$



Differently from the homogeneous case, the reference vectors  $l, m, n$  are arbitrary functions of the coordinates (subjected to the conditions associated with the  $0\alpha$ -components of the Einstein equations).

The behavior in Eq. (9.19) cannot last up to the singularity, unless a further condition is imposed on the vector corresponding to the negative index  $p_1$ ; without loss of generality, we can take  $p_l = p_1 < 0$  through the whole hypersurface, and such a condition reads as

$$l \cdot \nabla \wedge l = 0. \quad (9.22)$$

This restriction ensures that all terms in the three-dimensional Ricci tensor can be neglected toward the singularity (see below Sec. 9.2.2). This condition reduces the number of arbitrary functions to three, i.e. one less than the number required to deal with the general case. In fact, the metric (9.19) possesses 12 arbitrary functions of the coordinates (nine components of the Kasner axes and three indexes  $p_i(x^\gamma)$ ), and must satisfy the two Kasner relations (9.21), the three  $0\alpha$  Einstein equations, three conditions arising from three-dimensional coordinate transformations invariance, and Eq. (9.22).

### 9.2.2 Inhomogeneous BKL solution

Let us now generalize our scheme by investigating the implications of removing the condition (9.22). This analysis leads to the inhomogeneous BKL replacement map.

Taking the three-metric tensor in the form (9.19), we are summarizing the dynamical evolution of the model into the behavior of the scale factors  $a, b, c$ , while the Kasner vectors  $l, m$  and  $n$  fix generic directions. In general, the three-metric associated to a generic inhomogeneous model can be written, in analogy to Eq. (7.17), as

$$h_{\alpha\beta} = \eta_{ab}(t, x) e_\alpha^a(x^\gamma) e_\beta^b(x^\gamma), \quad (9.23)$$

where the matrix  $\eta_{ab}$  depends on the space coordinates because of inhomogeneity. In fact, the three linear independent vectors  $e_\alpha^a$  no longer define an isometry group but simply correspond to a generic choice of their components, and therefore the corresponding quantities  $\lambda^a_{bc}$  (2.111b) do not behave as constant terms. The metric (9.23) is mapped by simple identifications into metric (9.19) as far as  $\eta_{ab} = \text{diag}(a^2, b^2, c^2)$ ,  $e_\alpha^1 = l_\alpha, e_\alpha^2 = m_\alpha, e_\alpha^3 = n_\alpha$  and  $(a, b, c) = (l, m, n)$ .

The solution (9.19)-(9.21) is obtained neglecting the triadic projection of the three-dimensional Ricci tensor  ${}^3R_a^b$  into the vacuum Einstein equations

which read as

$$-R_0^0 = -\partial_t K_a^a + K_b^a K_a^b = 0, \quad (9.24a)$$

$$\begin{aligned} -R_a^0 &= \lambda^b{}_{fa} K_b^f - \lambda^f{}_{fd} K_a^d + \eta^{fg} \partial_f K_{ga} + \eta^{fg} \partial_f \eta_{gb} K_a^b \\ &\quad - \eta^{fg} \partial_a K_{fg} \frac{1}{2} \eta^{gb} \partial_d \eta_{gb} K_a^d - \frac{1}{2} K^{fg} \partial_a \eta_{fg} = 0 \end{aligned} \quad (9.24b)$$

$$-R_a^b = -\frac{1}{\sqrt{\eta}} \partial_t (\sqrt{\eta} K_a^b) + {}^3R_a^b = 0, \quad (9.24c)$$

where  $\lambda_{abc}$  are the Ricci coefficients given in Sec. 2.5. The validity of the Kasner behavior can be conveniently formulated in terms of the projections along the directions  $l, m, n$ , satisfying the conditions

$${}^3R_l^l, {}^3R_m^m, {}^3R_n^n \ll t^{-2}, \quad {}^3R_l^l \gg {}^3R_m^m, {}^3R_n^n. \quad (9.25)$$

In fact, the dominant terms should be the ones associated to the time derivatives of  $\eta_{ab}$  which, for the Kasner behavior, identically vanish but are potentially of order  $t^{-2}$ . In the Kasner-like behavior, the off-diagonal projections of Eq. (9.24c) determine the off-diagonal projections ( $\eta_{lm}, \eta_{ln}, \eta_{mn}$ ), which result to be small corrections to the leading diagonal terms of the metric. In this regime, the only non-vanishing projections are the diagonal terms ( $\eta_{ll}, \eta_{mm}, \eta_{nn}$ ) and satisfy

$$\eta_{lm} \ll \sqrt{\eta_{ll}\eta_{mm}}, \quad \eta_{ln} \ll \sqrt{\eta_{ll}\eta_{nn}}, \quad \eta_{mn} \ll \sqrt{\eta_{mm}\eta_{nn}}. \quad (9.26)$$

The three-dimensional Ricci tensor associated to the three-metric (9.19) reads in the form (all the vectorial operations are performed as in the Euclidean case)

$$\begin{aligned} {}^3R_{ll} &= \frac{a^2}{\Delta^2} \left\{ \frac{1}{2} (al\nabla \wedge al)^2 - \frac{1}{2} (bm\nabla \wedge bm)^2 - \frac{1}{2} (cn\nabla \wedge cn)^2 \right. \\ &\quad - (cn\nabla \wedge bm)^2 - (bm\nabla \wedge cn)^2 - (bm\nabla \wedge al)^2 - (cn\nabla \wedge al)^2 \\ &\quad + (cn\nabla \wedge cn) (bm\nabla \wedge bm) + (cn\nabla \wedge al) (al\nabla \wedge cn) \\ &\quad \left. + (al\nabla \wedge bm) (bm\nabla \wedge al) \right\} \\ &\quad + a^2 \left[ \frac{1}{b} \left( \frac{cn\nabla \wedge al}{\Delta} \right)_{,m} - \frac{1}{a} \left( \frac{cn\nabla \wedge bm}{\Delta} \right)_{,l} \right. \\ &\quad \left. + \frac{1}{a} \left( \frac{bm\nabla \wedge cn}{\Delta} \right)_{,l} - \frac{1}{c} \left( \frac{bm\nabla \wedge al}{\Delta} \right)_{,n} \right], \end{aligned} \quad (9.27a)$$

$$\begin{aligned}
{}^3R_{lm} = & \frac{ab}{\Delta^2} \left\{ (al\nabla \wedge al)(bm\nabla \wedge al) + (bm\nabla \wedge bm)(al\nabla \wedge bm) \right. \\
& + (al\nabla \wedge cn)(bm\nabla \wedge cn) - \frac{1}{2}(cn\nabla \wedge cn)[(al\nabla \wedge bm) + (bm\nabla \wedge al)] \\
& + \frac{1}{2}(bm\nabla \wedge cn)(cn\nabla \wedge al) + \frac{1}{2}(al\nabla \wedge cn)(cn\nabla \wedge bm) \left. \right\} \\
& + \frac{ab}{2} \left[ \frac{1}{b} \left( \frac{bm\nabla \wedge cn}{\Delta} \right)_{,m} - \frac{1}{a} \left( \frac{al\nabla \wedge cn}{\Delta} \right)_{,l} \right. \\
& \left. - \frac{1}{c} \left( \frac{bm\nabla \wedge bm}{\Delta} \right)_{,n} + \frac{1}{c} \left( \frac{al\nabla \wedge al}{\Delta} \right)_{,n} \right], \tag{9.27b}
\end{aligned}$$

where we have introduced the quantity  $\Delta = \sqrt{h} = abc(l \cdot m \wedge n)$  and the letters  $l, m, n$ , following the comma in the indices, denote differentiation along the corresponding direction. The other components may be obtained from those given by cyclic permutation of the letters  $l, m, n$  and, correspondingly, of  $a, b, c$ .

In view of these expressions, the condition (9.26) leads to the following inequalities

$${}^3R_{lm} \ll ab/t^2, \quad {}^3R_{ln} \ll ac/t^2, \quad {}^3R_{mn} \ll bc/t^2. \tag{9.28}$$

The diagonal projections  ${}^3R_l^l, {}^3R_m^m, {}^3R_n^n$  contain the terms

$$\frac{1}{2} \left( \frac{al\nabla \wedge (al)}{abc(l \cdot [m \wedge n])} \right) \sim \frac{k^2 a^2}{b^2 c^2} = \frac{k^2 a^4}{\Lambda^2 t^2}, \tag{9.29}$$

and analogous terms with  $al$  replaced by  $bm$  and  $cn$ ; here  $1/k$  denotes the order of magnitude of spatial distances over which the metric significantly changes and, dealing with a Kasner regime,  $\Lambda \equiv abc = \Lambda(t, x)$ . According to conditions (9.25), we get the inequalities

$$a\sqrt{k/\Lambda} \ll 1, \quad b\sqrt{k/\Lambda} \ll 1, \quad c\sqrt{k/\Lambda} \ll 1, \tag{9.30}$$

which are not only necessary, but also sufficient conditions for the existence of the generalized Kasner solution. As soon as the conditions (9.30) are satisfied, all other terms in  ${}^3R_l^l, {}^3R_m^m, {}^3R_n^n$ , as well as in  ${}^3R_{lm}, {}^3R_{ln}, {}^3R_{mn}$ , automatically satisfy Eq. (9.25) and Eq. (9.28) as well. In fact, as soon as one estimates the Ricci tensor projections, the inequalities (9.28) lead to the conditions

$$\frac{k^2}{\Lambda^2} (a^2 b^2, \dots, a^3 b, \dots, a^2 bc, \dots) \ll 1, \tag{9.31}$$

containing on the left-hand side the products of powers of two or three of the quantities which enter in Eq. (9.30). The inequalities (9.31) can be seen as the generalized version of the condition imposed when addressing the homogeneous Mixmaster model.

As  $t$  decreases, an instant  $t_{\text{tr}}$  may eventually occur when one of the conditions (9.30) is violated.<sup>1</sup> Thus, if during a given Kasner epoch the negative exponent refers to the function  $a(t)$ , i.e.  $p_l = p_1$ , then at  $t_{\text{tr}}$  we have

$$a_{\text{tr}} \sqrt{\frac{k}{\Lambda}} \sim 1. \quad (9.32)$$

Since during that epoch the functions  $b(t)$  and  $c(t)$  decrease with  $t$ , the other two inequalities in (9.30) remain valid and at  $t \sim t_{\text{tr}}$  we shall have

$$b_{\text{tr}} \ll a_{\text{tr}}, \quad c_{\text{tr}} \ll a_{\text{tr}}. \quad (9.33)$$

At the same time, all the conditions (9.31) continue to hold and all the off-diagonal projections of Eq. (9.24c) may be disregarded. In the diagonal projections (9.27), only the terms containing  $a^4/t^2$  become relevant. In such surviving terms we have

$$(a l \cdot \nabla \wedge (a l)) = a (l \cdot [\nabla a \times l]) + a^2 (l \cdot \nabla \wedge l) = a^2 (l \cdot \nabla \wedge l), \quad (9.34)$$

i.e. the spatial derivatives of  $a$  drop out. As a result, we obtain the following equations for the replacement of two Kasner epochs

$$-R_l^l = \frac{(\dot{a}bc)^\cdot}{abc} + \nu^2 \frac{a^2}{2b^2c^2} = 0, \quad (9.35a)$$

$$-R_m^m = \frac{(a\dot{b}c)^\cdot}{abc} - \nu^2 \frac{a^2}{2b^2c^2} = 0, \quad (9.35b)$$

$$-R_l^l = \frac{(ab\dot{c})^\cdot}{abc} - \nu^2 \frac{a^2}{2b^2c^2} = 0, \quad (9.35c)$$

$$-R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0, \quad (9.35d)$$

which differ from the corresponding ones of the homogeneous case (7.67-7.68) only for the quantity

$$\nu(x^\gamma) = \frac{l \cdot \nabla \wedge l}{l \cdot [m \wedge n]}, \quad (9.36)$$

which is no longer a constant, but a function of the space coordinates. Since Eq. (9.35) is a system of ordinary differential equations with respect

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<sup>1</sup>The case when two of these are simultaneously violated can happen when the exponents  $p_1$  and  $p_2$  are close to zero, corresponding to the case of small oscillations (see Sec. 7.4.3).

to time where space coordinates enter only parametrically, such difference does not affect at all the solution of the equations and the resulting BKL map, retaining in each space point the form as in Eq. (7.95). Similarly, the law of alternation of exponents derived for the homogeneous indices remains valid in the general inhomogeneous case.

### 9.2.3 Rotation of the Kasner axes

Even if the point-like dynamics is quite similar to that of the homogeneous case in vacuum, the new feature of the rotation of the Kasner axes emerges. If in the initial epoch the spatial metric is given by Eq. (9.19), then in the final one we have

$$h_{\alpha\beta} = a^2 l'_\alpha l'_\beta + b^2 m'_\alpha m'_\beta + c^2 n'_\alpha n'_\beta, \quad (9.37)$$

with  $a, b, c$  characterized by a new set of Kasner indexes, and some vectors  $l', m', n'$ . If we project all tensors (including  $h_{\alpha\beta}$ ) in both epochs onto the same directions  $l, m, n$ , the turning of the Kasner axes can be described as the appearance, in the final epoch, of off-diagonal projections  $\eta_{lm}, \eta_{ln}, \eta_{mn}$ , which behave in time as linear combinations of the functions  $a^2, b^2, c^2$ .

The main effects can be reduced to a rotation of the  $m$ - and  $n$ -axis by a large angle, and a rotation of the  $l$ -axis by a small one which can be neglected. The new Kasner axes are related to the old ones as

$$l' = l, \quad m' = m + \sigma_m l, \quad n' = n + \sigma_n l, \quad (9.38)$$

where the  $\sigma_m, \sigma_n$  are of order unity, and are given by

$$\sigma_m = -\frac{2}{p_2 + 3p_1} \left\{ [l \wedge m] \cdot \nabla \frac{p_1}{\lambda} + \frac{2p_1}{\lambda} m \cdot \nabla \wedge l \right\} \frac{1}{l \cdot [m \wedge n]}, \quad (9.39a)$$

$$\sigma_n = \frac{2}{p_2 + 3p_1} \left\{ [n \wedge l] \cdot \nabla \frac{p_1}{\lambda} - \frac{2p_1}{\lambda} n \cdot \nabla \wedge l \right\} \frac{1}{l \cdot [m \wedge n]}. \quad (9.39b)$$

These expressions can be inferred by linking the two Kasner epochs and by the  $0\alpha$  components of the Einstein equations which play the role of constraints over the space functions.

The rotation of the Kasner axes (that appears even for a matter-filled homogeneous space) is inherent in the inhomogeneous solution already in the vacuum case. The role played in the homogeneous case by the matter energy-momentum tensor can be mimicked by the terms due to inhomogeneities of the spatial metric. Furthermore, in analogy to the discussion presented in Sec. 7.2.1, the presence of matter does not influence the generalized Kasner solution to the leading order. Repeating in the inhomogeneous case the same analysis as in Sec. 7.2.1, it is possible to show that,

near the singularity, the perfect fluid exhibits a test-like behavior point by point. Its effect is mainly exhibited in modifying the relations between the arbitrary spatial functions which appear in the solution, now containing even matter degrees of freedom.

### 9.3 The Fragmentation Process

We will now qualitatively discuss a further mechanism that takes place in the inhomogeneous Mixmaster model in the limit towards the singular point: the so-called *fragmentation* process.

The extension of the BKL mechanism to the general inhomogeneous case contains the physical restriction of the “local homogeneity”: in fact, the general derivation is based on the assumption that the spatial variation of all spatial metric components possesses the same characteristic length, described by a unique parameter  $k$ , which can be regarded as an average wave number. Nevertheless, such local homogeneity could cease to be valid as a consequence of the asymptotic evolution towards the singularity. The conditions (9.21) do not require that the functions  $p_a(x^\gamma)$  have the same ordering in all points of space. Indeed, they can vary their ordering throughout space an infinite number of times without violating the conditions (9.21), in agreement with the oscillatory-like behavior of their spatial dependence. Furthermore, the most important property of the BKL map evolution is the strong dependence on initial values, which produces an exponential divergence of the trajectories resulting from its iteration.

Given a generic initial condition  $p_a^0(x^\gamma)$ , the continuity of the three-manifold requires that, at two nearby space points, the Kasner index functions assume correspondingly close values. However, for the mentioned property of the BKL map, the trajectories emerging from these two values exponentially diverge and, since the  $p_a(x^\gamma)$  vary within the interval  $[-1/3, 1]$  only, the spatial dependence acquires an increasingly oscillatory-like behavior.

In the simplest case, let us assume that, at a fixed instant of time  $t_0$ , all the points of the manifold are described by a generalized Kasner metric, the Kasner index functions have the same ordering point by point, and  $p_1(x^\gamma)$ ,  $p_2(x^\gamma)$ ,  $p_3(x^\gamma)$  are described throughout the whole space, by a narrow interval of  $u$ -values, i.e.  $u \in [K, K + 1]$  for a generic integer<sup>2</sup>  $K$ . We refer to this situation as a manifold composed by one “island”. We can

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<sup>2</sup>For the sake of simplicity, here we adopt  $X$  and  $K$  instead of  $x$  and  $k$  used in Sec. 7.4.2.

introduce the remainder part of  $u(x^\gamma)$  as (see also Sec. 7.4.2)

$$X(x^\gamma) = u(x^\gamma) - [u(x^\gamma)], \quad X \in [0, 1) \quad \forall x^\gamma \in \Sigma, \quad (9.40)$$

where the square brackets indicate the integer part. Thus, the values of the narrow interval can be written as  $u^0(x^\gamma) = K^0 + X^0(x^\gamma)$ . As the evolution proceeds, the BKL mechanism induces a transition from an epoch to another; the  $n$ th epoch is characterized by an interval  $[K - n, K - n + 1]$ , until  $K - n = 0$ , when the era comes to an end and a new one begins. The new  $u^1(x^\gamma)$  starts from  $u^1 = 1/X^0$ , i.e. takes value in the interval  $[1, \infty)$ . Only very close points can still be in the same “island” of  $u$  values; distant ones in space will be described by very different integer  $K^1$  and will experience eras of different lengths. As the singular point is reached, more and more eras take place, causing the formation of a greater and greater number of smaller and smaller “islands”, providing the “fragmentation” process. Our interest is focused on the value of the parameter  $K$ , which describes the characteristic wave number of the metric that increases as the islands get smaller. This implies the progressive increase of the spatial gradients and in principle could deform the BKL mechanism. By a qualitative analysis we can argue that this is not the case the progressive increase of the spatial gradients produces the same qualitative effects on all the terms present in the three-dimensional Ricci tensor, including the dominant ones. In other words, for each single value of  $K$  and in every island, a condition of the form

$$\frac{\text{inhomogeneous term}}{\text{dominant term}} \sim \frac{k^2 t^{2\delta_i} f(t)}{k^2 t^{4p_1}} \ll 1, \\ \delta_i = 1 - p_i \geq 0, \quad f(t) = \mathcal{O}(\ln t, \ln^2 t), \quad t \ll 1,$$

is still valid, where the inhomogeneous terms contain the spatial gradients of the scale factors, which are evidently absent from the dynamics of the homogeneous cosmological models. The possibility to neglect such gradients towards the cosmological singularity is equivalent to state that each space point evolves independently in agreement with the Mixmaster dynamics. It is indeed this feature to allow to extrapolate the notion of chaotic behavior in a local sense, when the generic cosmological solution is concerned.

From the analysis above, the fragmentation process does not produce any behavior capable of stopping the iterative scheme of the oscillatory regime.

### 9.3.1 Physical meaning of the BKL conjecture

The condition to deal with a Kasner-like regime even in the generic inhomogeneous case corresponds, as discussed above, to the inequality

$$\frac{a^4 k^2}{a^2 b^2 c^2} \ll \frac{1}{t^2}. \quad (9.41)$$

If we introduce the co-moving inhomogeneous scale  $l \equiv 1/k$  and replace each scale factor by the geometrical average  $R(t) \equiv \sqrt[3]{abc}$ , the inequality (9.41) rewrites as

$$R(t)l \gg t. \quad (9.42)$$

Let us observe that  $R(t)l$  gives the typical physical inhomogeneous scale  $l_{\text{in}}$  of the Universe, and  $t$  stands for the order of magnitude that the cosmological and Hubble horizons take in correspondence to  $R(t)$ . Therefore, the BKL conjecture holds if we can state, up to the singularity,

$$l_{\text{in}} \gg d_{\text{H}}. \quad (9.43)$$

In other words, we can say that the homogeneous Mixmaster behavior is recovered when the physical inhomogeneous scale is super-horizon sized. The mathematical notion of independent space points which dynamically decouple towards the initial singularity must be replaced, on a physical level, by independent causal regions evolving accordingly to the local BKL map.

## 9.4 The Generic Cosmological Solution in Misner Variables

When considering a cosmological solution containing a number of space functions such that a generic inhomogeneous Cauchy problem is satisfied on a non-singular hypersurface, we refer to it as a generic inhomogeneous model. In the ADM formalism, the corresponding line element reads as

$$ds^2 = N(t, x)^2 dt^2 - h_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt), \quad (9.44)$$

$$h_{\alpha\beta} = e^{q_a(t, x)} l_\alpha^a l_\beta^a, \quad (9.45)$$

where  $N$  is the lapse function,  $N^\alpha$  the shift-vectors and  $q_a(t, x)$  three scalar functions; the vectors  $l_\alpha^a$  have components which are generic functions of the spatial coordinates only, available for the Cauchy data. The general case, in which  $l_\alpha^a$  are time-dependent vectors, is addressed in the following Section to simplify the variational principle. It is convenient to introduce



also the reciprocal vectors  $l_a^\alpha$ , such that  $l_a^\alpha l_b^\alpha = \delta_b^a$  and  $l_a^\alpha l_a^\beta = \delta_\alpha^\beta$ . The dynamics of this system is summarized by the action

$$S_g = \int_{\Sigma \times \mathbb{R}} d^3x dt (p_a \partial_t q_a - N\mathcal{H} - N^\alpha \mathcal{H}_\alpha) \quad (9.46)$$

$$\mathcal{H} = \frac{2\kappa}{\sqrt{h}} \left( \frac{1}{2} \sum_a p_a^2 - \sum_{a < b} p_a p_b - \frac{h}{4\kappa^2} {}^3R \right) \quad (9.47)$$

$$\mathcal{H}_\alpha = -2h\partial_\beta \sum_m l_m^\beta l_\alpha^m p_m - h p_b \partial_\alpha q_b, \quad (9.48)$$

$p_a$  being the conjugate momenta to the variables  $q_a$ . By varying the action (9.46) with respect to the functions  $N$  and  $N^\alpha$ , we get the super-Hamiltonian and super-momentum constraints  $\mathcal{H} = 0$  and  $\mathcal{H}_\alpha = 0$ , respectively. Let us introduce the Misner-like variables  $\alpha(t, x)$ ,  $\beta_\pm(t, x)$  via the transformation (8.26). In terms of this set of configurational coordinates, the Hamiltonian constraints rewrite as

$$\mathcal{H} = \frac{\kappa}{12} e^{-3\alpha} (-p_\alpha^2 + p_+^2 + p_-^2 + \mathcal{V}) \quad (9.49)$$

$$\mathcal{H}_\gamma = -\kappa \left\{ \partial_\gamma \left( \frac{p_\alpha}{3} + \frac{p_+}{6} + \sqrt{3} \frac{p_-}{6} \right) + \frac{1}{6} \partial_\delta \left[ l_3^\delta l_\gamma^3 (p_+ - 2\sqrt{3} p_-) - 2\sqrt{3} l_2^\delta l_\gamma^2 p_- \right] \right\} - \kappa [(\partial_\gamma \alpha) p_\alpha + (\partial_\gamma \beta_+) p_+ + (\partial_\gamma \beta_-) p_-] \quad (9.50)$$

$$\mathcal{V} = -\frac{6}{\kappa^2} h {}^3R. \quad (9.51)$$

A detailed analysis of the potential term  $\mathcal{V}$  leads to

$$\mathcal{V} = -\frac{1}{4\kappa^2} e^{4\alpha} \left[ \lambda_1^2(x^\gamma) e^{-8\beta_+} + \lambda_2^2(x^\gamma) e^{4(\beta_+ + \sqrt{3}\beta_-)} + \lambda_3^2(x^\gamma) e^{4(\beta_+ - \sqrt{3}\beta_-)} + W(x^\gamma, \alpha, \beta_\pm, \partial_\gamma \alpha, \partial_\delta \beta_\pm) \right], \quad (9.52)$$

where  $\lambda_a$  refers to the space quantities (related to  $\nu$  in Eq. (9.36))

$$\lambda_a(x^\gamma) \equiv l_a \cdot \nabla \wedge l_a. \quad (9.53)$$

To outline the relative behavior of the two terms in the potential as the singularity is approached for  $\alpha \rightarrow -\infty$ , let us consider the quantities  $D \equiv \exp(3\alpha)$  and  $Q_a$  defined in Eq. (8.42). Taking into account these definitions, the potential  $\mathcal{V}$  rewrites as

$$\mathcal{V} \sim \sum_b (\lambda_b^2 D^{2Q_b}) + W \quad (9.54a)$$

$$W \sim \sum_{b \neq c} \mathcal{O}(D^{Q_b + Q_c}). \quad (9.54b)$$

Near the cosmological singularity  $D \rightarrow 0$ , so that the term  $W$  becomes negligible. Indeed this conclusion is supported by the behavior of the spatial gradients, which do not destroy the features outlined above (see Sec. 9.5.2). Through the canonical replacements

$$p_\alpha = \frac{\partial S}{\partial \alpha} \quad (9.55)$$

$$p_\pm = \frac{\partial S}{\partial \beta_\pm}, \quad (9.56)$$

the classical evolution is summarized by the Hamilton-Jacobi system

$$-\left(\frac{\partial S}{\partial \alpha}\right)^2 + \left(\frac{\partial S}{\partial \beta_+}\right)^2 + \left(\frac{\partial S}{\partial \beta_-}\right)^2 + \mathcal{V}(\alpha, \beta_+, \beta_-) = 0 \quad (9.57a)$$

$$\begin{aligned} & \frac{1}{6} \left\{ \partial_\gamma \left[ 2 \frac{\partial S}{\partial \alpha} + \frac{\partial S}{\partial \beta_+} + \sqrt{3} \frac{\partial S}{\partial \beta_-} \right] \right. \\ & \quad \left. + \partial_\delta \left[ l_3^\delta l_\gamma^3 \left( \frac{\partial S}{\partial \beta_+} - 2\sqrt{3} \frac{\partial S}{\partial \beta_-} \right) - 2\sqrt{3} l_2^\delta l_\gamma^2 \frac{\partial S}{\partial \beta_-} \right] \right\} \\ & \quad + \left\{ (\partial_\gamma \alpha) \frac{\partial S}{\partial \alpha} + (\partial_\gamma \beta_+) \frac{\partial S}{\partial \beta_+} + (\partial_\gamma \beta_-) \frac{\partial S}{\partial \beta_-} \right\} = 0. \end{aligned} \quad (9.57b)$$

Since sufficiently close to the cosmological singularity the potential term becomes step by step negligible, then, deep inside the potential well, the solution of Eq. (9.57a) reads as

$$S = -\sqrt{k_+^2 + k_-^2} \alpha + k_+ \beta_+ + k_- \beta_-, \quad (9.58)$$

where  $k_\pm = k_\pm(x^\gamma)$  are arbitrary functions of the coordinates and the minus sign in front of the square root has been taken considering an expanding Universe. According to the Jacobi prescription, the functional derivatives of the above action (9.58) with respect to  $k_\pm$  have to be set equal to stationary quantities  $c_\pm(x^\gamma)$  and therefore we get the following expressions for  $\beta_\pm$  in terms of  $\alpha$  as

$$\beta_\pm = \pi_\pm(x^\gamma) \alpha + c_\pm(x^\gamma), \quad (9.59)$$

where

$$\pi_\pm \equiv \frac{k_\pm}{\sqrt{k_+^2 + k_-^2}} \quad (9.60a)$$

$$\pi_+^2 + \pi_-^2 = 1. \quad (9.60b)$$

Substituting the solution (9.58) with (9.59) in the Hamilton–Jacobi equation (9.57b) corresponding to the super-momentum and taking into account the relations (9.60), the last sum cancels out leaving the equations

$$\begin{aligned}
 & -2 \frac{k_+ + k_-}{\sqrt{k_+^2 + k_-^2}} \partial_\gamma (k_+ + k_-) + \partial_\gamma k_+ + \sqrt{3} \partial_\gamma k_- \\
 & + \partial_\delta \left[ l_3^\delta l_\gamma^3 (k_+ - 2\sqrt{3}k_-) - 2\sqrt{3} l_2^\delta l_\gamma^2 k_- \right] = 0,
 \end{aligned} \tag{9.61}$$

which are constraints on the spatial functions only. The above mentioned functions  $c_\pm(x^\gamma)$  have been set equal to zero because their presence would simply correspond to a rescaling of the vectors  $l_\gamma^a(x^\delta)$ . Thus, our solution contains ten arbitrary functions of the spatial coordinates, namely the nine components of the vectors  $l_\gamma^a$  and one of the two functions  $\pi_\pm$ . Such ten free functions have to satisfy the three constraints (9.61); the choice of the coordinate frame eliminates the arbitrariness of three more degrees of freedom so that the solution is characterized by *four physically* arbitrary functions of the spatial coordinates and, in this sense, it is a generic one.

So far, we have neglected the role of the potential because it influences the point-Universe evolution only via the bounces producing the establishment of a new free motion (for a detailed discussion about the chaotic properties of the random behavior that the point-universe performs in the potential, see Chap. 8). This effect of the potential is summarized by the reflection law (8.50) of the point-Universe for a bounce on one of the three equivalent walls of the triangular potential  $\mathcal{V}(\alpha, \beta_+, \beta_-)$ . This Section shows how the generic cosmological solution toward the Big Bang is isomorphic, point by point in space, to the one of the Bianchi types VIII and IX models because the spatial coordinates are involved in the problem only as parameters.

## 9.5 Hamiltonian Formulation in a General Framework

Let us now extend the analysis of Sec. 9.4 by dealing with a more general representation of the three-metric tensor  $h_{\alpha\beta}$ . In the ADM formalism, a generic set of three-vectors on the spatial surface of the splitting can be defined as

$$e_\alpha^a = e^{q_a/2} O_b^a \partial_\alpha y^b \tag{9.62}$$

where  $y^b$  denotes three scalar functions,  $O_b^a = O_b^a(x^\gamma)$  a  $SO(3)$  matrix on the hypersurface, and  $q_a$  three scale factors. This representation of the

tetrads is equivalent to the following three-metric tensor

$$h_{\alpha\beta} = e^{q_a} \delta_{ad} O_b^a O_c^d \partial_\alpha y^b \partial_\beta y^c. \quad (9.63)$$

Thus the action for the gravitational field is

$$S_g = \int_{\Sigma \times \mathbb{R}} dt d^3x \left( p_a \partial_t q^a + \Pi_d \partial_t y^d - N\mathcal{H} - N^\alpha \mathcal{H}_\alpha \right), \quad (9.64a)$$

$$\mathcal{H} = \frac{2\kappa}{\sqrt{h}} \left( \frac{1}{2} \sum_a p_a^2 - \sum_{a < b} p_a p_b - \frac{h}{4\kappa^2} {}^3R \right) \quad (9.64b)$$

$$\mathcal{H}_\alpha = \Pi_a \partial_\alpha y^a + p_a \partial_\alpha q^a + 2p_a (O^{-1})_a^b \partial_\alpha O_b^a, \quad (9.64c)$$

where  $p_a$  and  $\Pi_d$  are the conjugate momenta to the variables  $q^a$  and  $y^d$ , respectively. The ten independent components of a generic metric tensor are represented by the three scale factors  $q^a$ , the three degrees of freedom  $y^a$ , the lapse  $N$  and the shift-vector  $N^\alpha$ ; by the variation of the action (9.64a) with respect to  $p_a$ ,  $\Pi_a$ , the relations

$$\partial_t y^d = N^\alpha \partial_\alpha y^d \quad (9.65)$$

$$N = \frac{\sqrt{h}}{3\kappa \sum_a p_a} \left( N^\alpha \partial_\alpha \sum_b q^b - \partial_t \sum_b q^b \right) \quad (9.66)$$

hold. In particular, Eq. (9.65) states that the functions  $y^a$  are strictly connected to gauge transformations. In fact, if we set  $N^\alpha = 0$ , we obtain that  $y^d = \text{const}$ , i.e. they are not true dynamical degrees of freedom of the theory. So we can try to remove them from the dynamics, by fixing the form of  $N^\alpha$  or by solving the super-momentum constraint. This can be explicitly done taking  $\eta = t$ ,  $y^a = y^a(t, x)$  (which is equivalent to perform a coordinate transformation) and getting

$$\Pi_b = -p_a \frac{\partial q^a}{\partial y^b} - 2p_a (O^{-1})_a^c \frac{\partial O_c^a}{\partial y^b} \quad (9.67)$$

and furthermore

$$q^a(t, x) \rightarrow q^a(\eta, y) \quad (9.68a)$$

$$p_a(t, x) \rightarrow p'_a(\eta, y) = p_a(\eta, y)/|J| \quad (9.68b)$$

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial y^b}{\partial t} \frac{\partial}{\partial y^b} + \frac{\partial}{\partial \eta} \quad (9.68c)$$

$$\frac{\partial}{\partial x^\alpha} \rightarrow \frac{\partial y^b}{\partial x^\alpha} \frac{\partial}{\partial y^b}, \quad (9.68d)$$

where  $|J|$  denotes the Jacobian of the transformation. The relation (9.68a) holds in general for all the scalar quantities, while Eq. (9.68b) for all the scalar densities. The action (9.64a) rewrites as

$$S = \int_{\Sigma \times \mathbb{R}} d\eta d^3y \left( p_a \partial_\eta q^a + 2p_a (O^{-1})_a^c \partial_\eta O_c^a - N\mathcal{H} \right). \quad (9.69)$$

### 9.5.1 Local dynamics

In this scheme, the potential term appearing in the super-Hamiltonian (9.51) in these new variables explicitly reads as

$$\mathcal{V} = -\frac{D}{|J|^2} {}^3R(\eta, y^a) = \sum_a \lambda_a^2 D^{2Q_a} + \sum_{b \neq c} D^{Q_b + Q_c} \mathcal{O} \left( \partial q, (\partial q)^2, y, \eta \right) \quad (9.70)$$

where  $D = \exp \sum_a q^a$ ,  $Q_a$  are the anisotropy parameters (8.16) and  $\lambda_a$  are the functions

$$\lambda_a^2 = \sum_{k,j} \left[ O_b^a \vec{\nabla} O_c^a \left( \vec{\nabla} y^c \wedge \vec{\nabla} y^b \right)^2 \right]. \quad (9.71)$$

Assuming the  $y^a(t, x)$  smooth enough (which implies the smoothness of the coordinates system as well), all the gradients appearing in the potential  $V$  are regular, in the sense that their behavior is not strongly divergent to destroy the billiard representation. It can be shown (see Sec. 9.5.2) that the spatial gradients logarithmically increase with the proper time along the billiard's geodesics and are of higher order. As  $D \rightarrow 0$  the spatial curvature  ${}^3R$  diverges and the cosmological singularity appears; in this limit, the first term of  $V$  dominates all the remaining ones and can be approximated by the infinite potential well

$$\mathcal{V} = \sum_a \Theta_\infty(Q_a), \quad (9.72)$$

resembling the behavior of the Bianchi VIII and IX models (see Eq. (8.18)). By Eq. (9.72), the Universe dynamics evolves independently in each space point; the point-Universe moves within the dynamically-closed domain  $\Pi_Q$  (see Sec. 8.3) and near the singularity, by virtue of the super-Hamiltonian vanishing, we have  $\partial p_a / \partial \eta = 0$ . Finally, the term  $2p_a (O^{-1})_a^c \partial_\eta O_c^a$  in Eq. (9.69) behaves as an exact time-derivative and can be ruled out from the variational principle.

Henceforth, the same analysis developed for the homogeneous Mixmaster model in Secs. 8.1 - 8.3 can be straightforwardly implemented in a covariant way (i.e. without any gauge fixing for the lapse function or for the shift vector).

Introducing the MCl variables  $(\tau, \xi, \theta)$  (see Eq. (8.53) with  $\Gamma(\tau) = \tau$ ), the super-Hamiltonian constraint is solved in the domain  $\Pi_Q$  as

$$-p_\tau \equiv \mathcal{H}_{\text{ADM}} = \sqrt{(\xi^2 - 1)p_\xi^2 + \frac{p_\theta^2}{\xi^2 - 1}}, \quad (9.73)$$

and the reduced action reads as

$$\delta S_{\Pi_Q} = \delta \int_{\Sigma \times \mathbb{R}} d\eta d^3 y (p_\xi \partial_\eta \xi + p_\theta \partial_\eta \theta - \mathcal{H}_{\text{ADM}} \partial_\eta \tau) = 0. \quad (9.74)$$

By the asymptotic limit (9.72) and by the Hamilton equations associated with Eq. (9.74) we get  $d\epsilon/d\eta = \partial\epsilon/\partial\eta = 0$  and therefore  $\mathcal{H}_{\text{ADM}} = \epsilon(y^a)$  is, point by point in space, a constant of motion even in the non-homogeneous case.

### 9.5.2 Dynamics of inhomogeneities

We will discuss how the spatial gradients of the dynamical variables evolve toward the singularity. The result is that they increase only logarithmically and thus are not able to destroy the BKL mechanism (we remind the reader that the time derivatives increase as the power law  $t^{-2}$ ).

Let us introduce a different representation of the Lobačevskij plane through the “isotropic” variables  $\vec{r}$

$$\vec{r} = (r_1, r_2) = \frac{1 + \xi}{\sqrt{\xi^2 - 1}} (\cos \theta, \sin \theta). \quad (9.75)$$

In terms of such variables, given the matrix  $\vec{A}_a = (A_a^1, A_a^2)$ ,  $A_a^1 = (-\sqrt{3}, \sqrt{3}, 0)$ ,  $A_a^2 = (1, 1, -2)$ , the anisotropy parameters (8.42) read as

$$Q_a = \left[ \left( \vec{r} + \vec{A}_a \right)^2 + 1 - \left( \vec{A}_a \right)^2 \right], \quad (9.76)$$

and the action (9.74), in the asymptotic limit toward the singularity, rewrites ( $d\tau/d\eta = 1$ ) as

$$S_g = \int_{\Pi_Q} d^3 y d\tau \left( \vec{P} \partial_\tau \vec{r} - \mathcal{H}_{\text{ADM}} \right), \quad (9.77a)$$

$$\mathcal{H}_{\text{ADM}} = \epsilon(\vec{r}, \vec{P}) = \frac{1}{2} (1 - \vec{r}^2) |\vec{P}|, \quad (9.77b)$$

$\vec{P}$  being the momentum conjugate to  $\vec{r}$ . The Jacobi metric associated to such variational principle rewrites as

$$ds^2 = \frac{4d\vec{r}^2}{(1 - r^2)^2}, \quad |r| < 1 \quad (9.78)$$

and could also be derived by a direct transformation of Eq. (8.70).

We have discussed in details how the asymptotic behavior of the gravitational field can be reduced to the direct product of infinite equivalent

and decoupled dynamical systems, each of them described as the geodesic motion on the Lobačevskij plane where such flow is characterized by exponential instability due to the negative curvature of the manifold (the norm of the vector connecting two nearby geodesics behaves as  $\propto \exp(s)$ ). Then, the scale of inhomogeneity  $l_{\text{in}}$  decreases as

$$l_{\text{in}} \sim \left( \frac{\partial r}{\partial y} \right)^{-1} \sim l_0 \exp(-s), \quad (9.79)$$

where  $l_0$  is an initial inhomogeneous scale for which, in vacuum and in the asymptotic limit toward the singularity,  $s$  is given by (see Sec. 8.3.1 and Eq. (8.75))

$$s = E \int_{\tau_0}^{\tau} ds = E \Delta \tau. \quad (9.80)$$

Thus, for  $\tau \rightarrow \infty$  the dynamical variables  $\vec{r}(y^\gamma)$  and  $\vec{P}(y^\gamma)$  become random functions of the spatial coordinates, in accordance with the point-like Mixmaster evolution. This means that the invariant measure of the whole system is given by the direct product of the infinite “point” measures

$$d\mu = \Pi_{y^\gamma} d\mu(y^\gamma, \vec{r}, \vec{P}), \quad (9.81)$$

where  $d\mu$  reads in these new variables (from Eq. (8.80)) as

$$d\mu(y^\gamma, \vec{r}, \vec{P}) = \text{const} \times \frac{d^2 r d^2 m}{(1 - r^2)^2}, \quad \vec{m} = \frac{\vec{P}}{\epsilon}. \quad (9.82)$$

Furthermore, one can use the following  $n$ -point distribution function (evaluated either on an initial distribution or on a volume of the space) in order to calculate different mean values, i.e.

$$\rho_{y_1^\gamma, \dots, y_n^\gamma}(r_1, \dots, r_n, m_1, \dots, m_n) = \left\langle \prod_{i=1}^n \delta(r_i - r(y_i^\gamma)) \delta(m_i - m(y_i^\gamma)) \right\rangle, \quad (9.83)$$

so obtaining that, for  $|y^\gamma - y^\delta| \gg l_0 \exp(-s)$ , the averaging and correlating functions of the dynamical variables take the forms

$$\langle \vec{r}(y^\gamma) \rangle = \langle \vec{P}(y^\gamma) \rangle = 0, \quad (9.84)$$

$$\langle r_a(y^\gamma) r_b(y^\delta) \rangle = r_a r_b \delta(y^\gamma - y^\delta). \quad (9.85)$$

In order to estimate the growth of the inhomogeneities, we can set  $\vec{r} = 0$ . In the synchronous time  $t$ , the variation of the time variable  $\tau$  can be

estimated by  $\sqrt{h} \sim \exp(-3/2e^{-\tau}) \sim t$ . By means of Eq. (9.79), the time dependence of the scale of inhomogeneity takes the form

$$l_{\text{in}} \sim l_0 \frac{\ln(1/h_0)}{\ln(1/h)}, \quad (9.86)$$

where  $h_0$  corresponds to an initial condition for the three-metric determinant. This inhomogeneous scale decreases towards the singularity but, being a coordinate length, its physical behavior is fixed also by the statistical properties of the typical scale factor contained in the metric, i.e. we have to evaluate  $l_{\text{phys}} \sim \langle h^{Q_a/2} l_{\text{in}} \rangle_Q \sim l_{\text{in}} \langle h^{Q_a/2} \rangle_Q$ , where the last relation stands because  $l_{\text{in}}$  does not depend on  $Q_a$ . The quantity  $\langle h^{Q_a/2} \rangle_Q$  is given by the integral

$$\langle h^{Q_a/2} \rangle_Q = \int_{Q_{\min}}^1 h^{Q_a/2} \rho(Q_a) dQ_a, \quad (9.87)$$

$\rho(Q_a)$  being the distribution function resulting from the invariant measure. The main contribution is given by  $Q_{\min} = 0$  and the explicit form of the distribution reads as

$$\rho(Q_a) = \frac{2}{\pi \sqrt{Q_a(1-Q_a)} (1+3Q_a)}. \quad (9.88)$$

For very small values of  $Q_a$ , we finally get the estimate of the growth of the spatial lengths  $l_{\text{phys}}$ , as

$$l_{\text{phys}} \sim l_{\text{in}} \langle h^{Q_a/2} \rangle_Q \simeq \sqrt{\frac{2}{\ln(1/h)}} \left( l_0 \frac{\ln(1/h_0)}{\ln(1/h)} \right) \sim [\ln(1/h)]^{-3/2} \sim \left[ \ln \left( \frac{1}{t} \right) \right]^{-3/2}. \quad (9.89)$$

This means that the physical inhomogeneous scale decreases towards the singularity ( $h \rightarrow 0$ ), but only logarithmically and the inhomogeneities become over-horizon-sized when approaching the singularity.

## 9.6 The Generic Cosmological Problem in the Iwasawa Variables

A different formulation<sup>3</sup> of the problem of generic cosmological singularity has been recently given in terms of the so-called *Iwasawa variables*. This particular parametrization of the spatial metric allows a unique decomposition of  $h_{\alpha\beta}$  in the product of two triangular matrices and a diagonal one, i.e.

$$h = \mathcal{N}^T D^2 \mathcal{N}, \quad (9.90a)$$

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<sup>3</sup>In this Section we adopt the signature  $(-, +, +, +)$  and  $\kappa = 1$ .



$$\mathcal{N} = \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad (9.90b)$$

$$D = \begin{pmatrix} e^{-\beta^1} & 0 & 0 \\ 0 & e^{-\beta^2} & 0 \\ 0 & 0 & e^{-\beta^3} \end{pmatrix}. \quad (9.90c)$$

This representation corresponds to the Gram-Schmidt orthogonalization of the initial coordinate coframe  $dx^\alpha$ , that is

$$h_{\alpha\beta} dx^\alpha dx^\beta = e^{-2\beta^a} \theta^a \theta^a \quad (9.91)$$

where<sup>4</sup>  $\theta^a = \mathcal{N}^a{}_\alpha dx^\alpha$ . The explicit form for  $h_{\alpha\beta}$  is

$$h_{\alpha\beta} = e^{-2\beta^a} \mathcal{N}^a{}_\alpha \mathcal{N}^a{}_\beta. \quad (9.92)$$

The Lagrangian  $\mathcal{L}_g$  (2.11) in vacuum can be rewritten as

$$\mathcal{L}_g = \frac{\sqrt{h}}{2\kappa N} \left[ \mathbf{g}_{ab} \dot{\beta}^a \dot{\beta}^b + \frac{1}{2} \sum_{a < b} e^{2(\beta^b - \beta^a)} (\mathcal{N}^a{}_\alpha \mathcal{N}^\beta{}_a)^2 \right] + \frac{N\sqrt{h}}{2\kappa} {}^3R, \quad (9.93)$$

where  $\mathcal{N}^\alpha{}_a$  denotes the inverse of  $\mathcal{N}^a{}_\alpha$  and we introduced the “metric”  $\mathbf{g}_{ab}$  with signature  $(-, +, +)$  in the  $\beta$ -space defined as

$$\mathbf{g}_{ab} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad \mathbf{g}^{ab} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \quad (9.94)$$

A standard Legendre transformation yields the canonical formulation we are interested in. Let  $p_a$  denote the momentum conjugate to  $\beta^a$  and  $P^\alpha{}_a$  a lower triangular matrix whose components  $P^a$  are the momenta conjugate to the Iwasawa variables  $n_a$ , i.e.

$$P^\alpha{}_a = \begin{pmatrix} 0 & 0 & 0 \\ P^1 & 0 & 0 \\ P^2 & P^3 & 0 \end{pmatrix}. \quad (9.95)$$

For  $N^\alpha = 0$ , Eq. (2.71) explicitly rewrites as

$$S_g = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x \left\{ p_a \partial_t \beta^a + P^a \partial_t n_a - \frac{2\kappa N}{\sqrt{h}} \left[ \frac{1}{4} \mathbf{g}^{ab} p_a p_b + \frac{1}{2} \sum_{a < b} e^{2(\beta^b - \beta^a)} (P^\alpha{}_a \mathcal{N}^b{}_\alpha)^2 - \frac{h}{4\kappa^2} {}^3R \right] \right\}. \quad (9.96)$$

In Eq. (9.96) the super-Hamiltonian can be identified as the term in square brackets.

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<sup>4</sup>It follows also that the frame vectors are given by  $e_a = \mathcal{N}^\alpha{}_a \partial_\alpha$ .

### 9.6.1 Asymptotic freezing of the Iwasawa variables

The asymptotic dynamics of the Iwasawa degrees of freedom can be studied in the simple case of a Kasner like solution in vacua. The details of such a model are discussed in Sec. 7.2 where, in particular, the line element is of the form (7.49). Adopting the parametrization as in Eq. (9.90), one can obtain the relation of the new variables ( $\beta^a$ ,  $n_b$ ) with the Kasner parameters  $p_a$  and with the frame vectors (denoted as  $l^{(b)}$  in order to avoid confusion with  $n_a$ ), i.e.

$$\begin{aligned}\beta^1(t) &= -\frac{1}{2} \ln X, \\ \beta^2(t) &= \frac{1}{2} (\ln X - \ln Y), \\ \beta^3(t) &= \frac{1}{2} \ln \left[ \frac{t^2 [l^{(1)} \cdot (l^{(2)} \wedge l^{(3)})]^2}{Y} \right],\end{aligned}\tag{9.97a}$$

$$\begin{aligned}n_1(t) &= \frac{t^{2p_1} l_1^{(1)} l_2^{(1)} + t^{2p_2} l_1^{(2)} l_2^{(2)} + t^{2p_3} l_1^{(3)} l_2^{(3)}}{X}, \\ n_2(t) &= \frac{t^{2p_1} l_1^{(1)} l_3^{(1)} + t^{2p_2} l_1^{(2)} l_3^{(2)} + t^{2p_3} l_1^{(3)} l_3^{(3)}}{X},\end{aligned}$$

$$\begin{aligned}n_3(t) &= \frac{1}{Y} \left( t^{2p_1+2p_2} (l_1^{(1)} l_2^{(2)} - l_2^{(1)} l_1^{(2)}) (l_1^{(1)} l_3^{(2)} - l_3^{(1)} l_1^{(2)}) \right. \\ &\quad + t^{2p_1+2p_3} (l_1^{(1)} l_2^{(3)} - l_2^{(1)} l_1^{(3)}) (l_1^{(1)} l_3^{(3)} - l_3^{(1)} l_1^{(3)}) \\ &\quad \left. + t^{2p_2+2p_3} (l_1^{(2)} l_2^{(3)} - l_2^{(2)} l_1^{(3)}) (l_1^{(2)} l_3^{(3)} - l_3^{(2)} l_1^{(3)}) \right)\end{aligned}\tag{9.97b}$$

where the functions  $X$  and  $Y$  read as

$$X(t) = t^{2p_1} (l_1^{(1)})^2 + t^{2p_2} (l_1^{(2)})^2 + t^{2p_3} (l_1^{(3)})^2,\tag{9.97c}$$

$$\begin{aligned}Y(t) &= t^{2p_1+2p_2} (l_1^{(1)} l_2^{(2)} - l_2^{(1)} l_1^{(2)})^2 + t^{2p_1+2p_3} (l_1^{(1)} l_2^{(3)} - l_2^{(1)} l_1^{(3)})^2 \\ &\quad + t^{2p_2+2p_3} (l_1^{(2)} l_2^{(3)} - l_2^{(2)} l_1^{(3)})^2.\end{aligned}\tag{9.97d}$$

Let us assume that the Kasner exponents  $p_a$  are ordered as  $p_1 \leq p_2 \leq p_3$ . When approaching the singularity ( $t \rightarrow 0$ ), on one hand we have that the  $\beta^a$  variables behave as combinations of the scale factors of the metric  $a(t)$ ,  $b(t)$ ,  $c(t)$  (i.e. the power law in Eq. (7.49)); on the other hand, the Iwasawa degrees of freedom asymptotically “freeze out” of the dynamics, being

$$\lim_{t \rightarrow 0} n_1 = \frac{l_2^{(1)}}{l_1^{(1)}}, \quad \lim_{t \rightarrow 0} n_2 = \frac{l_3^{(1)}}{l_1^{(1)}}, \quad \lim_{t \rightarrow 0} n_3 = \frac{l_1^{(1)} l_3^{(2)} - l_3^{(1)} l_1^{(2)}}{l_1^{(1)} l_2^{(2)} - l_2^{(1)} l_1^{(2)}},\tag{9.98}$$

i.e. they assume constant values.

This result is in agreement with the one obtained in Sec. 9.2.3, where we showed how the Kasner axes rotate when approaching the singularity. That law states that two of the three Kasner axes rotate of an angle of order of unity, even when getting closer and closer to the initial singularity, implying that the Kasner frame does not admit a stationary limit although the Iwasawa variables remain unchanged.

Let us consider the case of a BKL piecewise solution: in two consecutive epochs, the line element has the same functional form as in Eq. (9.19) with coefficients given by Eq. (9.20), i.e. in one epoch we have that

$$h_{\alpha\beta} = \sum_{a=1}^3 t^{2p_i(x^\gamma)} l_\alpha^{(i)} l_\beta^{(i)}, \quad (9.99a)$$

while in the other one

$$h_{\alpha\beta} = \sum_{a=1}^3 t^{2p'_i(x^\gamma)} l_\alpha^{(i)'} l_\beta^{(i)'}. \quad (9.99b)$$

It can be shown that the asymptotic relation  $n_a = n_a(l^{(b)})$  and  $n'_a = n'_a(l^{(b)'})$  is the same as in Eq. (9.98), thus a direct substitution of Eq. (9.38) in Eq. (9.98) yields  $n_a = n'_a$ .

### 9.6.2 Cosmological billiards

Let us now formulate the BKL dynamics in this framework. The starting point of the analysis of system (9.96) is that  $\beta^a$  is expected to become a time-like vector in the vicinity of the initial singularity, i.e.  $\mathbf{g}_{ab}\beta^a\beta^b < 0$ . This condition allows one to introduce a new set of configurational variables, the hyperbolic planar ones  $(\rho, \gamma^i)$  defined as follows

$$\rho^2 = -\mathbf{g}_{ab}\beta^a\beta^b, \quad (9.100a)$$

$$\beta^a = \rho\gamma^a, \quad \gamma^a\gamma_a = -1. \quad (9.100b)$$

In Eq. (9.100),  $\rho$  is a “radial” coordinate that diverge for  $t \rightarrow 0$ , while  $\gamma^a$  are the coordinates on the hyperbolic space  $\Pi$ . The line element associated to  $\mathbf{g}_{ab}$  takes the form

$$d\sigma^2 = -d\rho^2 + \rho^2 d\Pi^2, \quad (9.101)$$

$d\Pi^2$  being the standard metric on the hyperbolic space. Rescaling further the radial variable  $\rho$  as  $\lambda = \ln \rho$ , and taking the lapse function in the form  $N = \rho^2 \sqrt{h}$ , we obtain the super-Hamiltonian as

$$N\mathcal{H} = \frac{1}{4} [-\pi_\lambda^2 + \pi_\gamma^2] + \mathcal{V}_S + \mathcal{V}_G. \quad (9.102)$$

Here,  $\pi_\lambda$  is the momentum conjugate to  $\lambda$ ,  $\pi_\gamma$  denotes the collection of momenta conjugate to  $\gamma^a$ ,  $\mathcal{V}_S$  is related to the kinetic terms of the off-diagonal components (the second term in the square brackets in Eq. (9.96), also called the *symmetric* potential), while  $\mathcal{V}_G = -h^3 R/4\kappa^2$  is the standard gravitational potential.

In such formulation, the asymptotic dynamics is governed by the scale factors  $\beta^a$ , while the remaining Iwasawa degrees of freedom  $n_a$  freeze to constant values. The two potential terms in Eq. (9.102) become (asymptotically) functions of the  $\gamma^a$  variables only and take the form of the infinite wells encountered in Sec. 9.5. Indeed, we can model any of the potentials as

$$\mathcal{V}(\beta^a, n_a) = \sum_A c_A e^{-2w_A(\beta^a)}, \quad (9.103)$$

where  $c_A$  are some functions of all the variables except  $\beta^a$  and their conjugate momenta  $\pi_a$ , and  $w_A(\beta^a)$  are linear combinations of the  $\beta$  variables.

We can now recover the asymptotic behavior discussed previously. In the limit  $t \rightarrow 0$ , i.e.  $\rho \rightarrow \infty$ , we have that Eq. (9.102) takes the form<sup>5</sup>

$$\mathcal{H}_\infty = \frac{1}{4} (\pi_\lambda^2 + \pi_\gamma^2) + \sum_{A'} \Theta_\infty(-2w_{A'}(\gamma)), \quad (9.104)$$

see Eq. (8.18). In Eq. (9.104), the sum is over the restricted set of the so-called “dominant walls”, that is the minimal collection of walls sufficient to define the billiard table, and obtainable from the following condition

$$\{w_{A'}(\gamma) \geq 0\} \quad \Rightarrow \quad \{w_A(\gamma) > 0\}. \quad (9.105)$$

The asymptotic picture can be summarized as follows. Since the potentials  $\mathcal{V}$  depend only on the variables on the hyperbolic space, the remaining degrees of freedom become asymptotically constants of motion, together with  $\mathcal{H}_\infty$ . The dynamics is described by the free motion of a non-relativistic point particle within a billiard, whose surrounding walls are given by the relations  $w_{A'}(\gamma^a) = 0$ . These walls are time-like hyperplanes, with space-like normal vectors (in the  $\beta$  space).

The scheme described above can be extended to include any kind of matter (and also to any number of dimensions, see Sec. 9.7), and the resulting billiard can have finite or infinite volume. This is a difference of crucial importance because, from the standard theory of geodesic motion on hyperbolic billiard, it is well known that the motion is chaotic if the

<sup>5</sup>A key point in the reduction of the potential to a well is that  $c_{A'} \geq 0$ .

volume is finite, while it is not chaotic if the volume is infinite. In the first case we have an infinite sequence of bounces, like those characterizing the BKL evolution; in the latter, only a finite number of bounces takes place. It is worth noting that in the case of standard gravity in vacua, the volume of the billiard is finite, as we will discuss in detail in Sec. 10.9.

## 9.7 Multidimensional Oscillatory Regime

Let us consider a  $(d + 1)$ -dimensional space-time ( $d \geq 3$ ), whose associated metric tensor obeys a dynamics described by the generalized vacuum Einstein equations

$${}^{d+1}R_{ik} = 0, \quad (i, k = 0, 1, \dots, d), \quad (9.106)$$

where the  $(d + 1)$ -dimensional Ricci tensor takes its natural form in terms of the metric components  $g_{ik}(x^l)$ . It can be shown that the inhomogeneous Mixmaster behavior finds a direct generalization in correspondence to any value of  $d$ . Moreover, in correspondence to  $d > 9$ , the generalized Kasner solution acquires a generality character, in the sense of the number of arbitrary functions, i.e. without a condition analogous to the one in Eq. (9.22). In a synchronous reference (described by the usual coordinates  $(t, x)$ ), the time-evolution of the  $d$ -dimensional spatial metric  $h_{\alpha\beta}(t, x)$  singles out an iterative structure near the cosmological singularity ( $t = 0$ ). Each single stage consists of intervals of time (Kasner epochs) during which  $h_{\alpha\beta}$  takes the generalized Kasner form

$$h_{\alpha\beta}(t, x) = \sum_{a=1}^d t^{2p_a} l_{\alpha}^a l_{\beta}^a, \quad (9.107)$$

where the Kasner indexes  $p_a(x^\gamma)$  satisfy

$$\sum_{a=1}^d p_a(x^\gamma) = \sum_{a=1}^d p_a^2(x^\gamma) = 1, \quad (9.108)$$

and  $l_{\alpha}^1(x^\gamma), \dots, l_{\alpha}^d(x^\gamma)$  denote  $d$  linear independent vectors whose components are arbitrary functions of the spatial coordinates. In each point of space, the conditions (9.108) define a set of ordered indexes  $\{p_a\}$  ( $p_1 \leq p_2 \leq \dots \leq p_d$ ) which, from a geometrical point of view, fixes one point in  $\mathbb{R}^d$ , lying on a connected portion of a  $(d - 2)$ -dimensional sphere. We note that the conditions (9.108) require  $p_1 \leq 0$  and  $p_{d-1} \geq 0$ , where the equality takes place for the values  $p_1 = \dots = p_{d-1} = 0$  and only  $p_d = 1$ .

For two consequent Kasner epochs, the following  $d$ -dimensional BKL map, linking the old Kasner exponents  $p_a$  to the new ones  $q_a$ , holds

$$q_1 = \frac{-p_1 - P}{1 + 2p_1 + P}, \quad q_2 = \frac{p_2}{1 + 2p_1 + P}, \dots, \quad q_{d-2} = \frac{p_{d-2}}{1 + 2p_1 + P} \quad (9.109)$$

$$q_{d-1} = \frac{p_{d-1} + 2p_1 + P}{1 + 2p_1 + P}, \quad q_d = \frac{p_d + 2p_1 + P}{1 + 2p_1 + P} \quad (9.110)$$

where  $P = \sum_{a=2}^{d-2} p_a$ .

It can be shown that each single step of the iterative solution is stable, in a given point of the space, if

$$\lim_{t \rightarrow 0} t^2 {}^d R_\alpha^\beta = 0. \quad (9.111)$$

The limit (9.111) is a sufficient condition to disregard the dynamical effects of the spatial curvature in the Einstein equations. An elementary computation shows how the only terms capable to perturb the Kasner behavior in  $t^2 {}^d R$  contain the powers  $t^{2\alpha_{abc}}$ , where  $\alpha_{abc}$  are related to the Kasner exponents as

$$\alpha_{abc} = 2p_a + \sum_{d \neq a, b, c} p_d, \quad (a \neq b, a \neq c, b \neq d), \quad (9.112)$$

and for generic  $l^a$ , all possible powers  $t^{2\alpha_{abc}}$  appear in Eq. (9.111). This leaves two possibilities for the vanishing of  $t^2 {}^d R_\alpha^\beta$  as  $t \rightarrow 0$ . Either the Kasner exponents can be chosen in an open region of the Kasner sphere defined in (9.108), such to have  $\alpha_{abc}$  positive for all triples  $a, b, c$ , or the conditions

$$\alpha_{abc}(x^\gamma) > 0 \quad \forall x^1, \dots, x^d \quad (9.113)$$

are in contradiction with Eq. (9.108), and one must impose extra conditions on the functions  $l^a$  and their derivatives. The second possibility occurs, for instance, for  $d = 3$ , since  $\alpha_{abc}$  is given by  $2p_a$ , and one Kasner exponent is always negative, i.e.  $\alpha_{1,d-1,d}$ . It can be shown that, for  $3 \leq d \leq 9$ , at least the smallest of the quantities (9.112), i.e.  $\alpha_{1,d-1,d}$  is always negative (excluding isolated points  $\{p_i\}$  in which it vanishes). Thus, Eq. (9.107) is a solution of the vacuum Einstein equations to leading order if and only if at least the vector  $l^1 = l$ , associated with the negative  $\alpha_{1,d-1,d}$ , obeys the additional condition

$$l \cdot \nabla \wedge l = 0, \quad (9.114)$$

and this requirement kills one arbitrary function of the space coordinate, as we have seen in details in Sec. 9.2.2.

Finally, for  $d \geq 10$  an open region of the  $(d - 2)$ -dimensional Kasner sphere where  $\alpha_{1,d-1,d}$  takes positive values exists, the so-called Kasner Stability Region (KSR). For  $3 \leq d \leq 9$ , the evolution of the system to the singularity consists of an infinite number of Kasner epochs, while for  $d \geq 10$ , the existence of the KSR, implies a deep modification in the asymptotic dynamics. In fact, the indications presented by Demaret in 1986 and by Kirillov and Melnikov in 1995 in favor of the “attractivity” of the KSR, imply that in each space point (excluding sets of zero measure) a final stable Kasner-like regime appears.

In correspondence to any value of  $d$ , the considered iterative scheme contains the right number of  $(d + 1)(d - 2)$  physically arbitrary functions of the spatial coordinates, required to specify generic initial conditions (on a non-singular space-like  $d$ -hypersurface). In fact, we have  $d^2$  functions from the  $d$  vectors  $l$  and  $d - 2$  Kasner exponents; the invariance under spatial reparametrizations allows to eliminate  $d$  of these functions, and other  $d$  because of the  $0\alpha$  Einstein equations (which play also the role of constraints for the space functions). This piecewise solution describes the asymptotic evolution of a generic inhomogeneous multidimensional cosmological model.

### 9.7.1 *Dilatons, $p$ -forms and Kac-Moody algebras*

We summarize some properties about the insertion of  $p$ -forms and dilatons in the gravitational dynamics in the multidimensional case.

The inclusion of massless  $p$ -forms in a generic multi-dimensional model can restore chaos when it is otherwise suppressed. In particular, even though pure gravity is non-chaotic in  $d = 10$  space-times, the 3-forms of  $d + 1 = 11$  supergravity make the system chaotic. The billiard description in Iwasawa variables given in Sec. 9.6 in the four-dimensional case is quite general and can be extended to higher space-time dimensions, with  $p$ -forms and dilatons. If there are  $n$  dilatons, the billiard is a region of the hyperbolic space  $\Pi_{d+n-1}$ , and in the Hamiltonian each dilaton is equivalent to the logarithm of a new scale factor  $\beta^a$ . The other ingredients that enter the billiard definition are the different types of the walls bounding it: in addition to the symmetry and to the gravitational walls  $\mathcal{V}_S$ ,  $\mathcal{V}_G$ , in the general case  $p$ -form walls are also present (that can be divided in *electric* and *magnetic* walls). All of them are hyper-planar, and the billiard is a convex polyhedron with finitely many vertices, some of which are at infinity. We have seen in Sec. 8.7.2, how an Abelian 1-form can restore chaos in higher dimensional homogeneous models. The analysis performed in the case when

$p$ -forms are present can be thought as the maximal generalization of this scheme.

Finally, we want to stress that some of the billiards we have described can be associated with a Kac-Moody algebra; in this framework, the asymptotic BKL dynamics is equivalent to that of a one-dimensional non-linear  $\sigma$ -model based on a certain infinite dimensional space.

## 9.8 Properties of the BKL Map

In this Section we discuss from a mathematical point of view the main properties of the BKL map in a generic number of dimensions  $d$  as outlined by Elskens and Henneaux in 1987.

A set of Kasner indexes is a set of parameters  $p_a (a = 1, \dots, d)$  that satisfy the conditions

$$\sum_{i=a}^d p_a = \sum_{a=1}^d p_a^2 = 1, \quad (9.115)$$

$$p_1 \leq p_2 \leq \dots \leq p_{d-1} \leq p_d. \quad (9.116)$$

The constraints in Eq. (9.115) define the so-called *Kasner sphere* (in  $d - 2$  dimensions), while the inequalities (9.116) coincide with the request of an *ordered* set of Kasner parameters.

We call *BKL map* the application  $T : \{p_a\} \in \mathbb{R}^d \rightarrow \{p'_a\} \in \mathbb{R}^d$  such that

$$p'_a = \text{ordering of } q_a \quad (9.117a)$$

$$q_1 = -\frac{-p_1 - \Sigma}{1 + 2p_1 + \Sigma}, \quad (9.117b)$$

$$q_2 = \frac{p_2}{1 + 2p_1 + \Sigma}, \quad (9.117c)$$

...

$$q_{d-2} = \frac{p_{d-2}}{1 + 2p_1 + \Sigma}, \quad (9.117d)$$

$$q_{d-1} = \frac{p_{d-1} + 2p_1 + \Sigma}{1 + 2p_1 + \Sigma}, \quad (9.117e)$$

$$q_d = \frac{p_d + 2p_1 + \Sigma}{1 + 2p_1 + \Sigma}, \quad (9.117f)$$



where  $\Sigma$  is defined as

$$\Sigma = \sum_{a=2}^{d-2} p_a. \quad (9.118)$$

We have seen in the multidimensional case that the occurrence of such transition is possible if and only if at least one of the quantities  $\alpha_{abc}$  is negative, which are defined as

$$\alpha_{abc} = 2p_a + \sum_{d \notin \{a,b,c\}} p_d. \quad (9.119)$$

This situation always happens in a number of space-time dimensions  $n = d + 1 \leq 10$ . On the other hand, when  $n \geq 11$  a region KSR of non-zero measure exists where all the  $\alpha_{abc}$  are greater than zero, so that the BKL oscillations will stop as soon as the transition mechanism brings to a set of Kasner indexes in this region.

We have already studied in detail the three-dimensional case of this map (see Sec. 7.4) that exhibits stochastic features. In the higher dimensional case the following two properties hold:

- (i) The BKL map is chaotic when the number of spatial dimensions  $d$  is smaller than 10, and almost every set of initial  $p_a$  evolves visiting any parameter region of non-zero measure.
- (ii) The map is not chaotic in  $d > 10$  and, for almost every set of initial  $p_a$ , the evolution reaches the KSR.

Indeed, there is no contradiction between i) and ii). In fact, even if one can pass from  $d \geq 10$  to  $d \leq 9$  by a dimensional reduction, i.e. by taking some of the Kasner indices equal to zero, such sets correspond to regions of zero measure. This phenomenon ensures that in  $d \geq 10$  some subsets of points of zero measure exist with never ending oscillations of the scale factors.

### 9.8.1 *Parametrization in a generic number of dimensions*

We will now discuss a parametric representation of the Kasner exponents that generalizes the one expressed in terms of  $u$  for the three-dimensional case (see Sec. 7.2).

Let us assume<sup>6</sup>  $p_d < 1$ . The parametrization is then given by

$$p_a = \frac{v_a}{\Upsilon}, \quad a = 1, \dots, d-2, \quad (9.120a)$$

$$p_{d-1} = \frac{1 - \sum_{a=1}^{d-2} v_a}{\Upsilon}, \quad (9.120b)$$

$$p_d = \frac{\sum_{a=1}^{d-2} (v_a^2 - v_a)}{\Upsilon}, \quad (9.120c)$$

$$(9.120d)$$

where

$$\Upsilon = \sum_{a=1}^{d-2} \left( v_a^2 - v_a + v_a \sum_{b=1}^{a-1} v_b + 1 \right), \quad (9.121)$$

or, equivalently,

$$\Upsilon = \frac{1}{1 - p_d}, \quad 1 < \Upsilon < \infty. \quad (9.122)$$

This parametrization is valid in the range defined by the inequalities coming from the conditions  $p_{d-2} \leq p_{d-1} \leq p_d$  and then

$$2v_{d-2} \leq 1 - \sum_{a=1}^{d-3} v_a, \quad (9.123a)$$

$$2Z = \sum_{a=1}^{d-2} v_a^2 + \left( \sum_{b=1}^{d-2} v_b \right)^2 \geq 2. \quad (9.123b)$$

This new set of parameters  $v_a$  is ordered by construction. The corresponding inverse relations read as

$$v_a = \Upsilon p_a = \frac{p_a}{1 - p_d} \quad a = 1, \dots, d-2, \quad (9.124)$$

which provides the new parametrization plus the additional relations

$$v_{d-1} = 1 - \sum_{a=1}^{d-2} v_a, \quad (9.125a)$$

$$v_d = \frac{1}{2} \left( \sum_{b=1}^{d-1} v_b^2 - 1 \right). \quad (9.125b)$$

---

<sup>6</sup>The case  $p_d = 1$  corresponds to the fixed set  $\{0, 0, \dots, 0, 1\}$  and will be recovered as the point at infinity.

**Case  $d = 3$ .** We have only  $d - 2 = 1$  independent parameters, that is  $v \equiv v_1$ . From Eq. (9.124) we get

$$v = \frac{p_1}{1 - p_3}. \quad (9.126)$$

From Eq. (9.121) we have  $\Upsilon = v^2 - v + 1$ , so that from Eq. (9.120) we get

$$p_1 = \frac{v}{v^2 - v + 1}, \quad p_2 = \frac{1 - v}{v^2 - v + 1}, \quad p_3 = \frac{v^2 - v}{v^2 - v + 1}. \quad (9.127)$$

This set of relations is constrained by those in Eq. (9.123) that now read as

$$v \leq 1/2, \quad v^2 \geq 1, \quad (9.128)$$

that mean  $v \leq -1$ . This way, Eq. (7.55) is recovered by setting  $v = -u$ .

**Case  $d = 4$ .** We have  $4 - 2 = 2$  independent parameters,  $\delta_1 = v_1$  and  $\delta_2 = v_2$ , so that

$$p_1 = \frac{\delta_1}{\Upsilon}, \quad p_2 = \frac{\delta_2}{\Upsilon}, \quad (9.129a)$$

$$p_3 = \frac{1 - \delta_1 - \delta_2}{\Upsilon}, \quad p_4 = \frac{\Upsilon - 1}{\Upsilon}, \quad (9.129b)$$

$$\Upsilon = \delta_1^2 + \delta_2^2 + \delta_1\delta_2 - \delta_1 - \delta_2 + 1. \quad (9.130)$$

The range of validity of this parametrization is bidimensional and is given by the inequalities

$$\begin{cases} 2\delta_2 \leq 1 - \delta_1 \\ \delta_1^2 + \delta_2^2 + \delta_1\delta_2 \geq 1 \end{cases} \quad (9.131)$$

together with the condition  $\delta_1 < \delta_2$ . The solution is sketched in Fig. 9.1, while in Fig. 9.2 the tridimensional representation of  $p_1, p_2, p_3$  and  $p_4$  is presented in the allowed domain.

### 9.8.2 Ordering properties

In this Subsection we will study the relation existing between an *ordered* set of Kasner exponents and an *unordered* one.

Let us denote  $q_a$  as the coordinates of a generic point on the  $d$ -dimensional Kasner sphere, i.e.  $\{q_a\}$  is a generic unordered set of exponents. Let  $w_f$  be the corresponding parametrization *via* (9.122) and (9.124). If the  $q_a$  are not ordered, then one of the inequalities (9.123) will be violated. At the same time, let  $p_a$  be the ordered set associated to  $q_a$

$$\{p_a\}_{a=1,\dots,d} = \text{ordering of } \{q_b\}_{b=1,\dots,d} \quad (9.132)$$

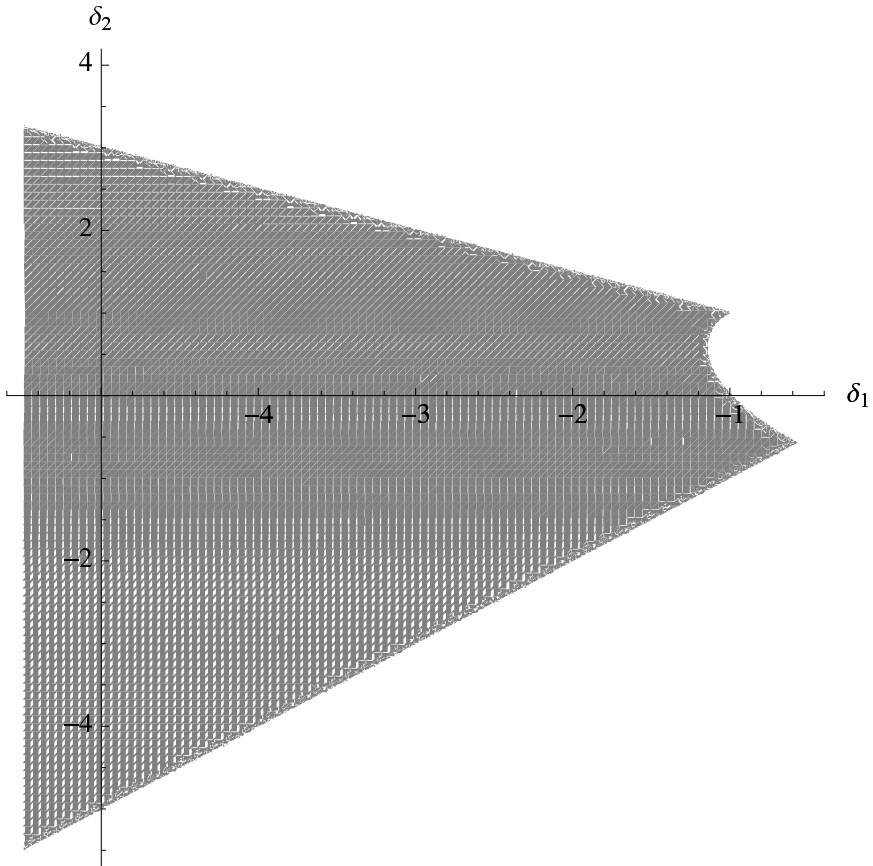


Figure 9.1 Domain of validity of the  $(\delta_1, \delta_2)$  parametrization for the four-dimensional Kasner parameters.

and  $\{v_g\}_{g=1,\dots,d}$  the corresponding parametrization. This way, the relation between the  $q_b$  and  $p_a$  is straightforward.

From an analysis of the parametrization (9.124) and (9.122), the relevant quantities are  $\Upsilon$ ,  $Z$ ,  $v_{d-1}$  and  $v_d$ , (defined in Eqs. (9.122), (9.123b), (9.125a) and (9.125b), respectively) which depend on  $q_{d-1}, q_d$  and on the relation with the other  $d - 2$  exponents. This implies that there are three different cases only:

- (i)  $q_d$  and  $q_{d-1}$  are the largest indexes;
- (ii)  $q_d$  is the highest but  $q_{d-1}$  is not the second highest;
- (iii)  $q_d$  is not the largest index.

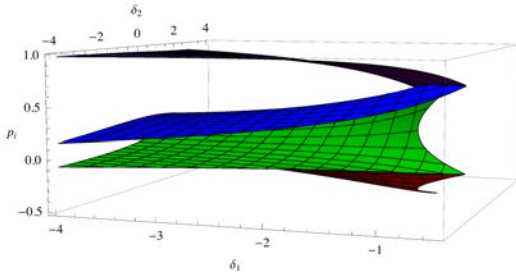


Figure 9.2 Three-dimensional representation of the ordered ( $p_1 < p_2 < p_3 < p_4$ ) Kasner parameters in the four-dimensional case as a function of  $\delta_1$  and  $\delta_2$ . One can see how  $p_4$  is always greater than zero, while  $p_1$  is always negative.

Let us discuss each case in more details.

$$(i) \quad q_d \geq q_{d-1} \geq q_a \quad \forall a = 1, \dots, d-2$$

In this case,  $p_d \equiv q_d$  and  $p_{d-1} \equiv q_{d-1}$ , while  $(p_{d-2}, \dots, p_1) \equiv$  ordering of  $(q_{d-2}, \dots, q_1)$ . These relations imply that

$$\{v_g\} \equiv \text{ordering of } \{w_f\} \quad (9.133)$$

and leave  $\Upsilon$ ,  $Z$ ,  $v_{d-1}$  and  $v_d$  unchanged. Finally, the transformation (9.133) is a permutation, and thus has unitary modulus.

$$(ii) \quad q_d \geq q_a \text{ and } q_{d-1} \not\geq q_a \quad \forall a = 1, \dots, d-2$$

In this case, we have that

$$\Upsilon(w) = \Upsilon(v), \quad (9.134)$$

so that we have to order the first  $d-1$  elements only; this can be done in three steps by applying (9.133) as

$$w'_f = \text{ordering of } w_f, \quad f = 1, \dots, d-2 \quad (9.135a)$$

$$w'_{d-1} = w_{d-1} \quad (9.135b)$$

$$w'_d = w_d \quad (9.135c)$$

then by exchanging  $w_{d-1}$  with  $w_{d-2}$

$$w''_f = w'_f, \quad f = 1, \dots, d-3 \quad (9.136a)$$

$$w''_{d-2} = w'_{d-1} = 1 - \sum_f^{d-2} w'_f \quad (9.136b)$$

$$w''_d = w'_d = w_d \quad (9.136c)$$

and finally by applying (9.133) we get

$$v_g = \text{ordering of } w_g'', \quad g = 1, \dots, d-2 \quad (9.137a)$$

$$v_{d-1} = w_{d-1}'' \quad (9.137b)$$

$$v_d = w_d'' \quad (9.137c)$$

It has been shown by Elskens and Henneaux (1987) that these combined transformations have unit modulus, thus implying that the volume in the  $v$ -space is preserved.

(iii)  $q_d \not\geq q_a \quad \forall a = 1, \dots, d-1$

Without loss of generality, we can assume  $q_{d-1}$  to be the greatest exponent (indexes can be rearranged this way following the previous procedure). If  $q_{d-1} > q_d$  then inequality (9.123b) is violated. In this case,  $\Upsilon(w)$  is given by

$$\Upsilon(w) = \frac{1}{1 - q_d} = \frac{1}{1 - p_{d-1}}. \quad (9.138)$$

Because  $w_i = \Upsilon q_i$ , to obtain the ordered set  $\{v\}$  we have to rescale  $\Upsilon$  by a factor  $(1 - p_{d-1})/(1 - p_d)$ , which is equal to

$$\frac{1 - p_{d-1}}{1 - p_d} = 1 + \frac{p_d}{1 - p_d} - \frac{p_{d-1}}{1 - p_d} = 1 + v_d - v_{d-1} = Z(v), \quad (9.139)$$

because  $Z(v) = 1 + v_d - v_{d-1}$ . From (9.139) follows that

$$Z(v) = \frac{1}{Z(w)} \quad (9.140)$$

and so the correct transformation is given by

$$v_f = \frac{w_f}{Z(w)}. \quad (9.141)$$

In this case the transformation is not unitary, but has modulus  $1/Z > 1$ .

### 9.8.3 Properties of the BKL map in the $v$ space

The BKL map assumes a simple expression in terms of the parametrization (9.120). A direct substitution yields that, if taking the  $q_a$  as in Eq. (9.117) where  $q_1$  and  $q_{d-1}$  have been exchanged,<sup>7</sup> the map  $T_R$  for the reduced

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<sup>7</sup>This re-statement of the map leads to the same set of Kasner exponents since it is just a permutation.

variables is given by

$$w_1 = v_1 + 1, \quad (9.142a)$$

$$w_2 = v_2, \quad (9.142b)$$

$$\dots$$

$$w_{d-2} = v_{d-2}. \quad (9.142c)$$

Together with Eq. (9.142), the following relations hold

$$w_{d-1} = v_{d-1} - 1, \quad (9.142d)$$

$$w_d = Z(v) + v_1, \quad (9.142e)$$

$$Z(w) = Z(v) - v_1 + v_{d-1}. \quad (9.142f)$$

The map  $T_R : \{v\} \rightarrow \{w\}$  defined by Eq. (9.142) exhibits several properties which are summarized below but we shall not prove here. Given a set of ordered  $\{v\}$ , then

- at least one set  $\{q\}$  such that  $\{v\}$  is the image of  $\{q\}$  through  $T_R$ , i.e.  $v_f = T_R q_a$ , always exists.
- If  $w_f = T_R v_a$ , then the new parameters need at least one rearrangement for  $d \leq 9$  if the exchange  $q_1 \leftrightarrow q_{d-1}$  is not performed.
- Such rearrangement necessarily involves  $q_{d-1}$  which cannot remain at the next-to-last place.
- The Kasner Stability Region KSR coincides with the following set of values available for the Kasner indices:

- (1) if  $d \leq 8$ ,  $\text{KSR} \equiv \{\infty\} = \{(0, 0, \dots, 0, 1)\}$
- (2) if  $d = 9$ ,  $\text{KSR} \equiv \{\infty, c, c'\}$  where  $c = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3})$   
and  $c' = (-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5})$
- (3) for  $d \geq 10$ , the KSR has non-zero measure.

## 9.9 Guidelines to the Literature

The original analysis on the Bianchi IX model stability presented in Sec. 9.1 was given by Regge & Hu in [245]; some developments can be found in [243, 244].

The first derivation of the generalized Kasner solution discussed in Sec. 9.2.1 was given by Lifshitz & Khalatnikov in [312]. For a complete presentation of the inhomogeneous BKL solution, as in Sec. 9.2.2, see [65] and for the specific case of the small oscillations, see [58]. A critical re-analysis of the BKL solution can be found in [44]. For a more recent study

on the inhomogeneous Mixmaster, see [236], while for numerical support to the Conjecture, see for example [78, 191, 459]. The mechanism of Kasner axes rotation, as presented in Sec. 9.2.3, is provided in [65]. More recently, Belinskii reviewed the main problems in the topic of cosmological singularity in [57].

A discussion of the fragmentation process, illustrated in Sec. 9.3, is addressed by [284] and [348].

The Hamiltonian formulation of the inhomogeneous Mixmaster model in terms of Misner variables, as in Sec. 9.4, can be found in [260]. For analysis of the inhomogeneous Mixmaster model when the Kasner vectors are time-dependent quantities as discussed in Sec. 9.5, see [283]. An extension of the formalism presented in Sec. 9.5.1, addressed in view of a generic choice of the gauge can be found in [70] where the covariance of the inhomogeneous Mixmaster chaos is outlined. A proof concerning the negligibility of the spatial gradients presented in Sec. 9.5.2 in the inhomogeneous Mixmaster (in more than 4 dimensions and also in the presence of a scalar field) can be found in [285].

For a complete discussion of the multidimensional inhomogeneous Mixmaster (in vacuum as well as in presence of matter) adopting Iwasawa variables as presented in Sec. 9.6, see the review [137] or the lecture [135]. A very recent result about the structure of the space-time in the BKL limit is given in [136]. For a discussion on the implication of such a framework on the de-emergence of space-time near the singularity, see [138].

The Dynamical System approach in the non-homogeneous case, that we did not present here, is reviewed in [236]. A rigorous attempt to formulate the BKL Conjecture in such framework was firstly given in [441] and developed for example in [12, 191]. For what concerns the BKL Conjecture in the connection formalism for GR, see [27].

The original discussion of the multidimensional oscillatory regime, provided in Sec. 9.7, can be found in [146, 147]. The ergodic theory of the multidimensional BKL map, as discussed in Sec. 9.8, is analyzed in [169].



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## PART 4

# Quantum Cosmology

In these Chapters, we provide a characterization of the quantum behavior of the cosmological models, as described by the implementation of the most viable quantum gravity approaches.

Chapter 10 concerns the derivation of canonical quantum gravity in the metric approach and its application to various cosmological contexts. We formulate the quantum cosmological problem and compare the Dirac quantization procedure with the path integral formalism.

Chapter 11 provides a discussion of generalized Heisenberg algebras, related to cut-off features of space-time. Particular attention is devoted to the so-called polymer quantization approach (which mimics Loop Quantum Cosmology features) and the generalized uncertainty principle prescription (related to the String theory paradigm).

Chapter 12 is focused on the derivation of the Loop Quantum Gravity theory. The quantum dynamical implications are developed on a cosmological setting, outlining successes and shortcomings of the minisuperspace formulation. Quantum cosmological model based on the extension to the minisuperspace of the generalized Heisenberg algebras are also presented for some relevant cases.

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## Chapter 10

# Standard Quantum Cosmology

In this Chapter we face the analysis of the quantum gravity problem in the framework of the metric approach and we implement this theory to the study of the quantum Universe morphology. Indeed, the functional nature of this scheme of quantization makes solvable the quantum dynamics problem just in correspondence to highly symmetric models, as those concerning the cosmological setting. In this respect, when treating the quantum Universe, we will be able to analyze systems with a finite number of degrees of freedom, i.e. the so-called *minisuperspace* models.

We start by deriving the Wheeler-DeWitt (WDW) equation describing the quantum dynamics of the gravitational field, as a direct consequence of implementing to operator level the classical Hamiltonian constraints. The resulting physical states are annihilated by the super-Hamiltonian and the super-momentum operators and by the momenta conjugate to the lapse function and the shift vector as well. The implications of these quantum constraints are clearly outlined with particular reference to the equivalence between the super-momentum conditions and the invariance of the state function under spatial diffeomorphisms.

The link between this Dirac approach to quantum gravity and the path integral formalism, as heuristically extended to the gravitational sector, is properly addressed aiming to show that the WDW scheme can be recovered by the path integral approach in some limit.

The annihilation of the state function by the super-Hamiltonian operator implies the absence of a time evolution of the quantum gravity configurations. We face this problem of the canonical quantum gravity, known as the *frozen formalism*, within two different frameworks:

- (i) fixing a time variable before implementing the quantization procedure;

- (ii) recognizing a physical time at quantum level, when the Dirac constraints have already been implemented as operators.

Finally we outline as main point the possibility to deal with a timeless quantum gravity, in which evolution is essentially a relational property between different system components.

We then face the quantum gravity problem under the cosmological hypotheses, leading to what is commonly called quantum cosmology. We discuss the form that the general theory takes in correspondence to the symmetry restrictions of the primordial Universe. The nature of the minisuperspace and how the cosmological singularity could be removed by the quantum evolution are analyzed in some detail.

The path integral representation of the isotropic Universe is given to derive the precise equivalence we mentioned above with the Dirac quantization procedure. The full mathematical equipment to achieve this equivalence statement is provided and the most relevant steps of the proof outlined.

The possibility to use a real scalar field as a good relational time is successfully explored, especially in view of the implementation we will make of this scheme in Chap. 12, when discussing the Big Bounce in Loop Quantum Cosmology.

Wide space is dedicated to the interpretation of the Universe wave function in the semiclassical approximation. According to the Vilenkin analysis, we show that a proper separation of the system into a classical part and a small quantum portion allows to recover a Schrödinger dynamics for the latter system component. An elucidating example concerning the Universe isotropization is presented, where the Universe volume is the classical time coordinate and the small Universe anisotropies evolve in a full quantum scheme.

We discuss how boundary conditions for the WDW equation can be properly fixed in order to make predictions from the theory. In particular, we analyze the important case of the no-boundary proposal and of the tunneling boundary problem, outlining and comparing their implications on the quantum Universe morphology.

After the setting of the full quantum cosmology theory, the Chapter ends with a series of important and meaningful applications, i.e. the isotropic Universe in the presence of a scalar field, the Taub Universe and the Mixmaster model, both in the Misner and Misner-Chitré variables representation. In particular, the quantum Mixmaster in the half Poincaré plane is finely described to fix some relevant features as the spectrum discreteness

and the existence of a point-zero energy.

## 10.1 Quantum Geometrodynamics

In this Section we will give an overview of canonical quantum gravity in the metric formalism. In this framework, the three-metric of the Cauchy surfaces is adopted as the configuration variable. This approach, known as the WDW theory, will also be related to the path integral formulation of quantum gravity.

### 10.1.1 The Wheeler-DeWitt Theory

As we have seen in Sec. 2.3, the Einstein theory of gravity can be written as a dynamical system subjected to first-class constraints with a Dirac algebra (2.80)-(2.80c). The configuration space of canonical gravity, on which the constraints are defined, is the space of all the Riemannian three-metrics  $\text{Riem}(\Sigma)$  modulo the spatial diffeomorphisms group  $\text{Diff}(\Sigma)$  of the slicing surface  $\Sigma$ . Explicitly, it reads as

$$\{h_{\alpha\beta}\} = \frac{\text{Riem}(\Sigma)}{\text{Diff}(\Sigma)}. \quad (10.1)$$

This is the space of all the three-geometries and is known as the *Wheeler superspace* which is infinite-dimensional, but of course there is a finite number of degrees of freedom at each space point.

To implement the quantization of such a constrained system there are essentially two ways. The first one relies on solving the constraints (2.77) at a classical level in order to deal with a formulation based on unconstrained physical variables only. This approach is the so-called *reduced quantization procedure* and has several faults even in more simple frameworks, as for example in quantum electrodynamics it is consistent in the non-interacting case only.

Thus, one usually follows the second approach to quantize first-class constrained systems, known as the *Dirac scheme* (discussed in more details in Sec. 12.1). In this scheme the quantum theory is constructed without solving the classical constraints. The Poisson brackets are then implemented as commutators and the constraints select the physically allowed states. In particular, given a (first-class) constraint  $\mathcal{C} = 0$ , a physical state must remain unchanged when one performs (gauge) transformations generated by  $\mathcal{C}$ . This consideration has to be implemented at a quantum level. The

physical states are thus the ones annihilated by the quantum operator constraints, i.e. by imposing the relation

$$\hat{\mathcal{C}}|\Psi\rangle = 0. \quad (10.2)$$

It is worth noting that the reduced and the Dirac quantizations are formally equivalent to each other but, in general, it may break down because of factor-ordering problems.

The first step of the canonical quantization à la Dirac of GR in the metric formulation relies on implementing the Poisson algebra (2.74a) - (2.74c) in the form of the canonical commutation relations

$$\left[\hat{h}_{\alpha\beta}(x, t), \hat{h}_{\gamma\delta}(x', t)\right] = 0 \quad (10.3a)$$

$$\left[\hat{\Pi}^{\alpha\beta}(x, t), \hat{\Pi}^{\gamma\delta}(x', t)\right] = 0 \quad (10.3b)$$

$$\left[\hat{h}_{\gamma\delta}(x, t), \hat{\Pi}^{\alpha\beta}(x', t)\right] = i\delta_{(\gamma}^{\alpha}\delta_{\delta)}^{\beta}\delta^3(x - x'). \quad (10.3c)$$

This is only a formal prescription and requires some remarks:

- (i) Eq. (10.3a) is a kind of microcausality condition for the three-metric field, though the functional form of the constraints is independent of any foliation of space-time. This confirms that the points of the three-manifold  $\Sigma$  are space-like separated.
- (ii) The above relations are not compatible with the requirement that the operator  $\hat{h}_{\alpha\beta}(x, t)$  has a positive definite spectrum. In fact, the classical quantity  $h_{\alpha\beta}(x, t)$  is a Riemannian metric tensor, i.e. it is positive definite. Such property should also be implemented at a quantum level. More precisely, for any (non-vanishing) vector field  $\xi^\alpha(x)$ , the classical relation

$$h(\xi \otimes \xi) = \int_{\Sigma} d^3x \xi^\alpha \xi^\beta h_{\alpha\beta} > 0 \quad (10.4)$$

holds. It is reasonable to require that this feature is implemented at a quantum level as

$$\hat{h}(\xi \otimes \xi) > 0. \quad (10.5)$$

However, we know that if  $\hat{\Pi}^{\alpha\beta}$  is a self-adjoint operator it can be exponentiated as an unitary operator. The spectrum of this operator takes negative values, similarly to the spectrum of the translation operator in quantum mechanics being the entire real axis. The problem is to give a physical meaning to these negative values. The self-adjoint property of the momentum operator is therefore

not compatible with the positive definite requirement of the three-metric operator. The positiveness of the configuration operator can be recovered by restricting the Hilbert space (for an explicit example, see Sec. 10.8), but this implies that the momentum operator is no longer self-adjoint.

Proceeding in a formal way, one imposes the constraint equations (2.77) as operators to select the physically allowed states, that is

$$\hat{\mathcal{H}}(\hat{h}, \hat{\Pi}) \Psi = 0, \quad (10.6a)$$

$$\hat{\mathcal{H}}_\alpha(\hat{h}, \hat{\Pi}) \Psi = 0. \quad (10.6b)$$

Here,  $\Psi$  is known as *the wave function of the Universe*. As we have seen, the Hamiltonian for the Einstein theory (2.75) reads as

$$\mathbf{H} \equiv \int_{\Sigma} d^3x (N\mathcal{H} + N^\alpha \mathcal{H}_\alpha), \quad (10.7)$$

and therefore, considering Eqs. (10.6) in a putative Schrödinger-like equation such as

$$i \frac{\partial}{\partial t} \Psi_t = \hat{\mathbf{H}} \Psi_t = 0, \quad (10.8)$$

the state functional  $\Psi_t$  results independent of “time”. This is the so-called *frozen formalism* because it apparently implies that nothing evolves in a quantum theory of gravity. Loosely speaking, an identification of the quantum Hamiltonian constraint as the zero-energy Schrödinger equation  $\hat{\mathbf{H}}\Psi = 0$  holds. Such feature is known as the *problem of time* and deserves to be treated in some details in Sec. 10.2. It is worth noting that, by the expression (10.7), we assumed the primary constraints

$$C(x, t) \equiv \Pi(x, t) = 0, \quad C^\alpha(x, t) \equiv \Pi^\alpha(x, t) = 0, \quad (10.9)$$

to be satisfied. The wave functional  $\Psi = \Psi(h_{\alpha\beta}, N, N^\alpha)$  then becomes functional of the three-metric only, i.e.  $\Psi = \Psi(h_{\alpha\beta})$ .

Let us now explicitly discuss the meaning of the quantum constraints (10.6). First of all, a representation of the canonical algebra can be chosen as

$$\hat{h}_{\alpha\beta} \Psi = h_{\alpha\beta} \Psi, \quad \hat{\Pi}^{\alpha\beta} \Psi = -i \frac{\delta \Psi}{\delta h_{\alpha\beta}}, \quad (10.10)$$

which is the widely used representation of the canonical approach to quantum gravity in the metric formalism. However, the above equations do not define proper self-adjoint operators because of the absence of any Lebesgue



measure on  $\Sigma$  and moreover they are not compatible with the positivity requirement  $\hat{h}_{\alpha\beta} > 0$ . Ignoring at this level such problems, we proceed further in a formal way.

The easiest constraint to be addressed is (10.6b), which is the so-called diffeomorphism (or kinematic) one. Considering Eqs. (2.72b) and (10.10), it reads as

$$\hat{\mathcal{H}}_\alpha \Psi = 2i h_{\alpha\gamma} \nabla_\beta \frac{\delta \Psi}{\delta h_{\gamma\beta}} = 0. \quad (10.11)$$

As shown in Sec. 2.3, the functional  $\vec{H}(\vec{f})$  generates the Lie algebra  $\text{diff}(\Sigma)$  and this feature must also be implemented at a quantum level. The relation (10.11) implies that the state functional  $\Psi$  is a constant on the orbits of the spatial diffeomorphism group  $\text{Diff}(\Sigma)$ . The functional  $\Psi(h_{\alpha\beta})$  is thus defined on the whole class of three geometries  $\{h_{\alpha\beta}\}$  (invariant under spatial diffeomorphisms) and not only on  $\text{Riem}(\Sigma)$ , i.e.  $\Psi = \Psi(\{h_{\alpha\beta}\})$ . More explicitly, under the infinitesimal transformation

$$x_\alpha \rightarrow x_\alpha + \delta N_\alpha, \quad (10.12)$$

the three-metric  $h_{\alpha\beta}$  becomes

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - (\nabla_\alpha \delta N_\beta + \nabla_\beta \delta N_\alpha). \quad (10.13)$$

The wave functional  $\Psi(h_{\alpha\beta})$  thus transforms as

$$\Psi(h_{\alpha\beta}) \rightarrow \Psi(h_{\alpha\beta}) - 2 \int_\Sigma d^3x \frac{\delta \Psi}{\delta h_{\alpha\beta}} \nabla_\alpha \delta N_\beta. \quad (10.14)$$

By integrating by parts<sup>1</sup> (assuming that  $\delta N_\alpha$  vanishes at infinity) this expression, we obtain that the condition

$$\nabla_\alpha \frac{\delta \Psi}{\delta h_{\alpha\beta}} = 0 \quad (10.15)$$

implies that  $\Psi$  is invariant under infinitesimal coordinate transformations, i.e. the constraint (10.11) holds. The wave functional  $\Psi$  does not depend on the particular form of the three-metric, but on the three-geometry only (namely all the three-metrics related by a coordinate transformation). We have thus recovered that the configuration space of the canonical quantum gravity is exactly the Wheeler superspace defined as in Eq. (10.1).

The dynamics is generated via the scalar constraint (10.6a), providing the famous WDW equation obtained by DeWitt (following the idea of Wheeler) in 1967, which explicitly reads as

$$\hat{\mathcal{H}}\Psi = -\mathcal{G}_{\alpha\beta\gamma\delta} \frac{\delta^2 \Psi}{\delta h_{\alpha\beta} \delta h_{\gamma\delta}} - \frac{\sqrt{h}}{2\kappa} {}^3R\Psi = 0, \quad (10.16)$$

---

<sup>1</sup>The covariance of this integral is ensured by the fact that  $\Pi^{\alpha\beta}$  is a densitized tensor, i.e. it contains  $\sqrt{h}$  in its definition (2.69a).

where  $\mathcal{G}_{\alpha\beta\gamma\delta}$  is the supermetric (2.72c). The factor-order ambiguity is not addressed at this stage. We have chosen the simplest factor ordering, i.e. the one with the momenta placed to the right of  $h_{\alpha\beta}$ . This ordering however is not self-adjoint in the kinematic Hilbert space associated with the representation (10.10).

Equation (10.16) is not a single equation, but actually one at each space point  $x \in \Sigma$ . It is a second-order hyperbolic functional differential equation which is *not* defined on the space-time, but on the configuration space (10.1). At a quantum level the space-time itself has disappeared as the particle trajectories are absent in quantum mechanics. In fact, in GR the space-time is the analogous to a particle trajectory in classical (non-relativistic) mechanics.

The WDW equation is at the heart of the Dirac constraint quantization approach and the key aspects of the canonical quantum gravity are all connected with it. Mathematical and conceptual problems emerge in the WDW approach to quantum gravity, and the most relevant ones can be summarized as:

- (i) The WDW constraint (10.6a) is not polynomial, neither analytical in the three-metric. Moreover, since Eq. (10.16) contains products of functional differential operators evaluated at the same spatial point, it is hopelessly divergent. Distributions in the denominator are also not clearly defined.
- (ii) Although ignoring the problems at point i), it is not possible to find a formal solution to the WDW equation. As a matter of fact, not even the constant state

$$\Psi(h_{\alpha\beta}) = \text{const} \quad (10.17)$$

is a solution. The WDW equation implies that a physical state  $\Psi$  should be an eigenvector of the operator  $\hat{\mathcal{H}}(x, t)$  with eigenvalue 0 and thus some sort of boundary conditions have to be imposed on  $\Psi$ , but the theory does not give any information about how to set them.

- (iii) Understanding the physical meaning of the WDW equation is one of the most challenging problems, as for example it requires a notion of “time” (or “time-evolution”) at a quantum level. Such feature can be eventually related to the fact that the classical slicing is performed before the quantization procedure.

### 10.1.2 Relation with the Path Integral Quantization

It is interesting to analyze the relation between the canonical (à la Dirac) quantization of gravity and the path integral quantization framework. The latter approach is also known as *covariant quantum gravity* since the space-time covariance is manifestly preserved. In fact, one integrates over the whole space-time metric in analogy with the path integral in quantum mechanics. The main ideas of this scheme can be summarized as follows and its implementation in the minisuperspace arena is given in Sec. 10.4.

The Feynman amplitude between an initial configuration (a state with an intrinsic metric  $h_{\alpha\beta}$  on  $\Sigma$ ) and a final configuration (a state with an intrinsic metric  $h'_{\alpha\beta}$  on  $\Sigma'$ ) is given by

$$\langle h'_{\alpha\beta}, \Sigma' | h_{\alpha\beta}, \Sigma \rangle = \sum_{\mathcal{M}} \int \mathcal{D}g e^{iS_g}. \quad (10.18)$$

Here  $\mathcal{M}$  denotes the space-time manifold,  $S_g$  denotes the Einstein-Hilbert action (2.11) and the integration over  $\mathcal{D}g$  includes an integration over the three-metric  $h_{\alpha\beta}$ , the lapse function  $N$  and the shift vector  $N^\alpha$ . In analogy with ordinary Quantum Field Theory (QFT), one performs the Wick rotation  $t \rightarrow -i\tau$  and takes into account the Euclidean action  $I_g = -iS_g$ , with the sum over metrics with signature  $(+ + + +)$ . This way one deals with the so-called Euclidean approach to quantum gravity, mainly due to Hawking and his group in Cambridge.

However, also this elegant approach to the quantum gravity problem suffers significant drawbacks as, in particular

- the gravitational action is not positive definite. Thus, differently from the Yang-Mills theory, the path integral does not converge if the sum is considered only on real metric with Euclidean signature. To deal with this feature a complex metric in the sum have to be added, though a unique prescription does not exist.
- The measure  $\mathcal{D}g$  in equation (10.18) is ill-defined and up to now there is no rigorous definition.

Disregarding these shortcomings, the wave function of the Universe  $\Psi$  (on the surface  $\Sigma$  with three-metric  $h_{\alpha\beta}$ ) can be defined by the functional path-integral as

$$\Psi(h_{\alpha\beta}, \Sigma) = \sum_{\mathcal{M}} \int \mathcal{D}g e^{-I_g}, \quad (10.19)$$

where the sum is over a class of four-metrics  $g_{ij}$  taking values  $h_{\alpha\beta}$  on the boundary  $\Sigma$ . The convergence of this integral is ensured by including

complex manifolds in the sum, i.e. by imposing boundary conditions which restricts the manifold where the integration is performed.

The Euclidean theory can then be considered as the quantum gravity sector where the “initial boundary conditions” on  $\Psi$  should be imposed. In particular, in order to evaluate the path integral (10.19), a saddle-point approximation has to be taken into account. By means of this approximation, the action  $I_g$  is described by the dominating classical solutions only. The Euclidean world is thus considered as the fundamental one, while the Lorentzian world is regarded as an emergent phenomenon where the saddle point is complex. This argument implies that Eq. (10.19) can be regarded as the “starting point” and boundary conditions have not been foisted (*no-boundary proposal*). In fact, one integrates over such metrics where the only boundary is given by that corresponding to the actual Universe. The question of boundary conditions is of fundamental importance in primordial cosmology and will be discussed in Sec. 10.7.

The Euclidean wave function (10.19) is consistent with the canonical framework. In fact, it is possible to show that it satisfies the WDW equation (10.16) provided that the action, the measure and the class of paths summed over are invariant under four-diffeomorphisms. This way a connection between the covariant and the canonical approaches to quantum gravity is established, although making these formal arguments rigorously defined is far from being trivial.

As a last point, a boundary term in the Einstein-Hilbert action (2.11) has to be taken into account. In fact, the Ricci scalar contains terms which are linear in the second derivatives of the metric. The path integral approach requires an action which depends on the first derivatives only, which can be accomplished by removing the second derivatives by means of integration by parts. More precisely, the Einstein-Hilbert action is consistent only when the underlying space-time manifold is closed, i.e. it is compact without boundary. In the event that the manifold has a boundary, the action should be supplemented by a boundary term so that the variational principle is well-defined. Such term, known as the Gibbons-Hawking-York boundary term, reads as

$$S_{\text{GHK}} = \mp \frac{1}{\kappa} \int_{\partial\mathcal{M}} d^3x \sqrt{h} K, \quad (10.20)$$

where  $\mp$  refer to a space-like or time-like boundary  $\partial\mathcal{M}$ , respectively, and  $K$  is the trace of the extrinsic curvature (2.66).

## 10.2 The Problem of Time

One of the major conceptual problems in quantum gravity is: *what is time?* Indeed, as we have previously seen, in the canonical formulation of quantum gravity the Schrödinger equation is replaced by the WDW one. An external time coordinate does not explicitly appear in the formalism. This feature distinguishes the quantization of a diffeomorphism-invariant field theory (like non-perturbative canonical quantum gravity) from an ordinary quantum field theory (regarded as a quantum theory over a fixed background metric structure). In fact, GR is a fully parametrized theory, recognized in the canonical framework via the appearance of infinitely many constraints as standing in the structure of the Hamiltonian (see Sec. 2.3).

The task of appropriately define the notion of time at a fundamental level is deeply connected with the role assigned to temporal concepts in all theories of physics different from GR. For example, in Newtonian physics, as well as in non-relativistic quantum mechanics, time is an external parameter to the system itself and is treated as a background degree of freedom. In ordinary QFT the situation is conceptually the same since a Minkowski background is fixed and the Newtonian time is replaced by the time measured in a set of relativistic inertial frames. On the other hand, the key aspect of GR is the diffeomorphisms invariance of the physical laws. *There is no background space-time metric (flat or curved) over which phenomena happen.* The space-time metric itself is a dynamical entity (it is the gravitational field) and a space-time location can thus be only relational. In the non-general-relativistic physics, the location is given in terms of reference-system objects which, defining a fixed background, are decoupled from the field under consideration. In GR, objects (fields) are localized only with respect to each other and points of the space-time are not a priori distinguishable.

The notion of “time” (or a fixed background metric) plays a crucial role in the formulation of the quantum theory. In fact, the conventional Copenhagen interpretation of quantum mechanics, as well as the whole framework of QFT, breaks down as soon as the metric is no longer fixed. Concepts like probability and measurement are highly non-trivial in a timeless physics since, for example, the inner product in quantum mechanics is a quantity conserved in time. Moreover, the Wightman axioms of QFT (which lead to the notions of canonical commutation relations, microcausality, propagators, etc.) are based on a fixed causal structure which is no longer available in a diffeomorphism-invariant theory like as GR. Finally, as showed by Un-

ruh and Wald in 1989, a perfect clock (in the sense of a quantum observable  $T$  whose values monotonically grow with abstract time  $t$ ) is not compatible with the physical requirement of an energy positive spectrum. This behavior, absent in classical physics, can be understood as a peculiar feature of the quantum theory.

Keeping in mind all these considerations when addressing the problem of time in quantum gravity, there are essentially three ways to face it: introduce time *before* the quantization, *after* the quantization, or dealing with a *timeless* framework.

### 10.2.1 Time before quantization

This approach is essentially based on three steps:

- (i) identify a time coordinate as a functional of canonical variables,
- (ii) rewrite (solve) the classical scalar constraint in the form

$$P_A + h_A = 0, \quad (10.21)$$

- (iii) quantize the new expression leading to a Schrödinger-like equation

$$i \frac{\delta \Psi}{\delta q_A} = \hat{h}_A \Psi. \quad (10.22)$$

The evolution of the so-called “physical Hamiltonian”  $h_A$  is therefore described with respect to the “time”  $q_A$ , i.e. the variable canonically conjugate to  $P_A$ . This way a notion of time can be implemented before the quantization procedure, by two possible formulations.

In the first one a time variable is constructed from the phase space gravitational ones. The true degrees of freedom of GR (up to gauge fixing) are recovered by a canonical transformation, the constraints are classically solved and then they are quantized leading to Eq. (10.22). This is the so-called *internal time approach* and it is nothing but the implementation at quantum level of the ADM reduction of the dynamics described in Sec. 2.3. This scheme is also known in its general form as the multi-time approach since one deals with an infinite set of Schrödinger equations, one for each space point.

The second possibility relies on adding matter fields to the gravitational dynamics and then regarding the evolution with respect to these *matter clocks*. The seminal work in this direction was made by Brown and Kuchař in 1995 in which an incoherent dust, i.e. a dust with the gravitational interaction only, is included in the dynamics. This procedure leads to the

new constraints  $\tilde{\mathcal{H}}(x^i)$  and  $\tilde{\mathcal{H}}_\alpha(x^i)$  in which the dust plays the role of time and the true Hamiltonian does not depend on the dust variables.

Let us introduce the variables  $(T, Z^\alpha)$  and the corresponding conjugate momenta  $(M, W_\alpha)$ , so that the values of  $Z^\alpha$  be the co-moving coordinates of the dust particles and  $T$  be the proper time along their worldlines. In this scheme the new constraints read as

$$\tilde{\mathcal{H}} = P(x^i) + h(x^i, h_{\alpha\beta}, \Pi^{\alpha\beta}) = 0 \quad (10.23a)$$

$$\tilde{\mathcal{H}}_\alpha = P_\alpha(x^i) + h_\alpha(x^i, T, z^\alpha, h_{\alpha\beta}, \Pi^{\alpha\beta}) = 0 \quad (10.23b)$$

where

$$h = -\sqrt{G(x^i)}, \quad G(x^i) = \mathcal{H}^2 - h^{\alpha\beta} \mathcal{H}_\alpha \mathcal{H}_\beta, \quad (10.24a)$$

$$h_\alpha = Z_\alpha^\beta \mathcal{H}_\beta + \sqrt{G(X)} \partial_\beta T Z_\alpha^\beta, \quad (10.24b)$$

where  $P$  is the projection of the rest mass current of the dust onto the four-velocity of the observers and  $P_\alpha = -PW_\alpha$ . This way the Hamiltonian  $h$  does not depend on the dust. As we can immediately recognize, the form of Eq. (10.23a) is exactly the desired one. Therefore, when such constraint is implemented at a quantum level, a Schrödinger equation for the wave functional  $\Psi = \Psi(\hat{T}, \hat{h})$  as

$$i \frac{\delta \Psi}{\delta T} = \hat{h} \Psi \quad (10.25)$$

is recovered. The central point of this procedure is the independence of the effective Hamiltonian  $h$  on the dust because this allows a well-posed spectral analysis formulation. In fact,  $h$  commutes with itself and furthermore the Schrödinger equation can be split into a dust-(time)-dependent part and a truly gravitational one.

It is worth stressing that the Brown-Kuchař mechanism relies on a duality between time evolution and matter fields (in particular a dust fluid). Let us analyze in some details this feature. As a starting point we suppose that the state functional  $\Psi$  is defined on the Wheeler superspace of the three-geometries  $\{h_{\alpha\beta}\}$  (i.e. it is annihilated by the super-momentum operator  $\hat{\mathcal{H}}_\alpha$ ) and that the theory evolves along the space-time slicing so that  $\Psi = \Psi(t, \{h_{\alpha\beta}\})$ . Thus, the quantum evolution of the gravitational field is governed by a smeared Schrödinger equation

$$i \partial_t \Psi = \hat{\mathbf{H}} \Psi \equiv \int_{\Sigma_t} d^3x \left( N \hat{\mathcal{H}} \right) \Psi. \quad (10.26)$$

As usual,  $\hat{\mathcal{H}}$  is the super-Hamiltonian operator,  $N = N(t)$  the lapse function and  $\Sigma_t$  the one-parameter family of Cauchy surfaces which fills the space-time. Due to the evolutionary character of the dynamics, this approach is

known as *evolutionary quantum gravity*. Considering the expansion of the wave functional as

$$\Psi = \int D\epsilon \psi(\epsilon, \{h_{ij}\}) \exp \left\{ -i(t - t_0) \int_{\Sigma_t} d^3x (N\epsilon) \right\}, \quad (10.27)$$

where  $D\epsilon$  is the Lebesgue measure in the space of the functions  $\epsilon(x^i)$ , an eigenvalue problem for the stationary wave function  $\psi$  appears. Explicitly, we have

$$\hat{\mathcal{H}}\chi = \epsilon\chi, \quad \hat{\mathcal{H}}_\alpha\chi = 0, \quad (10.28)$$

which outline the appearance of a non-zero super-Hamiltonian eigenvalue, differently from the WDW framework.

In order to address the meaning of the time-variable  $t$  in Eq. (10.26), we have to analyze the classical limit of the theory. This step is formulated by means of the WKB paradigm, i.e. the wave functional  $\Psi$  is replaced by its corresponding zero-order WKB approximation  $\Psi \sim e^{iS}$ . Thus the eigenvalues problem (10.28) reduces to the following classical counterpart

$$\widehat{HJ}S = \epsilon = -2\sqrt{h}T_{00}, \quad \widehat{HJ}_\alpha S = 0 \quad (10.29)$$

where  $\widehat{HJ}$  and  $\widehat{HJ}_\alpha$  denote operators which, acting on the phase  $S$ , reproduce the super-Hamiltonian and super-momentum Hamilton-Jacobi equations, respectively (see Sec. 2.3). The classical limit of the adopted Schrödinger quantum dynamics is then characterized by the appearance of a new matter contribution (associated with the non-zero eigenvalue  $\epsilon$ ) whose energy density reads as

$$\rho \equiv T_{00} = -\frac{\epsilon(x^i)}{2\sqrt{h}}. \quad (10.30)$$

The quantity  $T_{00}$  refers to the 00-component of the induced matter energy-momentum tensor  $T_{ij}$ . Since the spectrum of the super-Hamiltonian has in general a negative component, we can infer that, when the gravitational field is in the ground state, such matter arising in the classical limit can have a positive energy density. The explicit form of Eq. (10.30) is that of a dust fluid co-moving with the slicing three-hypersurfaces, i.e. the field  $n^i$  coincides with the four-velocity normal to the Cauchy surfaces. In other words, we deal with an energy-momentum tensor  $T_{ij} = \rho n_i n_j$ . However, such a matter field with a negative energy density cannot be regarded as an ordinary one since it does not satisfy the strong energy condition (2.153).

A dualism between time evolution and matter fields can now be established analyzing the problem from the opposite perspective. We consider



a gravitational system in the presence of a macroscopic matter field, described by a perfect fluid having a generic equation of state (2.15). The energy-momentum tensor associated to this system reads as

$$T_{ij} = \gamma \rho u_i u_j - (\gamma - 1) \rho g_{ij}. \quad (10.31)$$

To fix the constraints when matter is included in the dynamics, let us make use of the relations

$$G_{ij} n^i n^j = -\kappa \frac{\mathcal{H}}{2\sqrt{h}} \quad (10.32a)$$

$$G_{ij} n^i \partial_\alpha y^j = \kappa \frac{\mathcal{H}_\alpha}{2\sqrt{h}}, \quad (10.32b)$$

where  $G_{ij}$  is the Einstein tensor (2.10) and  $\partial_\alpha y^i$  are the vectors tangent to the Cauchy surfaces, i.e.  $n_i \partial_\alpha y^i = 0$ . Equations (10.32), by (10.31) and identifying  $u_i$  with  $n_i$  (i.e. the physical space is filled by the fluid), rewrite as

$$\rho = -\frac{\mathcal{H}}{2\sqrt{h}}, \quad \mathcal{H}_\alpha = 0. \quad (10.33)$$

Furthermore, we get the equations

$$G_{ij} \partial_\alpha y^i \partial_\beta y^j \equiv G_{\alpha\beta} = \kappa(\gamma - 1) \rho h_{\alpha\beta}. \quad (10.34)$$

We now observe that the conservation law  $\nabla_j T_i^j = 0$  implies the following two conditions

$$\gamma \nabla_i (\rho u^i) = (\gamma - 1) u^i \partial_i \rho \quad (10.35a)$$

$$u^j \nabla_j u_i = \left(1 - \frac{1}{\gamma}\right) (\partial_i \ln \rho - u_i u^j \partial_j \ln \rho). \quad (10.35b)$$

With the space-time slicing, looking at the dynamics in the fluid frame (i.e.  $n^i = \delta_0^i$ ), by the relation  $n^i = (1/N, -N^\alpha/N)$ , the co-moving constraint implies the synchronous nature of the reference frame. Since a synchronous reference is also a geodesic one, the right-hand side of Eq. (10.35b) must identically vanish and, for a generic inhomogeneous case, this implies  $\gamma = 1$ . Hence, Eq. (10.35a) yields  $\rho = -\bar{\epsilon}(x^i)/2\sqrt{h}$  and substituted into (10.33), we get the same Hamiltonian constraints as above in Eq. (10.28), as soon as the function  $\bar{\epsilon}$  is turned into the eigenvalue  $\epsilon$ . In this respect, while  $\bar{\epsilon}$  is positive by definition, the corresponding eigenvalue can also be negative because of the structure of  $\mathcal{H}$ .

We can conclude that a dust fluid is a good choice to realize a clock in quantum gravity, because it induces a non-zero super-Hamiltonian eigenvalue into the dynamics. Moreover, for vanishing pressure ( $\gamma = 1$ ),

Eq. (10.34) reduces to the proper vacuum evolution equation for  $h_{\alpha\beta}$ , thus outlining a real dualism between time evolution and the presence of a dust fluid.

Both approaches described above (the multi-time and the dust-clock) give an evolutionary quantum dynamics for the gravitational field. Although they could seem to overlap each other, this is not the case. The latter framework is based on a full quantization of the system while the multi-time scheme relies on a quantization of some degrees of freedom only. In fact, the constraints are classically solved before implementing the quantization procedure, thus violating the geometrical nature of the gravitational field in favor of real physical degrees of freedom. This fundamental difference between the two approaches is evident, for example, in a cosmological context. When a minisuperspace model is quantized in the ADM formalism (see below), the scale factor of the Universe is usually chosen as an internal time coordinate. On the other hand, in a dust-like approach, the scale factor is treated on the same footing of the other variables (for example the anisotropies) and the evolution of the system is considered with respect to a privileged reference frame (i.e. the dust one).

Let us point out some problems that emerge when identifying a time before quantization. Firstly, it should be noted that the resulting Schrödinger-like Eq. (10.22) is in general inequivalent to the original WDW equation. In particular, the choice of the time variable is not unique and the conditions a variable has to satisfy in order to stand as a good time are not univocally defined. Moreover, it is well known that different choices of time lead to different (unitarily inequivalent) quantum descriptions and it is unclear how these predictions can be related to each other. In fact, in non-linear systems, most of the (classical) canonical transformations cannot be represented as unitary operators while maintaining the irreducibility of the canonical commutation relations. A second, more technical, question is due to the impossibility to obtain a global solution of the constraint in GR (this result was obtained by Torre in 1992). This feature forces to privilege the choice of a matter field as time coordinate. However, such a matter clock has to satisfy two basic requirements: (i) its Hamiltonian has to be linear in the momentum variables and (ii) it has to describe physical clocks, i.e. they should run forward. A natural (and simple) matter field which accomplishes these basic assumptions is represented by a massless scalar field. Its role as a clock time will be analyzed in Sec. 10.5 and implemented in different frameworks below.

### 10.2.2 Time after quantization

This approach is exactly the Dirac scheme to quantize a constrained system, i.e. the theory we have described in the previous Section (together with all its connected problems). The result is then the *frozen formalism* of the WDW equation in which it seems that no evolution takes place. To recover a time notion at this level (and what it implies) there are mainly two ways.

The first idea relies on the observed similarity between the WDW Eq. (10.16) and the Klein-Gordon one (2.22). In fact, the WDW equation can be seen as a Klein-Gordon equation with a varying mass. The Hilbert space can then be naturally obtained by constructing a Klein-Gordon-like inner product for quantum gravity. It is worth noting that, as usual, the corresponding probability can be negative. A priori this feature can be overcome by looking at the hyperbolic features of the WDW equation: it allows to characterize an internal time variation and, in some cases, to pursue a separation between positive and negative frequencies. There are however some crucial differences with respect to the scalar field theory. The mass-like term in Eq. (10.16), i.e.  $\sqrt{\hbar} {}^3R$ , can take both positive and negative values, while the standard potentials for the scalar field are only positive by definition. This feature makes it impossible to prove the positivity of the Klein-Gordon scalar product even when positive-frequency solutions are selected. Moreover, a suitable Killing vector field on  $\text{Riem}(\Sigma)$  (which permits the frequency decomposition) is not in general available. From this point of view, a general prescription able to define a Hilbert space for the WDW theory is far from being stated.

The second possibility to recover a time notion after the quantization is based on a *semiclassical interpretation*. In this approach, time is a meaningful concept only in some semiclassical sectors of the full WDW theory. The main idea is that time, and thus space-time, does not exist at fundamental level but emerges as an approximate feature only under some suitable conditions. In practice, one usually expands the wave functional  $\Psi$  in a WKB-like form from which a time variable is extracted. To leading order of this approximation, the WDW equation is replaced by a Schrödinger one and the system can be probabilistically described using the associated inner product. This paradigm appears to be very useful in quantum cosmology and will be described in detail in Sec. 10.6. Of course this approach suffers the limit of choosing by hand a preferred state and it is not fully defined how to describe the system once approaching the real Planck regime.

### 10.2.3 *Timeless physics*

The last approach to solve the problem of time in quantum gravity is based on the idea that there is no need of time at a fundamental level. The quantum theory of gravity can be constructed without a notion of time and such concept may arise only in some special situations, i.e. in specific approximations of the theory. The physical point of view behind this reasoning relies in taking seriously some lessons from GR:

- in (general) relativistic physics there is not an independent observable quantity which plays the role of parameter for the evolution;
- *any motion is the relative evolution between observables*. Such a situation is also true in the pre(general)-relativistic context.

In fact, what we measure in Newtonian physics are the relative changes of a system quantity (e.g. the elongation of a pendulum) and of a clock one (e.g. the motion of the second hand). It is then assumed (because it is more convenient) that there is a background quantity (time) with respect to which the former can be evolved.

It turns out that a timeless classical mechanics can be univocally formulated. This framework is based on observables and states which are meaningful also in a general relativistic scheme. Let us give only the main ideas of this approach without entering the details. Any classical system can be described by a triple  $(\mathcal{C}, \Gamma, f)$  as follows.

- $\mathcal{C}$  is the configuration space: the space spanned by the (partial) observables which we are interested in. The physical relevant object is the relational measurement of these observables (e.g. the elongation of the pendulum with respect to the second hand clock or vice versa).
- $\Gamma$  is the phase space: the space spanned by quantities which coordinate the relative motion (e.g. the initial conditions).
- $f : \mathcal{C} \times \Gamma \rightarrow V$  ( $V$  being a linear space) gives the evolution via the equation  $f = 0$ : the motion is a relation between partial observables in  $\mathcal{C}$  with suitable boundary conditions in  $\Gamma$ .

No usual notion of time has been used. Kinematics is here given by  $\mathcal{C}$ , while dynamics is contained in  $\Gamma$  and  $f$ . A non-relativistic system appears as a peculiar case of this framework when a partial observable plays a special role. Such a variable is called time  $t$  and the Hamiltonian  $\mathcal{H}$  takes the

particular form

$$\mathcal{H} = p_t + \mathcal{H}_0(q_i, p^i, t) = 0, \quad (10.36)$$

where  $p_t$  is the momentum conjugate to  $t$ . A timeless quantum mechanics (or a quantum version of a relativistic classical mechanics) can also be constructed although this framework is probably not yet complete. However, the ordinary quantum mechanics is, by construction, not complete as it can describe non-relativistic systems only.

Let us now analyze these issues in quantum gravity. The simplest way to deal with a timeless interpretation of the WDW theory relies on constructing the inner product as

$$\langle \Psi | \Phi \rangle = \int_{\text{Riem}(\Sigma)} \mathcal{D}h \Psi^\dagger(h) \Phi(h), \quad (10.37)$$

where  $\mathcal{D}h$  is a formal integration measure over the three-geometries. This choice seems to be the more natural one and the Hilbert space, on which the operators (10.10) are self-adjoint, is given by  $L^2(\text{Riem}(\Sigma), \mathcal{D}h)$ . However, these relations are completely ill-defined, purely formal and any probabilistic interpretation based on Eq. (10.37) is meaningless.

A more interesting framework to be analyzed is the so-called *evolution of constants of motion*. This approach is mainly due to Rovelli. We have seen that in a generally covariant system, like GR, observables of the theory (not to be confused with the partial observables mentioned above) are constants of motion.

Let us consider a toy quantum gravity model described by only one scalar constraint  $\mathcal{H}(q, p) = 0$  defined on a finite-dimensional phase space  $S$ . A (classical) physical observable  $\mathcal{O}$  is then a function Poisson-commuting with all the constraints, i.e. in this case  $\{\mathcal{O}, \mathcal{H}\} = 0$ .  $\mathcal{O}$  remains defined as a constant of motion (this is the frozen formalism of classical gravity). We introduce a function  $\mathcal{T} = \mathcal{T}(q, p)$  such that for any  $t \in \mathbb{R}$  the surface

$$S_t = \{(q, p) \in S | \mathcal{T}(q, p) = t\} \quad (10.38)$$

intersects any dynamical trajectories (generated by  $\mathcal{H}$ ) only once. In other words we require that

$$\{\mathcal{T}, \mathcal{H}\} \neq 0, \quad (10.39)$$

which means that  $\mathcal{T}$  is *not* an observable of the theory. The key idea is then to associate any function  $F$  on the phase space  $S$  a one-parameter family of observables  $F_t$  (i.e.  $\{F_t, \mathcal{H}\} = 0$ ) such that  $F_t = F$  on the subspace  $S_t$ . *Evolution is then described by the dependence of the observables  $F_t$  on the*

parameter  $t$ . We then deal with the classical analogous of the Heisenberg picture in the quantum mechanics. An explicit computation shows that the dynamics of these functions is given by the equation

$$\{\mathcal{T}, \mathcal{H}\} \frac{dF_t}{dt} = \{F, \mathcal{H}\}. \quad (10.40)$$

In the particular case as  $\{\mathcal{T}, \mathcal{H}\} = 1$  (when  $\mathcal{T}$  is called a perfect clock), the Hamiltonian  $\mathcal{H}$  decomposes as  $\mathcal{H} = p_{\mathcal{T}} + \mathcal{H}_0$ , i.e. like (10.36). Thus  $\{F_t, \mathcal{H}_0\} = \{F, \mathcal{H}_0\}$  and therefore we obtain the standard equation of motion

$$\frac{dF_t}{dt} = \{F_t, \mathcal{H}_0\} \quad (10.41)$$

for the one-parameter family of observables  $F_t$ . We have then formulated a solid implementation of the general framework discussed above.

The following step is then to quantize such a system. More precisely, the algebra generated by the classical functions  $F_t$  has to be represented in a suitable Hilbert space. Although this task is well posed, some problems still arise. In particular, an operator formulation of the observables  $F_t$  is far from being trivial and it is not clear if a single Hilbert space can account for all the possible choices of the internal time function  $\mathcal{T}$  or not.

### 10.3 What is Quantum Cosmology?

Quantum cosmology denotes the application of the quantum theory to the entire Universe. However, the following question arises: why implementing the quantum physics to the Universe as a whole? At first sight, quantum physics seems to be applicable and relevant only at microscopical scales. On the other hand, near enough to the Big Bang, the Universe should be treated like a quantum object as a whole. In fact, it is the only state which entangles system and environment, i.e. we can say that the system itself is coupled to its environment. Strictly speaking, the whole Universe is the only closed quantum system in Nature. Indeed, the so-called decoherence is a possible quantum mechanism able to lead to the manifestation of the Universe as a macroscopic classical object. Loosely speaking decoherence describes the process of entanglement of a system with its natural environment. A system assumes classical features through the unavoidable and irreversible interaction with the environment. From this perspective, as any macroscopic object, the Universe is at the same time of quantum nature and of classical appearance in most of its stages. However, there exist

regimes where the latter does not hold and the quantum nature is revealed. It is expected that the primordial cosmological regime (namely near the Big Bang) resembles such peculiar situation.

Quantum cosmology is not necessarily related with a quantum gravity. In fact, quantum gravity (intended as the quantum formulation of the gravitational field only) is the theory of one field among the many degrees of freedom of the entire Universe and, in this aspect, it is not different from the quantum theory of the electromagnetic field. However, since gravity is the dominant interaction at large scales, any realistic formulation of quantum cosmology should be based on a quantum theory of gravity. Quantum cosmology is therefore a natural arena to investigate quantum gravity as part of a more general context.

### 10.3.1 *Minisuperspace models*

Let us apply from an operative point of view the quantum framework to cosmological models. This way the fields are restricted to a finite dimensional subspace of the (infinite dimensional) Wheeler superspace. In fact, the cosmological models arise as soon as spatially homogeneous (or also isotropic) space-times are taken into account and, as we said, for each point  $x^\alpha \in \Sigma$  there is a finite number of degrees of freedom in superspace. All but a finite number of degrees of freedom are frozen out by imposing such symmetries and the resulting finite dimensional configuration space of the theory is known as *minisuperspace*. From this perspective, quantum cosmology is the minisuperspace quantization of a cosmological model. The diffeomorphism constraint  $\mathcal{H}_\alpha = 0$  is automatically satisfied and one deals with a purely constrained quantum mechanical system (no longer a field theory) described by a single WDW equation for all the spatial points.

However, it is not yet demonstrated that the truncation to minisuperspace can be regarded as a rigorous approximation of the full superspace. Strictly speaking, setting most of the field modes and their canonically conjugate momenta to zero violates the uncertainty principle, thus can be considered as an ad hoc procedure. On the other hand, classical cosmology is based on these symmetric models and their quantization should give answers to the fundamental questions like the fate of the classical singularity, the inflationary expansion and the chaotic behavior of the Universe toward the singularity. Moreover, as we have seen in Chap. 9, in the general context of inhomogeneous cosmology, the spatial derivatives of the Ricci scalar are negligible with respect to the temporal ones, toward the singularity (BKL

conjecture). A minisuperspace model can be relevant for the description of a generic Universe toward the classical singularity when restricted to each cosmological horizon. Such quantum cosmology can be regarded as a toy model which hopefully may capture some of the essence of the full quantum cosmology.

Let us now define the model. A generic  $n$ -dimensional homogeneous minisuperspace system involves the following assumptions:

- (i) the lapse function is taken to be space-independent, i.e.  $N = N(t)$ ;
- (ii) the shift vector is taken to be zero, i.e.  $N^\alpha = 0$ . The line element (2.64) then reads as

$$ds^2 = N^2(t)dt^2 - h_{\alpha\beta}(x, t)dx^\alpha dx^\beta. \quad (10.42)$$

- (iii) The three-metric  $h_{\alpha\beta}$  is described by a finite number  $n$  of homogeneous coordinates  $q^A(t)$ . This is the crucial assumption. In the Hamiltonian framework their conjugate momenta are given by  $p_B(t)$ , where  $A, B = 1, \dots, n$ . This way we deal with an  $n$ -dimensional mechanical system.

We will discuss here only the vacuum case which can be straightforwardly generalized if matter fields are included into the dynamics, i.e.  $q^A$  should include matter variables also. Of course, the FRW and the Bianchi models are particular cases of such a framework. The action for this model is given by

$$S_g = \int dt (p_A \dot{q}^A - N\mathcal{H}) = \int dt [p_A \dot{q}^A - N (\mathcal{G}^{AB} p_A p_B + U(q))] , \quad (10.43)$$

where  $\mathcal{G}^{AB}$  is called *the minisupermetric*. The variation with respect to the lapse function leads to the scalar constraint

$$\mathcal{H}(q^A, p_A) = \mathcal{G}^{AB} p_A p_B + U(q) = 0, \quad (10.44)$$

and the equations of motion read as

$$\dot{q}^A = N\{q^A, \mathcal{H}\}, \quad \dot{p}_A = N\{p_A, \mathcal{H}\}. \quad (10.45)$$

The minisupermetric  $\mathcal{G}_{AB}$  is the reduced version of the supermetric  $\mathcal{G}_{\alpha\beta\gamma\delta}$ , where the indices  $A, B = \{\alpha\beta\}, \{\gamma\delta\}$  run over the independent components of the three-metric  $h_{\alpha\beta}$ . It has Lorentzian signature  $(+, -, -, -, -, -)$  and explicitly defined as

$$\mathcal{G}_{AB} dq^A dq^B = \int d^3x \mathcal{G}^{\alpha\beta\gamma\delta} \delta h_{\alpha\beta} \delta h_{\gamma\delta}. \quad (10.46)$$



In action (10.43)  $U(q)$  denotes the potential term given by

$$U = -\frac{1}{2\kappa} \int d^3x \sqrt{h} {}^3R. \quad (10.47)$$

*A minisuperspace model can be regarded as a relativistic particle moving in an  $n$ -dimensional curved space-time with metric  $\mathcal{G}_{AB}$  subjected to a potential  $U(q)$ . The Hamiltonian constraint (10.44) reflects the parametrization invariance of the theory. This symmetry is the residual of the four-dimensional diffeomorphisms invariance of the full theory.*

The canonical quantization à la Dirac of this model is straightforward (the path integral quantization of a minisuperspace model is given in Sec. 10.4). The WDW equation of such a system reads as

$$\hat{\mathcal{H}}\Psi = (-\nabla^2 + U)\Psi = 0, \quad (10.48)$$

where  $\Psi = \Psi(q)$  denotes the wave function of the Universe. Here  $\nabla_A$  is the covariant derivative constructed from the metric  $\mathcal{G}_{AB}$  and the Laplacian  $\nabla^2 = \nabla_A \nabla^A$  is given by

$$\nabla^2 = \frac{1}{\sqrt{\mathcal{G}}} \partial_A \left( \sqrt{\mathcal{G}} \mathcal{G}^{AB} \partial_B \right), \quad (10.49)$$

where  $\mathcal{G} \equiv |\det \mathcal{G}_{AB}|$ . The factor ordering in Eq. (10.48) has been fixed by Eq. (10.49) and this choice is peculiar because the WDW equation has the same form in any (minisuperspace) coordinate systems and it is invariant under the redefinitions of the three-metric fields  $q^A \rightarrow q'^A(q^A)$ .

### 10.3.2 Interpretation of the theory

Let us discuss the important feature of quantum cosmology regarding the interpretation of the wave function of the Universe  $\Psi$  for the extraction of physical properties and considering the differences with respect to ordinary quantum mechanics. Let us firstly list the assumptions at the basis of the standard interpretation of quantum mechanics, i.e. the Copenhagen School proposal.

- QM1 A clear distinction between the classical and the quantum world is assumed. In particular, there exists an external (classical) observer to the quantum system. The model under investigation is not genuinely closed.
- QM2 Predictions are probabilistic in nature and performed by measurements of an external observer. These measurements are performed on a large ensemble of identical systems or many times on the same system (in the same state).

QM3 Time plays a central and peculiar role (see Sec. 10.2).

On the other hand, quantum cosmology is defined up to the following assumptions.

- QC1 There is no longer an *a priori* splitting between classical and quantum worlds. The analyzed quantum model is the Universe as a whole, i.e. it is closed and isolated without external classical observers.
- QC2 No external measurement crutch is available, and an internal one cannot play the observer-like role due to the Planck conditions to which a very early Universe is subject. The Universe is unique by definition and it is not possible to perform many measurements on it arranged in the same state.
- QC3 The time coordinate is not an observable in GR and at a quantum level the problem of time appears.

The most accepted idea to face these features relies in accepting that a meaningful interpretation of the wave function of the Universe can be recovered at a semiclassical level only. A quantum-mechanical interpretation is possible only for a small subsystem of the entire Universe, i.e. in the domain where at least some of the minisuperspace variables can be treated as semiclassical in the sense of a WKB approximation. This framework will be analyzed in Sec. 10.6.

### 10.3.3 *Quantum singularity avoidance*

An expected natural result of any quantum cosmology should be to tame the classical cosmological singularities. In fact, as the classical fall of the electron on the nucleon is tamed by quantum effects, it is widely expected that a quantum Universe should be singularity-free. However, to examine the behavior of a classical singularity at a quantum level, a general criterion for determining whether the quantized model actually collapses or not has still to be fixed, constituting a non-trivial task due to the lack of a complete quantum theory of gravity.

As we have seen in Sec. 2.7, a space-time singularity in GR can be defined using two criteria:

- (i) the causal geodesic incompleteness (global criterion);
- (ii) the divergence of the scalars built up from the Riemann tensor (local criterion).

Although the second one is useful to characterize a singularity, it is unsatisfactory since a space-time can be singular without any pathological character of these scalars. Furthermore, not all singularities have large curvature and, most importantly, a diverging curvature is not the basic mechanism behind the singularity theorems.

At a quantum level, the task of defining a quantum singularity is more challenging. In fact, the classical (smooth) space-time can only be approached as a “low-energy” limit of the quantum theory “far enough” from the singularities. In this sense, criterion (i) cannot be a valid measure for a singularity in quantum gravity, since the space-time itself cannot be clearly defined at this level, i.e. differential geometry is expected to hold only to the classical approximation of the full quantum theory of gravity. In quantum cosmology, the original idea (proposed by DeWitt) to deal with a singularity-free Universe is to impose that the wave function of the Universe vanishes in correspondence to the singularity. For example, in the FRW case, where the Big Bang singularity appears for  $a = 0$ , this prescription is realized by demanding

$$\Psi(a = 0) = 0. \quad (10.50)$$

Unfortunately, this is a boundary condition that does not guarantee the quantum singularity avoidance since it does not bring any physical information on the Universe dynamics. It seems better to study the expectation values of the observables which classically vanish at the singularity.

Let us suppose to construct a Hilbert space for the theory and that  $|\Psi(q, t)|^2$  represents merely a probability density. In this way one might have an evolving state that “bounces” even if  $|\Psi(a = 0, t)| \neq 0$  for all  $t$ . A bouncing state clearly describes a nonsingular quantum Universe dynamics. On the other hand, if one were able to construct a wave packet with probability

$$P_\delta \equiv \int_0^\delta |\Psi(q, t)|^2 dq \simeq 0, \quad (10.51)$$

where  $\delta$  is a very small but finite quantity, then he could reasonably claim to have a no-collapse scenario.

Such an approach is in agreement with the so-called *principle of quantum hyperbolicity* recently formulated by Bojowald. This principle postulates that a quantum state which evolves in a unique and well-defined manner through a (classical) singular configuration can be considered as an evidence of the singularity avoidance. The persistence of a singularity at a quantum level is then manifest if the quantum dynamics brakes down

without extending the domain of applicability toward the classical singular regime. These arguments will be clearer below when specific models will be analyzed.

It is worth noting that not all the approaches to quantum cosmology lead to a singularity-free Universe. In particular, the WDW theory is not able to solve the cosmological singularity even in the simpler models. This task is successfully accomplished by loop quantum cosmology in which the Big Bang is replaced by a Big Bounce. Both the WDW and the LQC frameworks will be discussed in details below.

#### 10.4 Path Integral in the Minisuperspace

As we have seen in Sec. 10.3, the minisuperspace quantization corresponds to freeze out all dynamical degrees of freedom but a finite number, as allowed by the symmetries of the model. The invariance of the full theory under four-dimensional diffeomorphisms is then translated into an invariance under some specific reparametrizations. At the level of canonical theory, the diffeomorphism invariance of GR is guaranteed by the appearance of four constraints. The three super-momentum constraints (2.72b) are linear in the momenta and generate diffeomorphisms within the Cauchy hypersurfaces. This symmetry resembles one of the ordinary gauge theories. On the other hand, the scalar constraint (2.72a) is quadratic in the momenta, a feature not present in the usual gauge theories. Such constraint expresses the invariance of the theory under time reparametrizations, but it also generates the dynamics. Therefore, in the Einstein theory symmetry and dynamics are unavoidably entangled, a feature usually paraphrased as the background independence of the theory.

The objective of this Section is to analyze the relation between the canonical (*à la* Dirac) and the covariant (*à la* Feynman) quantization methods in reparametrization-invariant theories described by the action (10.43), a task firstly accomplished by Halliwell in 1988.

Let us consider an arbitrary function  $\epsilon = \epsilon(t)$  and the transformations generated as

$$\delta_\epsilon q^A = \epsilon \{q^A, \mathcal{H}\} = \epsilon \frac{\partial \mathcal{H}}{\partial p_A} \quad (10.52a)$$

$$\delta_\epsilon p_A = \epsilon \{p_A, \mathcal{H}\} = -\epsilon \frac{\partial \mathcal{H}}{\partial q^A}$$

$$\delta_\epsilon N = \dot{\epsilon}, \quad (10.52b)$$

where  $\mathcal{H}$  is explicitly given in Eq. (10.44). Let the time interval be  $t \in [t_0, t_1]$ , thus the action (10.43) changes under the transformation (10.52) by the amount

$$\delta_\epsilon S_g = \int_{t_0}^{t_1} dt \left( \dot{q}^A \delta_\epsilon p_A + p_A \delta_\epsilon \dot{q}^A - \mathcal{H} \delta_\epsilon N - N \delta_\epsilon \mathcal{H} \right). \quad (10.53)$$

The last term is zero since  $\delta_\epsilon \mathcal{H} = \epsilon \{\mathcal{H}, \mathcal{H}\} = 0$ . Performing a partial integration of the second term and considering the transformations (10.52) we obtain

$$\delta_\epsilon S_g = \left[ \epsilon \left( p_A \frac{\partial \mathcal{H}}{\partial p_A} - \mathcal{H} \right) \right]_{t_0}^{t_1}. \quad (10.54)$$

The term in the round brackets gives

$$\mathcal{G}^{AB} p_A p_B - U(q) \neq 0. \quad (10.55)$$

The action (10.43) thus remains unchanged if and only if we impose

$$\epsilon(t_0) = 0 = \epsilon(t_1), \quad (10.56)$$

that is the boundaries must not be transformed. In fact only in this case the term in Eq. (10.54) vanishes. The transformations (10.52), with the appropriate boundary conditions (10.56), are thus the reparametrizations under which the minisuperspace action is invariant. Note that this condition is not imposed in gauge theories where one deals with linear constraints of the form  $\alpha(q)p = 0$  (e.g. the Gauss constraint). In such a case the term in round brackets in Eq. (10.54) vanishes. The main difference between gravitation and gauge theories is that the constraint  $\mathcal{H} = 0$  is quadratic in the momenta.

In order to construct the path integral, one has to note the nature of the constrained system. In particular, a gauge fixing ensures that equivalent histories are counted only once. The symmetry of the theory is broken by the gauge fixing condition

$$G \equiv \dot{N} - f(p_A, q^A, N) = 0, \quad (10.57)$$

where  $f$  is an arbitrary function. This kind of gauge is often called “non-canonical” since it depends on  $N$  (otherwise is called “canonical”), and is the analogous of the Lorentz gauge  $\partial_i A^i = 0$  used in electrodynamics. In fact,  $A^0$  behaves as a Lagrangian multiplier in gauge theories (see Sec. 2.2.4), i.e. plays the role of  $N$ . On the other hand, the Coulomb gauge  $\partial_\alpha A^\alpha = 0$  is a familiar example of a “canonical” gauge.

We are now able to write down the path integral for the model. Let us consider the paths  $\{q^A(t), p_A(t), N(t)\}$  such that  $p_A$ ’s and  $N$  are free at

the end points  $t_0$  and  $t_1$ . On the other hand, the  $q^A$ 's satisfy the boundary conditions  $q^A(t_1) = \tilde{q}^A$ . The wave function of the Universe thus reads as

$$\Psi(\tilde{q}^A) = \int \mathcal{D}p_A \mathcal{D}q^A \mathcal{D}N \delta(G) \Delta_G e^{i S_g(q,p,N)}, \quad (10.58)$$

where  $\mathcal{D}$  denotes the usual functional integration measure and  $\Delta_G$  is the Faddeev-Popov determinant associated to the gauge-fixing condition (10.57). In particular, the latter ensures the path integral to be independent of the gauge-fixing functional  $G$ . It turns out that it is more convenient to work in the gauge

$$\dot{N} = 0 \quad \Rightarrow \quad f = 0. \quad (10.59)$$

In fact, skipping technical details,  $\Delta_G$  is the determinant of the operator  $\delta_\epsilon G / \delta \epsilon$ . In the gauge (10.59) such operator becomes  $d^2/dt^2$ , whose determinant is a constant and the Faddeev-Popov measure  $\Delta_G$  is indeed a constant.

As a result, the functional integration over  $N$  reduces to an ordinary integration leading to the path integral

$$\Psi(\tilde{q}^A) = \int dN \int \mathcal{D}p_A \mathcal{D}q^A e^{i S_g(q,p,N)} = \int dN \psi(\tilde{q}^A, N). \quad (10.60)$$

This formula is exactly the time-integration of an ordinary quantum mechanical propagator in which the lapse function  $N$  plays the role of time coordinate. As in quantum mechanics, the function  $\psi(\tilde{q}^A, N)$  satisfies the Schrödinger equation with  $N$  as time coordinate, that is

$$i \frac{\partial \psi}{\partial N} = \hat{\mathcal{H}} \psi. \quad (10.61)$$

Let us now act on Eq. (10.60) with the operator  $\hat{\mathcal{H}}$  defined in Eq. (10.48) (this is the minisuperspace version of the scalar constraint operator (10.6a)). Then, using Eq. (10.61), we obtain

$$\hat{\mathcal{H}} \Psi(\tilde{q}^A) = \int dN i \frac{\partial \psi}{\partial N} = i \psi(\tilde{q}^A, N)|_{\text{b}}, \quad (10.62)$$

where  $\psi(\tilde{q}^A, N)|_{\text{b}}$  stands for  $\psi(\tilde{q}^A, N)$  evaluated at the end points (boundary) of the  $N$  integral. Obviously, the wave function of the Universe satisfies the WDW Eq. (10.48) if the condition

$$\psi(\tilde{q}^A, N)|_{\text{b}} = 0 \quad (10.63)$$

is satisfied. These end points (or equivalently the contour on which the wave function is integrated over) are chosen to ensure such a condition.

We have shown the existence of a precise relation between the canonical and the covariant quantization frameworks, for a minisuperspace model, providing a specific implementation of what discussed in Sec. 10.1.

As a last point, we mention that one usually performs a rotation to the Euclidean time. However, differently from the case of ordinary matter fields, the minisuperspace action (10.43) is not positive definite. Complex integration contours are then necessary to give a precise meaning to the path integral. This feature will be discussed in Sec. 10.7.

## 10.5 Scalar Field as Relational Time

In this Section we analyze in some details the role of a matter field as a time clock. In particular, we focus on a massless scalar field  $\phi$ , employed as a time-like variable for the quantum dynamics of the gravitational field. As we have seen, when the canonical quantization procedure is applied to GR, the usual Schrödinger equation is replaced by a WDW one in which the time coordinate is dropped out from the formalism. One possible solution to this problem can be provided by including in the dynamics a matter field and letting evolve the physical degrees of freedom of the gravitational system with respect to it. This way the dynamics is described from a relational point of view, i.e. the matter field behaves as a relational clock.

In quantum cosmology, the choice of a scalar field appears as the most natural one. In fact, near the classical singularity, a monotonic behavior of  $\phi$  as a function of the isotropic scale factor always appears. It is worth remembering (see Chap. 5) that the behavior of a massless scalar field well approximates the one of an inflaton field when its potential is negligible at high enough temperature.

Let us consider the case of the Bianchi IX model in the presence of such a field. By considering the ( $m = 0$ ) Lagrangian density (2.21) over a homogeneous space-time, it is immediate to show that the energy density of  $\phi = \phi(t)$  is given by

$$\rho_\phi = p_\phi^2/a^6, \quad (10.64)$$

where  $p_\phi = p_\phi(t)$  denotes the momentum canonically conjugate to  $\phi$ . Hereafter  $\phi$  and  $p_\phi$  have been rescaled, with respect to Eq. (2.21), of a factor  $\sqrt{32\pi^2}$ . The scalar constraint in the Misner variables ( $a \equiv e^\alpha, \beta_\pm$ ) has the

form (see Eq. (8.35))

$$\begin{aligned} \mathcal{H}_{\text{IX}} + \mathcal{H}_\phi = \frac{\kappa}{3(8\pi)^2} \left[ -\frac{p_a^2}{a} + \frac{1}{a^3} (p_+^2 + p_-^2) \right] \\ + \frac{4\pi^2}{\kappa} a U_{\text{IX}}(\beta_\pm) + \frac{p_\phi^2}{a^3} = 0, \end{aligned} \quad (10.65)$$

where  $U_{\text{IX}}(\beta_\pm)$  is the potential term given by the curvature scalar as in Eq. (8.37b). The phase space of this system is eight-dimensional with coordinates  $(a, p_a, \beta_\pm, p_\pm, \phi, p_\phi)$  where  $p_\phi$  is a constant of motion because of the absence of a potential term  $V(\phi)$ . Thus each classical trajectory can be specified with respect to  $\phi$ , i.e. the scalar field  $\phi$  can be regarded as an internal clock for the dynamics. This condition can be imposed requiring the time gauge to be

$$\dot{\phi} = N \frac{\partial \mathcal{H}_\phi}{\partial p_\phi} = 1, \quad (10.66)$$

namely fixing the lapse function as

$$N = a^3/2p_\phi. \quad (10.67)$$

By adopting such a gauge, we deal with an effective Hamiltonian  $\mathcal{H}_e$  in the  $\phi$  time that explicitly stands as

$$p_\phi = \mathcal{H}_e \equiv \sqrt{\frac{\kappa}{3(8\pi)^2}} \left[ a^2 p_a^2 - (p_+^2 + p_-^2) - \frac{3(4\pi)^4}{\kappa^2} a^4 U_{\text{IX}}(\beta_\pm) \right]^{\frac{1}{2}}. \quad (10.68)$$

When this model is canonically quantized, the associated WDW equation describes the wave function  $\Psi = \Psi(a, \beta_\pm, \phi)$  evolution with respect to  $\phi$ . More precisely, from Eq. (10.65) it follows that<sup>2</sup>

$$\begin{aligned} (\partial_\phi^2 + \Theta) \Psi = 0, \\ \Theta \equiv \hat{\mathcal{H}}_e^2 = \frac{\kappa}{3(8\pi)^2} \left[ -a^2 \partial_a^2 + \partial_+^2 + \partial_-^2 - \frac{3(4\pi)^4}{\kappa^2} a^4 U_{\text{IX}}(\beta_\pm) \right]. \end{aligned} \quad (10.69)$$

As usual the WDW equation can be thought of as a Klein-Gordon like equation where  $\phi$  plays the role of (relational) time and  $\Theta$  of the spatial Laplacian. In order to have an explicit Hilbert space, the natural frequencies decomposition of the solutions of Eq. (10.69) is performed and the positive frequency sector is considered. The wave function

$$\Psi(a, \beta_\pm, \phi) = e^{i\omega\phi} \psi(a, \beta_\pm) \quad (10.70)$$

corresponds to positive frequencies with respect to  $\phi$  and  $\omega^2$  denotes the spectrum of  $\Theta$ . The function in Eq. (10.70) satisfies the positive frequency

<sup>2</sup>Since the normal ordering doesn't affect what follows, we adopt the simplest one.



(square root) of the quantum constraint in Eq. (10.69) and we deal with a Schrödinger-like equation

$$-i\partial_\phi\Psi = \sqrt{\Theta}\Psi, \quad (10.71)$$

with a non-local Hamiltonian  $\sqrt{\Theta}$ .

We now analyze how the massless scalar field can be regarded as an appropriate time parameter for the gravitational dynamics. Let us consider the dynamics toward the cosmological singularity, i.e. in the purely quantum era described by

$$a \ll \sqrt{\kappa} \sim \mathcal{O}(l_P). \quad (10.72)$$

In this region the potential term in Eq. (10.68)  $a^4 U_{\text{IX}}(\beta_\pm)$  can be neglected. Notably it is possible to show that, by means of a WKB expansion, the quasi-classical limit of the Universe dynamics is reached before the potential term becomes relevant.

The classical equations of motion are obtained from Eq. (10.68) and are given by

$$\frac{da}{d\phi} = \sqrt{\frac{\kappa}{3(8\pi)^2}} \frac{a^2 p_a}{\sqrt{a^2 p_a^2 - p_\beta^2}} \quad (10.73a)$$

$$\frac{dp_a}{d\phi} = -\sqrt{\frac{\kappa}{3(8\pi)^2}} \frac{ap_a^2}{\sqrt{a^2 p_a^2 - p_\beta^2}} \quad (10.73b)$$

$$\frac{dp_+}{d\phi} = \frac{dp_-}{d\phi} = 0, \quad (10.73c)$$

where  $p_\beta^2 \equiv p_+^2 + p_-^2$ . A solution to the system (10.73) has the form

$$\begin{aligned} a(\phi) &= B \exp \left( \sqrt{\frac{\kappa}{3(8\pi)^2}} \frac{A\phi}{\sqrt{A^2 - p_\beta^2}} \right), \\ p_a(\phi) &= \frac{A}{B} \exp \left( -\sqrt{\frac{\kappa}{3(8\pi)^2}} \frac{A\phi}{\sqrt{A^2 - p_\beta^2}} \right), \end{aligned} \quad (10.74)$$

$A$  and  $B$  being integration constants and  $p_\beta^2 = \text{const.}$ , providing a monotonic dependence of the isotropic variable of the Universe  $a$  with respect to the scalar field  $\phi$ . The field  $\phi$  shows to be a satisfactory (relational) time for the gravitational dynamics, a feature which remains valid for isotropic models, i.e. when  $\beta_\pm = 0$ . This way a massless scalar field is largely used in quantum cosmology as matter clock and we will see below some applications of this framework.

## 10.6 Interpretation of the Wave Function of the Universe

In this Section we will discuss in details the *semiclassical approximation* of quantum cosmology. It deserves interest because it leads to a probabilistic interpretation of the theory, i.e. some of the problems previously addressed (see Sec. 10.3) can be solved. In particular, a meaningful wave function of the Universe is constructed, although probability and unitarity result approximate concepts only.

Let us analyze the definition of probability in minisuperspace outlining the differences with respect to ordinary quantum mechanics. As we have seen above, a probabilistic interpretation of quantum cosmology cannot be clearly formulated due to the nonexistence of external or internal classical (or at least semiclassical) observers and furthermore the probability density in minisuperspace is ill defined. In quantum mechanics, given a wave function  $\Psi(x, t)$  describing a system, the probability to find the system in a configuration-space element  $d\Omega_x$  at time  $t$  is given by

$$dP = |\Psi(x, t)|^2 d\Omega_x, \quad (10.75)$$

providing a positive semidefinite probability, i.e.  $dP \geq 0$ . On the other hand, the wave function of the Universe  $\Psi$  generically depends on the three-metric, on the possible matter fields and no dependence on time explicitly appears. In analogy to quantum mechanics, the associated probability in quantum cosmology can be defined as

$$dP = |\Psi(q)|^2 \sqrt{\mathcal{G}} d^n q, \quad (10.76)$$

an expression that is however not normalizable since its integral over the whole minisuperspace diverges. Such behavior can be considered as the analogue of the quantum mechanical feature

$$\int |\Psi(x, t)|^2 d\Omega_x dt = \infty. \quad (10.77)$$

In fact, in quantum cosmology time is included among the set of variables  $q_A$  and the element  $\sqrt{\mathcal{G}} d^n q$  corresponds exactly to  $d\Omega_x dt$ . This way, it remains unclear how the probability conservation can be recovered.

To avoid these undesirable features, an alternative definition of the Universe probability can be formulated in terms of conserved currents  $j^A$  such that

$$\nabla_A j^A = 0, \quad j^A = -\frac{i}{2} \mathcal{G}^{AB} (\Psi^\dagger \nabla_B \Psi - \Psi \nabla_B \Psi^\dagger). \quad (10.78)$$

This approach arises from the analogy between the WDW and the Klein-Gordon theories. In fact, the WDW equation (10.48) can be seen as a Klein-Gordon equation with a variable mass  $U(q)$ . The corresponding probability to find the Universe in a surface element  $d\Sigma_A$  is given by

$$dP = j^A d\Sigma_A \quad (10.79)$$

and the conservation of the current  $j^A$  ensures the conservation of probability. The main problem relies on the fact that it can still be negative, similarly to the problem of negative probabilities in the Klein-Gordon framework. A possible route, following again the analogy, can be a second quantization of the system (note that this case would actually correspond to a third quantization), but such approach leads to several difficulties and it will not be discussed here.

The semiclassical framework is introduced exactly to solve this puzzle. The underlying idea is that a correct definition of probability (positive semidefinite) in quantum cosmology can be formulated by distinguishing between semiclassical and quantum variables. In particular, the variables which satisfy the Hamilton-Jacobi equation are regarded as semiclassical. It is also assumed that the quantum variables do not affect the dynamics generated by the semiclassical ones. In this respect, the quantum variables describe a small subsystem of the Universe while the semiclassical variables play the role of an external observer for the purely quantum dynamics. Such approach allows one to match the assumptions underlying quantum cosmology (QC1, QC2, QC3) with the ones of ordinary quantum mechanics (QM1, QM2, QM3).

We will firstly discuss the general framework and then consider a specific implementation.

### 10.6.1 *The semiclassical approximation*

For pedagogical reasons we firstly analyze a purely semiclassical model, where all the configuration variables  $q_A$  are semiclassical and the wave function  $\Psi(q)$  is given by

$$\Psi = A(q) e^{iS(q)}. \quad (10.80)$$

This state admits a WKB expansion and to lowest order it leads to the Hamilton-Jacobi equation for  $S$

$$\mathcal{G}^{AB}(\nabla_A S)(\nabla_B S) + U = 0. \quad (10.81)$$

Considering the expansion to next order, one obtains the continuity equation for the amplitude  $A$  and it leads to the conserved current

$$j^C = |A|^2 \nabla^C S. \quad (10.82)$$

As usual, the classical action  $S(q)$  describes a congruence of classical trajectories and a probability distribution on the  $(n - 1)$ -dimensional equal-time surfaces can be defined. More precisely, considering that  $p_A = \nabla_A S$ , the vector tangent to the classical path is given by

$$\dot{q}^A = N \frac{\partial \mathcal{H}}{\partial p_A} = 2N \nabla^A S. \quad (10.83)$$

By requiring only single crossings between trajectories and such equal-time surfaces, formulated as

$$\dot{q}^A d\Sigma_A > 0, \quad (10.84)$$

the probability (10.79) results to be positive semidefinite. The wave function  $\Psi$  can eventually be rescaled so that the probability is normalized to unity. It is worth stressing that, as showed by Halliwell in 1987, a wave function of the form  $e^{iS}$  corresponds to a classical space-time which can be predicted when a wave function of the Universe is peaked on a classical configuration. A correlation of the form  $p_A = \partial S / \partial q^A$ , where  $S$  is a solution of the Hamilton-Jacobi Eq. (10.81), is exactly expected when considering a wave function as  $e^{iS}$ .

Let us now consider the case in which not all the minisuperspace variables are semiclassical. We assume that there are  $m$  quantum variables labeled by  $\rho_I$  ( $I = 1, \dots, m$ ) and  $n - m$  semiclassical variables  $q_A$  ( $A = 1, \dots, n - m$ ). We also demand that the effect of the quantum variables on the dynamics of the semiclassical ones can be neglected, similarly to considering negligible the effect of electrons on the dynamics of nuclei in the Born-Oppenheimer approximation. The semiclassical degrees of freedom are thus treated as the “heavy” nuclei and the quantum ones as the “light” electrons. This way, the WDW Eq. (10.48) can be decomposed in a semiclassical and in a quantum part. The semiclassical operator

$$\hat{\mathcal{H}}_0 = -\nabla_0^2 + U(q) \quad (10.85)$$

is obtained neglecting all the quantum variables  $\rho_I$  and the corresponding momenta  $\pi^I$ , corresponding to the part previously analyzed. The quantum operator is denoted by  $\hat{\mathcal{H}}_\rho$  and the smallness of the quantum subsystem can be formulated requiring that its Hamiltonian  $\mathcal{H}_\rho$  be of order  $\mathcal{O}(\epsilon^{-1})$ , where  $\epsilon$  is a small parameter proportional to  $\hbar$ . Since the action of the semiclassical

Hamiltonian operator  $\hat{\mathcal{H}}_0$  on the wave function  $\Psi$  is of order  $\mathcal{O}(\epsilon^{-2})$ , the idea that the quantum subsystem does not influence the semiclassical one can be formulated as

$$\frac{\hat{\mathcal{H}}_\rho \Psi}{\hat{\mathcal{H}}_0 \Psi} = \mathcal{O}(\epsilon). \quad (10.86)$$

Such requirement is physically reasonable since the semiclassical properties of a cosmological model as well as the smallness of a quantum subsystem are both expectably linked to the fact that the Universe is large enough.

The minisuperspace metric can consequently be expanded in terms of  $\epsilon$  as

$$\mathcal{G}_{AB}(q, \rho) = \mathcal{G}_{AB}^0(q) + \mathcal{O}(\epsilon), \quad (10.87)$$

and the Universe wave function  $\Psi(q, \rho)$  is assumed (notice that this is an *ansatz* for the solution) to be

$$\Psi = \Psi_0(q) \chi(q, \rho) = A(q) e^{iS(q)} \chi(q, \rho). \quad (10.88)$$

The wave function is WKB-like in the  $q$  coordinates, i.e. the amplitude  $A$  and the phase  $S$  depend on the semiclassical variables only. On the other hand, the additional function  $\chi$  describes the quantum subsystem and it depends on  $\rho$  and only parametrically on the  $q$  variables, in the sense of the Born-Oppenheimer approximation. The function  $\Psi_0$  satisfies the WKB equations analyzed above and the function  $\chi$  has to be a solution of

$$\left[ \nabla_0^2 + 2(\nabla_0(\ln A))\nabla_0 + 2i(\nabla_0 S)\nabla_0 - \hat{\mathcal{H}}_\rho \right] \chi = 0, \quad (10.89)$$

where the operator  $\nabla_0$  is built using the metric  $\mathcal{G}_{AB}^0(q)$  as before.

Such an equation describes the evolution of the quantum subsystem. It is worth noting that the first two terms are of higher order in  $\epsilon$  with respect to the third one and can be neglected, resulting in

$$2i(\nabla_0 S)\nabla_0 \chi = \hat{\mathcal{H}}_\rho \chi. \quad (10.90)$$

In order to obtain a purely Schrödinger equation for the wave function  $\chi$  we need to redefine a time variable and using the classical relation (10.83) we obtain

$$i \frac{\partial \chi}{\partial t} = N \hat{\mathcal{H}}_\rho \chi. \quad (10.91)$$

A time parameter thus arises only at a semiclassical level where the wave function is oscillatory, i.e. it is a consequence of the initial assumption on  $\Psi$  as in Eq. (10.88). From this perspective such an approach represents a possible implementation of the idea that time is an emerging feature on a

classical background. This way the problem of time is solved in the spirit of the ordinary quantum theory.

The last point to address is to express the probability distribution. From the peculiarity of this framework, two different probability currents can be obtained, one for the semiclassical set and the other for the quantum one. The (total) probability distribution can be written as

$$\sigma(q, \rho, t) = \sigma_0(q, t) \sigma_\chi(q, \rho, t), \quad (10.92)$$

where  $\sigma_0$  is the probability distribution for the semiclassical variables and

$$\sigma_\chi = |\chi(q, \rho, t)|^2 \quad (10.93)$$

denotes the probability distribution for the quantum variables on the classical trajectories  $q_A(t)$  where the wave function  $\chi$  can be normalized. Let us consider the surface element on equal-time surfaces  $d\Sigma = d\Sigma_0 d\Omega_\rho$ ,  $d\Sigma_0$  remaining defined from the metric  $\mathcal{G}^0(q)$ . The probability distribution  $\sigma_0$  is normalized as  $\int \sigma_0 d\Sigma_0 = 1$  and therefore  $\chi(q, \rho, t)$  can be normalized as

$$\int |\chi|^2 d\Omega_\rho = 1, \quad d\Omega_\rho = \sqrt{|\det \mathcal{G}_{IJ}|} d^m \rho. \quad (10.94)$$

The whole probability distribution in Eq. (10.92) is normalizable.

Summarizing, we have recovered the standard interpretation of the wave function for a small subsystem of the Universe (only) in agreement with the intrinsic approximate interpretation of the Universe wave function. In fact, in the interpretation of quantum mechanics, all realistic measuring devices have some quantum uncertainty. The bigger the apparatus, the smaller the quantum fluctuations. In this sense, we are able to give a meaningful interpretation of the wave function of the Universe only in a semiclassical domain where the conventional law of physics apply.

In conclusion, we recall the two assumptions underlying this model:

- (i) the analysis has been developed within the minisuperspace regime;
- (ii) the fundamental requirement of existence of a family of equal-time surfaces is taken as a general feature.

### 10.6.2 *An example: A quantum mechanism for the isotropization of the Universe*

We now discuss how the scenario above described can be implemented to find a mechanism able to isotropize a quantum Universe which is weakly anisotropic. The minisuperspace cosmological model we consider is the

quasi-isotropic Mixmaster Universe with a cosmological constant, a quite general system exactly solvable for which an isotropization mechanism naturally arises.

Such dynamics is summarized by the scalar constraint (see Sec. 8.2)

$$\tilde{\kappa} \left[ -\frac{p_a^2}{a} + \frac{1}{a^3} (p_+^2 + p_-^2) \right] + \frac{a}{2\tilde{\kappa}} (\beta_+^2 + \beta_-^2) + U(a) = 0, \quad (10.95)$$

where  $\tilde{\kappa} = \kappa/3(8\pi)^2$  and the quadratic  $\beta$ -term is the first-order expansion given in Eq. (8.45). The isotropic potential  $U(a)$  explicitly reads

$$U(a) = \frac{1}{4\tilde{\kappa}} a \left( -\frac{1}{4} + \frac{\Lambda}{3} a^2 \right). \quad (10.96)$$

We remember that the Misner variables  $a = a(t)$  and  $\beta_{\pm} = \beta_{\pm}(t)$  describe the isotropic expansion and the shape changes (anisotropies) of the Universe, respectively. The phase space of this model is six-dimensional and the cosmological singularity appears for  $a \rightarrow 0$ . As usual  $\rho_{\Lambda} \equiv \Lambda/\kappa$  is the energy density associated with a cosmological constant<sup>3</sup> and, as discussed in Sec. 5.4.1, far enough from the singularity this term dominates the ordinary matter fields, a necessary condition for the emergence of the inflationary scenario.

In order to consider the semiclassical scheme, it is natural to regard the isotropic expansion variable  $a$  as the semiclassical one while considering the anisotropy coordinates  $\beta_{\pm}$  (the two physical degrees of freedom of the Universe) as the purely quantum variables. We are assuming *ab initio* that the radius of the Universe plays a different role with respect to the anisotropies. The wave function of the Universe  $\Psi = \Psi(a, \beta_{\pm})$  then reads as (see Eq. (10.88))

$$\Psi = \Psi_0 \chi = A(a) e^{iS(a)} \chi(a, \beta_{\pm}). \quad (10.97)$$

The Hamilton-Jacobi equation for  $S$  and the continuity equation for the amplitude  $A$  are respectively given by

$$-\tilde{\kappa} A (S')^2 + a U A + \mathcal{V}_q = 0 \quad (10.98)$$

$$\frac{1}{A} (A^2 S')' = 0, \quad (10.99)$$

where the prime denotes differentiation with respect to the scale factor  $a$  and  $\mathcal{V}_q = \tilde{\kappa} A''$  is the so-called quantum potential, which in this model is negligible far from the classical singularity even if the  $\hbar \rightarrow 0$  limit is not taken into account (see below). The evolutionary equation (10.90) for the

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<sup>3</sup>The cosmological constant  $\Lambda$  has dimension  $[\Lambda] = 1/\kappa$ .

quantum state  $\chi$  (i.e. neglecting higher order correction terms in  $\epsilon$ ) reads as

$$-2ia^2 S' \partial_a \chi = \hat{\mathcal{H}}_\rho \chi, \quad \mathcal{H}_\rho = p_+^2 + p_-^2 + \frac{a^4}{2\tilde{\kappa}^2} (\beta_+^2 + \beta_-^2). \quad (10.100)$$

The Schrödinger Eq. (10.91) for the wave function  $\chi$  is obtained by taking into account the vector tangent to the classical path. Using  $p_a = S'$ , the equations of motion (10.98) and considering the time gauge  $da/dt = 1$ , it is possible to define the new time variable  $\tau$  such that

$$d\tau = N \frac{\tilde{\kappa}}{a^3} da. \quad (10.101)$$

Far from the singularity (namely in the asymptotic interesting region  $a \gg 1/\sqrt{\Lambda}$ ) the evolution equation (10.100) rewrites as

$$i\partial_\tau \chi = \frac{1}{2} [-\Delta_\beta + \omega^2(\tau)(\beta_+^2 + \beta_-^2)] \chi, \quad (10.102)$$

where

$$\tau = \frac{\tilde{\kappa}}{12\sqrt{\Lambda}} \frac{1}{a^3} + \mathcal{O}\left(\left(\frac{1}{\Lambda a^2}\right)^{\frac{5}{2}}\right) \quad (10.103)$$

and  $\omega^2(\tau) = C/\tau^{4/3}$ ,  $C$  being a constant given by  $2C = 1/6^{4/3}(\tilde{\kappa}\Lambda)^{2/3}$ . The dynamics of the Universe anisotropies subsystem can then be regarded as a time-dependent bi-dimensional harmonic oscillator with frequency  $\omega(\tau)$ .

The construction of a quantum theory for a time-dependent, linear, dynamical system has remarkable differences with respect to the time-independent one. If the Hamiltonian fails to be time-independent, solutions which oscillate with purely positive frequency do not exist at all, i.e. the dynamics of the wave function is not carried out by a unitary time operator. In particular, in the absence of a time translation symmetry, no natural preferred choice of the Hilbert space is available. However, in the finite-dimensional case, the Stone-Von Neumann theorem holds (see Sec. 11.1). This way, the theory is unitarily equivalent to the standard (namely Schrödinger) one for any choice of the Hilbert space.

The quantum theory of the harmonic oscillator with time-dependent frequency is well known and the solution of the Schrödinger equation (10.102) can be analytically obtained. The analysis is mainly based on the use of the “exact invariants method” and on some time-dependent transformations. An exact invariant  $J(\tau)$  is a constant of motion, namely

$$\dot{J} = \frac{dJ}{d\tau} = \partial_\tau J - i[J, \hat{\mathcal{H}}_\rho] = 0, \quad (10.104)$$



and is Hermitian ( $J^\dagger = J$ ). For the Hamiltonian  $\mathcal{H}_\rho$  as in Eq. (10.102) it explicitly reads as

$$J_\pm = \frac{1}{2} \left( \xi^{-2} \beta_\pm^2 + (\xi p_\pm - \dot{\xi} \beta_\pm)^2 \right), \quad (10.105)$$

where  $\xi = \xi(\tau)$  is any function satisfying the auxiliary non-linear differential equation

$$\ddot{\xi} + \omega^2 \xi = \xi^{-3}. \quad (10.106)$$

The goal for the use of the invariants (10.105) relies on the fact that they match the wave function of a time-independent harmonic oscillator with the time-dependent one. Let  $\phi_n(\beta, \tau)$  be the eigenfunctions of  $J$  forming a complete orthonormal set corresponding to the time-independent eigenvalues  $k_n = n + 1/2$ . These states are related to the eigenfunctions  $\tilde{\phi}_n = \tilde{\phi}_n(\beta/\xi)$  of a time-independent harmonic oscillator via the unitary transformation

$$T = \exp(-i\dot{\xi}\beta^2/2\xi) \quad (10.107)$$

as  $\tilde{\phi}_n = \xi^{1/2} T \phi_n$ . The non-trivial (and in general non-available) step in this construction is an exact solution of the auxiliary equation (10.106). However, in our case it can be explicitly constructed as

$$\xi = \sqrt{\frac{\tau}{\sqrt{C}}} \left( 1 + \frac{\tau^{-2/3}}{9C} \right). \quad (10.108)$$

Finally, the solution to the Schrödinger equation (10.102) is connected to the  $J$ -eigenfunctions  $\phi_n$  by the relation

$$\chi_n(\beta, \tau) = e^{i\alpha_n(\tau)} \phi_n(\beta, \tau). \quad (10.109)$$

The general solution to (10.102) can thus be written as the linear combination  $\chi(\beta, \tau) = \sum_n c_n \chi_n(\beta, \tau)$ ,  $c_n$  being constants. Here, the time-dependent phase  $\alpha_n(\tau)$  is given by

$$\alpha_n = - \left( n + \frac{1}{2} \right) \int_0^\tau \frac{d\tau'}{\xi^2(\tau')}. \quad (10.110)$$

The wave function  $\chi$  is given by  $\chi_n = \chi_+ \chi_-$ , where

$$\chi_\pm = \chi_n(\beta_\pm, \tau) = C \frac{e^{i\alpha_n(\tau)}}{\sqrt{\xi}} H_n(\beta_\pm/\xi) \exp \left[ \frac{i}{2} \left( \dot{\xi} \xi^{-1} + i \xi^{-2} \right) \beta_\pm^2 \right], \quad (10.111)$$

in which  $H_n$  are the usual Hermite polynomials of order  $n$ . It is immediate to verify that, when  $\omega(\tau) \rightarrow \omega_0 = \text{const.}$  and  $\xi(\tau) \rightarrow \xi_0 = 1/\sqrt{\omega_0}$  (namely  $\alpha(\tau) \rightarrow -\omega_0(n + 1/2)\tau$ ), the wave function of a time-independent harmonic oscillator is recovered.

Let us investigate the probability density to find the quantum subsystem of the Universe in a given state. As a result, the anisotropies appear to be probabilistically suppressed as soon as the Universe expands enough far from the cosmological singularity (it appears for  $a \rightarrow 0$  or  $\tau \rightarrow \infty$ ). Such a feature can be realized from the behavior of the squared modulus of the wave function (10.111) which is given by

$$|\chi_n|^2 \propto \frac{1}{\xi^2} |H_{n_+}(\beta_+/\xi)|^2 |H_{n_-}(\beta_-/\xi)|^2 e^{-\beta^2/\xi^2}, \quad (10.112)$$

where  $\beta^2 = \beta_+^2 + \beta_-^2$ . Notice that such a probability density is still time-dependent through  $\xi = \xi(\tau)$  since the evolution of the wave function  $\chi$  is not traced by a unitary time operator. As we can see from (10.112), when a large enough cosmological region (namely as soon as  $a \rightarrow \infty$  or  $\tau \rightarrow 0$ ) is considered, the probability density to find the Universe is sharply peaked at the isotropic configuration, i.e. for  $|\beta_{\pm}| \simeq 0$ . In this limit (which corresponds also to  $\xi \rightarrow 0$ ) the probability density  $|\chi_{n=0}|^2$  of the ground state ( $n = n_+ + n_- = 0$ ) is given by

$$|\chi_{n=0}|^2 \xrightarrow{\tau \rightarrow 0} \delta(\beta). \quad (10.113)$$

The probability density is then proportional to the Dirac  $\delta$ -distribution centered on  $(\beta_+, \beta_-) = (0, 0)$  (see Fig. 10.1).

Summarizing, when the Universe moves away from the cosmological singularity, the probability density is asymptotically peaked around the closed FRW configuration. Near the initial singularity all values of anisotropies  $\beta_{\pm}$  are almost equally favored from a probabilistic point of view while, as the radius of the Universe grows, the isotropic state becomes the most probable one. This result relies on considering the isotropic scalar factor  $a$  as a semiclassical variable. Furthermore, we can write a positive semidefinite probability density and provide a clear interpretation of the model. The validity of such assumption can be verified from the analysis of the Hamilton-Jacobi equations (10.98). In particular, the WKB function  $\Psi_0 = \exp(iS + \ln A)$  approaches the quasi-classical limit  $e^{iS}$  as soon as the limit  $a \gg 1/\sqrt{\Lambda}$  is considered.

## 10.7 Boundary Conditions

Boundary, or initial, conditions are usually regarded in the context to arrange a physical system to perform an experiment. If the system we are analyzing is the entire Universe, boundary conditions can be arbitrarily

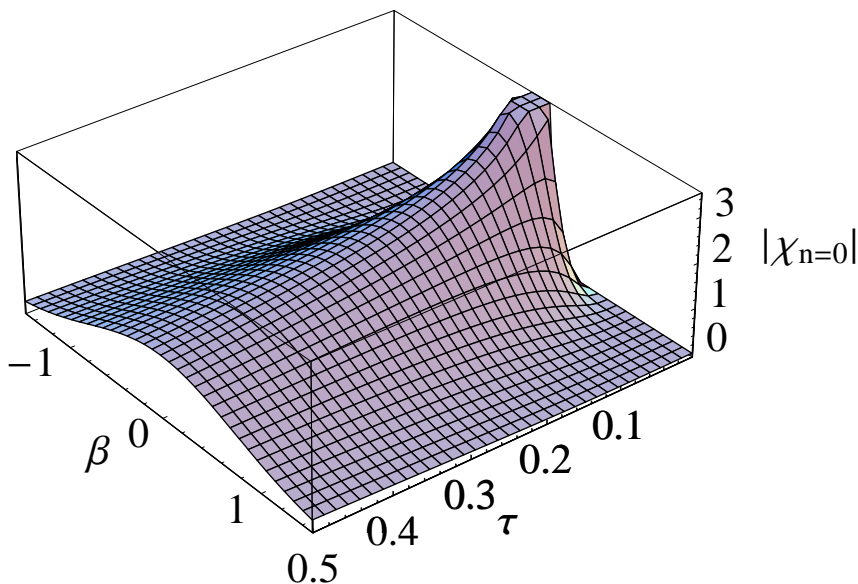


Figure 10.1 The wave function of the ground state  $\chi_{n=0}(\beta_{\pm}, \tau)$  far from the cosmological singularity, i.e. in the  $\tau \rightarrow 0$  limit. In the plot we take  $C = 1$ .

chosen. Thus, a proper choice (together with the removal of the cosmological singularity) can be considered as the main goal of any satisfactory quantum cosmology. Initial conditions are fundamental in cosmology since they determine the further evolution of the Universe as a whole. For example, whether the Universe has been subjected to an inflationary phase consistent with the observations is one of the questions that can be addressed. At a quantum level, in principle, initial (boundary) conditions have to select just one wave function of the Universe from the many allowed by the dynamics, in order to select a particular solution of the WDW equation. *A priori*, such state should contain all the information to describe our Universe. However, there is not a physical hint to choose appropriate boundary conditions and only motivations like mathematical consistency and simplicity may be invoked.

The two most studied boundary conditions are the *no-boundary* and the *tunneling* proposals, discussed here at a pedagogical level.

### 10.7.1 No-boundary proposal

The no-boundary proposal has been formulated by Hartle and Hawking in 1983 and it is essentially of a topological nature, based on the Euclidean path integral wave function of the Universe (10.19). This proposal consists of two parts: (i) the sum in Eq. (10.19) is restricted to include *only* compact Euclidean four-dimensional manifolds  $\mathcal{M}$  and (ii) the Cauchy surface  $\Sigma$ , on which  $\Psi$  is defined, forms the *only* boundary for these geometries. Therefore, no additional boundary conditions need to be imposed.

From an operative point of view, one usually works with the lowest-order WKB wave function (see Sec. 10.6)

$$\Psi = A e^{-I_g}, \quad (10.114)$$

where  $I_g = -iS_g$  is the classical action evaluated along the solution to the Euclidean field equations. The task is thus to find appropriate initial conditions which correspond to the no-boundary proposal at the classical level. This way one imposes that: (i) the four-geometry closes, (ii) the saddle points in the functional integral (10.19) correspond to regular metrics which are solution to the field equations.

In the simplest case the geometry is described by the Hartle-Hawking instanton, see Fig. 10.2. Half of the Euclidean four-sphere  $S^4$  (for small scale factor) is matched with a de Sitter space as the analytical continuation of  $S^4$ . The three-geometry matching of the two spaces has vanishing extrinsic curvature. Since  $S^4$  is compact, there is no boundary at the pole  $\tau = 0$  ( $\tau$  is the imaginary time obtained by the Wick rotation  $t \rightarrow -i\tau$ ). This is a non-singular four-geometry in which the Euclidean regime (imaginary time) describes the epoch where the scale factor is small while, as soon as the Universe expands enough, the dominating regime is the Lorentzian one (real time). The Lorentzian world can be regarded as an emergent phenomenon, i.e. there is a transition between the imaginary time and the standard one once the Lorentzian regime is approached from the Euclidean one.

As we mentioned in Sec. 10.4, the path integral to convergence requires a complex contour of integration in Eq. (10.19). The main problem is that, although convergent contours can be found, they are not univocally fixed. The wave function depends on which contour is considered and the no-boundary proposal is not able to give a unique physical prediction and therefore some extra information to determine the contour must be added.

Let us now show the results of this scheme when implemented in a simple minisuperspace model, describing a closed FRW Universe with a scalar field

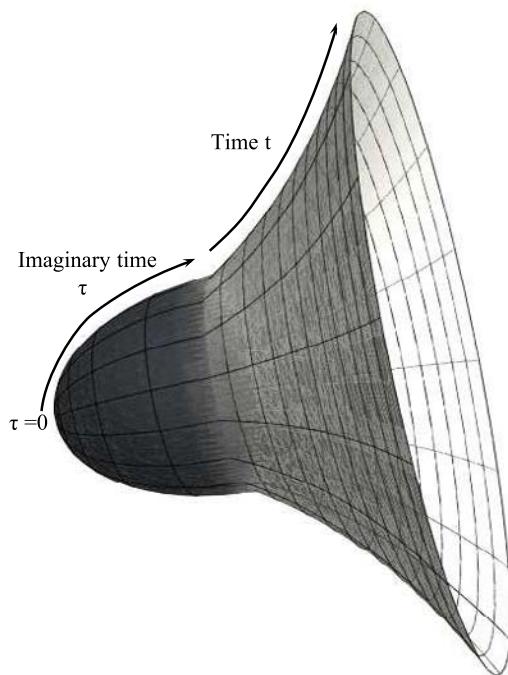


Figure 10.2 The Hartle-Hawking instanton

with potential  $V = V(\phi)$ . Such computation is performed considering the semiclassical (namely saddle-point) approximation of the functional integral and choosing a particular integration contour. Introducing the quantity

$$S = \frac{1}{3V} (a^2 V - 1)^{3/2} - \frac{\pi}{4}, \quad (10.115)$$

where  $a$ , as usual, denotes the scale factor, the no-boundary wave function is given by

$$\Psi_{\text{NB}} \sim \exp\left(+\frac{1}{3V}\right) (e^{iS} + e^{-iS}), \quad (10.116)$$

which is real being a sum of the WKB zeroth order wave function  $e^{iS}$  and its complex conjugate.

Before analyzing some of its cosmological implications, let us discuss the other boundary proposal, showing at the end a comparison between these approaches.

### 10.7.2 Tunneling proposal

The tunneling proposal, in its final version, has been formulated by Vilenkin in 1988 and predicts the existence (by a tunneling mechanism) of the Universe from *nothing*. In analogy with the Klein-Gordon theory, it is proposed to divide the solutions of the WDW equation in positive and negative frequency solutions. Then only the wave function consisting in outgoing modes has to be considered. More precisely, this proposal can be formulated as: (i) the wave function is *everywhere bounded* and (ii) it consists solely of *outgoing modes* at singular boundaries of superspace, except for the boundaries corresponding to vanishing three-geometries.

The problem now is to define the meaning of outgoing. In quantum mechanics, having the reference phase  $e^{-i\omega t}$ , an outgoing plane wave is  $e^{ikx}$ . In general, such a paradigm is of course vague due to the absence of Killing vectors in the superspace. In fact, it is not always possible to clearly define incoming and outgoing modes. On the other hand, in a WKB minisuperspace context, such a decomposition can be exactly formulated. As we have seen, a WKB oscillatory wave function  $\Psi \sim e^{iS}$  leads to a conserved current (see Eq. (10.82))

$$j \sim \nabla S. \quad (10.117)$$

The incoming and outgoing modes can be defined with respect to the direction of  $\nabla S = p$  on the considered surface. In particular, a mode is defined to be outgoing at the boundary if the quantity  $\nabla S$  points outward. Intuitively, the tunneling proposal reduces the possible ensemble of the Universes described by  $\Psi$ , retaining only those with positive momentum  $\nabla S$ , i.e. able to emerge from *nothing*.

In the minisuperspace model considered above (FRW closed Universe plus a massive scalar field) the tunneling wave function reads as

$$\Psi_T \sim \exp\left(-\frac{1}{3V}\right) e^{iS}, \quad (10.118)$$

where  $S$  is given by (10.115). Such solution is complex, different from the Hartle-Hawking one (10.116), consisting of just one WKB component.

### 10.7.3 Comparison between the two approaches

Let us discuss some physical implications of the above two proposals. The main difference between the tunneling wave function (10.118) and the no-boundary solution (10.116) is the sign in the non-oscillating exponential term

$$\exp\left(\pm\frac{1}{3V(\phi)}\right), \quad (10.119)$$

which has deep implications on the inflationary dynamics. Since we are interested in discussing such cosmological era, we will focus on the slow-roll approximation (see Sec. 5.4.1), i.e. we consider the case

$$\phi(t) \sim \phi_0 = \text{const.} \quad (10.120)$$

Both wave functions are strongly peaked around the classical solutions, i.e. those satisfying the first integral  $p = \nabla S$ . These solutions are of inflationary type

$$a(t) \sim e^{\sqrt{V}t}. \quad (10.121)$$

However, the strength of the inflationary phase depends on the value  $\phi_0$  which can, or cannot, lead to a sufficient inflation. The two frameworks favor different values of  $\phi_0$ , i.e. different inflationary scenarios.

Let us now discuss the two proposals. The component  $e^{iS}$  of the no-boundary wave function must be compared with the tunneling one, focusing our attention on the real exponential terms (10.119) only. The task is to show which wave function inflates for the correct amount. This can be realized by integrating the probability flux (10.79) on the surface separating the oscillating and the tunneling regions. From (10.115), this surface is defined by

$$a^2 V(\phi) = 1. \quad (10.122)$$

Because during the slow-rolling phase we have  $\phi \sim \text{const.}$ , the conserved current (10.82) points to the  $a$ -direction. A natural choice for the surface  $\Sigma$  is then  $a = \text{const.}$ , with a current  $j$  given by

$$j \sim |A|^2 = \exp\left(\pm\frac{2}{3V}\right), \quad (10.123)$$

and therefore the probability measure reads as

$$dP = j \cdot d\Sigma \sim \exp\left(\pm\frac{2}{3V}\right) d\phi. \quad (10.124)$$

The  $\pm$  signs refer to the no-boundary and the tunneling wave functions, respectively.

Let the range of the initial values of the scalar field  $\phi \sim \phi_0$  be  $\phi_0 \in [\phi_m, \phi_M]$ . Let then  $\phi_s \in [\phi_m, \phi_M]$  be the value for a sufficient inflation, i.e. the right amount appears if  $\phi_0 > \phi_s$ , while if  $\phi_0 < \phi_s$  the inflation will not be strong enough. Therefore, the probability to obtain a sufficient inflation is given by the probability to have the right behavior over all the possibilities, explicitly as

$$P(\phi_0 > \phi_s / \phi_0 \in [\phi_m, \phi_M]) = \frac{\int_{\phi_s}^{\phi_M} dP}{\int_{\phi_m}^{\phi_M} dP} = \frac{\int_{\phi_s}^{\phi_M} \exp(\pm \frac{2}{3V}) d\phi}{\int_{\phi_m}^{\phi_M} \exp(\pm \frac{2}{3V}) d\phi}. \quad (10.125)$$

Such a quantity shows that an inflation strong enough seems to be favored by the tunneling wave function  $(-)$ , which favors large values of  $\phi_0$  over the small ones. On the other hand the no-boundary wave function  $(+)$  seems to disfavor it. A sufficient inflation seems to be a prediction of the tunneling wave function (10.118) only. However, no definitive answer on this issue has been given yet.

## 10.8 Quantization of the FRW Model Filled with a Scalar Field

In this Section we discuss the canonical quantization of the FRW Universe with a scalar field in the WDW framework. The general prescription to quantize a minisuperspace model has been previously described and is now implemented to the isotropic models analyzed in Sec. 3.2. The FRW models are described by the line element (3.77) and have a two-dimensional phase space where the only non-vanishing Poisson brackets are

$$\{a, p_a\} = 1. \quad (10.126)$$

The dynamics is summarized by the scalar constraint (3.81) which we rewrite for convenience

$$\mathcal{H}_{\text{RW}} = -\frac{\kappa}{24\pi^2} \frac{p_a^2}{a} - \frac{6\pi^2 K}{\kappa} a + 2\pi^2 \rho a^3 = 0. \quad (10.127)$$

We are interested to analyze a flat model ( $K = 0$ ) filled with a massless scalar field  $\phi$  whose energy density reads as  $\rho_\phi = p_\phi^2/a^6$  (in our conventions  $\phi$  has dimension of an energy). This is an interesting model since it is exactly solvable and represents for quantum cosmology what the harmonic oscillator is for quantum field theory. The scalar constraint (10.127)



rewrites as

$$\mathcal{H}_{\text{RW}+\phi} = -\frac{\kappa}{24\pi^2} \frac{p_a^2}{a} + 2\pi^2 \frac{p_\phi^2}{a^3} = 0. \quad (10.128)$$

Before quantizing this model we have to define a time coordinate for the dynamics. As discussed in detail in Sec. 10.5, an appropriate choice is to take  $\phi$  as a relational time variable, i.e. the time gauge  $\dot{\phi} = 1$  by fixing the lapse function as

$$N = \frac{a^3}{4\pi^2 p_\phi}. \quad (10.129)$$

In this case the Friedmann Eq. (3.46) is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = B^2, \quad B = \sqrt{\frac{\kappa}{3(4\pi)^2}}, \quad (10.130)$$

and its solutions reads as

$$a(\phi) = a_0 e^{\pm B(\phi - \phi_0)}, \quad (10.131)$$

where  $a_0$  and  $\phi_0$  are integration constants. The plus sign describes an expanding Universe from the Big Bang, while the minus sign a contracting one into the Big Crunch. The classical cosmological singularity is reached at  $\phi = \pm\infty$  by every classical solution.

Fixing the lapse function as said, one deals with an effective Hamiltonian  $\mathcal{H}_e$  with respect to  $\phi$ , given by

$$p_\phi = \pm B |p_a a| \equiv \mathcal{H}_e. \quad (10.132)$$

The momentum  $p_\phi$  plays the same role as the constant energy in classical mechanics and the  $\pm$  signs select the direction of time. Such a Hamiltonian  $\mathcal{H}_e$  is known as the Berry-Keating-Connes Hamiltonian. Although the direct quantization of this Hamiltonian is not trivial due to the presence of the absolute value function, we focus only on the positive term  $p_a a$ . In fact, since  $\mathcal{H}_e$  is a conserved quantity, it is sufficient to analyze initial states which are superposition of positive eigenstates only.

The main quantum features of this model can be immediately obtained in the Heisenberg picture. The time evolution of any observable  $\mathcal{O}$  can be realized with respect to the Hamiltonian (10.132), i.e. the equation of motion for the expectation value

$$\frac{d}{d\phi} \langle \mathcal{O} \rangle = -i \langle [\mathcal{O}, \mathcal{H}_e] \rangle \quad (10.133)$$

holds. Equation (10.133) for the scale factor  $a$  and its conjugate momentum  $p_a$  read as

$$\frac{d}{d\phi}\langle a \rangle = B\langle a \rangle, \quad \frac{d}{d\phi}\langle p_a \rangle = -B\langle p_a \rangle. \quad (10.134)$$

These trajectories are in exact agreement with the classical ones. In order to discuss the fate of the cosmological singularity at quantum level, we have to analyze the evolution of a semiclassical initial state. In particular, we refer to the requirements that

- (i) its expectation value is close to the classical one
- (ii) the fluctuations  $(\Delta\mathcal{O})^2$  are small enough, that is

$$(\Delta\mathcal{O})^2 \ll \langle \mathcal{O} \rangle^2. \quad (10.135)$$

The scale factor fluctuations  $(\Delta a)^2 = \langle a^2 \rangle - \langle a \rangle^2$  obey the equation

$$\frac{d}{d\phi}(\Delta a)^2 = -i\langle [a^2, \mathcal{H}_e] \rangle - 2\langle a \rangle \frac{d}{d\phi}\langle a \rangle = 2B(\Delta a)^2 \quad (10.136)$$

and therefore they are neither constants nor bounded during the evolution. On the other hand, the relative fluctuation  $(\Delta a)^2 / \langle a \rangle^2$  is a conserved quantity which satisfies the equation of motion

$$\frac{d}{d\phi} \left( \frac{(\Delta a)^2}{\langle a \rangle^2} \right) = \frac{1}{\langle a \rangle^2} \left( \frac{d}{d\phi}(\Delta a)^2 - \frac{2}{\langle a \rangle}(\Delta a)^2 \frac{d}{d\phi}\langle a \rangle \right) = 0. \quad (10.137)$$

Such a behavior ensures that an initial semiclassical state remains semiclassical.

Let us consider the dynamics backward in time (toward the singularity) of an initial state sharply peaked on the expanding (plus sign) classical trajectory (10.131). By an initial state we refer to a state which is peaked at late times, i.e. at an energy much smaller than the Planck one. Roughly speaking it can be considered peaked around the observed classical Universe configuration. By means of the equations of motion (10.134) and (10.137), it remains sharply peaked along the whole classical trajectory until the *unavoidable* fall into the classical Big Bang singularity. This kind of dynamics undoubtedly indicates that *the classical singularity is not tamed by the WDW formalism*.

It is now instructive to analyze this model in the Schrödinger representation. Regarding the massless scalar field  $\phi$  as a relational time, the WDW associated to the constraint (10.128) takes the form

$$(\partial_\phi^2 + \Theta)\Psi = 0, \quad (10.138)$$

where the action of the operator  $\Theta = \hat{\mathcal{H}}_e^2$  reads as

$$\Theta\Psi \equiv -B a \partial_a^2 (a\Psi) \quad (10.139)$$

for the wave function of the Universe  $\Psi = \Psi(a, \phi)$ . The operator  $\Theta$  is self-adjoint even if  $p_a \rightarrow -i\partial_a$  is not. In fact, since the classical range of  $a$  is  $(0, \infty)$ , the natural choice for the Hilbert space in the quantum theory is  $L^2(\mathbb{R}_+, da)$  where the symmetric operator  $-i\partial_a$  has no self-adjoint extensions. This feature (which arises also in the ordinary quantum mechanics of a particle confined in the semi-axis) is exactly the minisuperspace reminiscence of the problem (ii) regarding the commutation relations (10.3). As in Sec. 10.5, considering the positive frequency modes

$$\Psi(a, \phi) = e^{i\omega\phi} \psi_\omega(a), \quad (10.140)$$

we obtain the eigenvalues problem

$$(\Theta - \omega^2)\psi_\omega = 0. \quad (10.141)$$

The solution of this equation is given by

$$\psi_\omega(a) = A_- a^{-(1-\gamma)/2} + A_+ a^{-(1+\gamma)/2}, \quad (10.142)$$

where  $\gamma^2 = 1 - 4\omega^2/\sqrt{B}$  and the spectrum  $\omega^2$  is purely continuous. As before,  $\Psi(a, \phi)$  satisfies the Schrödinger-like equation  $-i\partial_\phi\Psi = \sqrt{\Theta}\Psi$ . Such wave function is diverging at the singularity ( $a \rightarrow 0$ ) and the probability (10.51) is also diverging. It is now possible to construct a wave packet peaked at late times and analyze its dynamics toward the classical singularity. A wave packet is a superposition of the eigenfunctions

$$\Psi(a, \phi) = \int d\omega A(\omega) e^{i\omega\phi} \psi_\omega(a) \quad (10.143)$$

and, for example, we can take  $A(\omega)$  as a Gaussian weighting function centered in  $\omega_0 \gg l_P$ . As a result, these wave packets remain localized around the classical trajectory and fall into the cosmological singularity. The singularity non-avoidance is recovered in the Schrödinger framework too.

Let us conclude by stressing the fate of the cosmological singularity at a quantum level. As showed by Gotay and Demaret in 1983, the avoidance of the singularity in the WDW framework crucially depends on the choice of time. In general, a clock can be of two different kinds. It can be *slow* (its corresponding classical dynamics is incomplete) or *fast* (its corresponding classical dynamics is complete). More precisely, a time variable  $t$  is a fast time if the singularity occurs at either  $t = -\infty$  or  $t = +\infty$ . If this is not the case,  $t$  is called a slow time. In this terminology, the massless scalar field we

have used above is a fast time since the Big Bang singularity appears at  $\phi = \pm\infty$ . The conjecture is the following: *any quantum dynamics in a fast-time clock is always singular*. This is exactly what we obtained. On the other hand, as we will discuss in Sec. 12.2, the LQC framework clearly avoids a (fast-time) Big Bang singularity replacing it by a Big Bounce dynamics. Although such behavior seems to contradict this conjecture, the conflict is solved since in LQC we deal with a unitarily inequivalent representation, with respect to the Schrödinger one, of the canonical commutation relations.

## 10.9 The Poincaré Half Plane

This Section is devoted to introduce the Poincaré half plane. By means of this framework we will discuss the quantum dynamics of the Taub cosmological model in Sec. 10.10 as well as of the Mixmaster Universe (see Sec. 10.12). In particular, we here introduce a suitable form of the MCI variables (characterized by static potential walls) known as the Poincaré half plane representation. The choice of such a parametrization of the Lobačevskij plane allows to deal with a simple geometry which reduces the differences between the Bianchi I model and the Mixmaster type to a problem of boundary conditions.

The so-called Poincaré variables  $(u, v)$  are defined as

$$\xi = \frac{1 + u + u^2 + v^2}{\sqrt{3}v}, \quad (10.144a)$$

$$\theta = -\arctan\left(\frac{\sqrt{3}(1+2u)}{-1+2u+2u^2+2v^2}\right). \quad (10.144b)$$

In the vicinity of the initial singularity, we have seen that the potential term behaves as a potential well and as soon as we restrict the dynamics to  $\Pi_Q$ ,  $\mathcal{H}_{\text{ADM}} = \epsilon$  and we can rewrite Eq. (8.60) and Eq. (8.58) respectively as

$$\delta S_{\Pi_Q} = \delta \int d\tau (p_u \dot{u} + p_v \dot{v} - \mathcal{H}_{\text{ADM}}) = 0 \quad (10.145a)$$

$$\mathcal{H}_{\text{ADM}} = v \sqrt{p_u^2 + p_v^2}. \quad (10.145b)$$

The asymptotic dynamics is defined in a portion  $\Pi_Q$  of the Lobačevskij

plane, delimited by inequalities

$$Q_1(u, v) = -u/d \geq 0 \quad (10.146a)$$

$$Q_2(u, v) = (1 + u)/d \geq 0 \quad (10.146b)$$

$$Q_3(u, v) = (u(u + 1) + v^2)/d \geq 0 \quad (10.146c)$$

$$d = 1 + u + u^2 + v^2, \quad (10.146d)$$

whose boundaries are composed by geodesics of the plane, i.e. two vertical lines and one semicircle centered on the absolute  $v = 0$ . This region is sketched in Fig. 10.3.

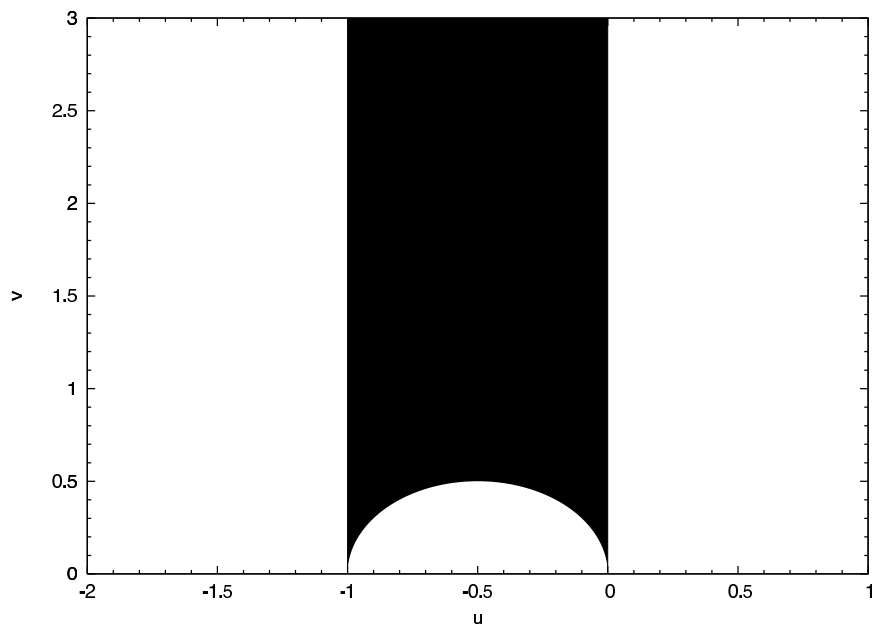


Figure 10.3  $\Pi_Q(u, v)$  is the available portion of the configuration space in the Poincaré upper half plane. It is bounded by three geodesics  $u = 0$ ,  $u = -1$ , and  $(u + 1/2)^2 + v^2 = 1/4$ , and has a finite measure  $\mu = \pi$ .

The billiard has a finite measure and its region open at infinity together with the two points on the absolute  $(0, 0)$  and  $(-1, 0)$  correspond to the three cusps of the potential in Fig. 8.3. It is easy to show that, in the  $(u, v)$  plane, the measure in Eq. (8.80) becomes

$$d\mu = \frac{1}{\pi} \frac{du dv}{v^2}. \quad (10.147)$$

## 10.10 Quantum Dynamics of the Taub Universe

In this Section we focus on the quantum features of the Taub Universe in the WDW framework. Such a cosmological model is a natural step toward the quantization of the more interesting case of Bianchi IX Universe. We will firstly analyze the classical model and then quantize it.

### 10.10.1 Classical framework

The Taub cosmological model is homogeneous and its symmetry group is  $SO(3)$ , i.e. the same as for Bianchi IX. However, this Universe is rotationally invariant about one axis of the three-dimensional space. The case of Taub is thus the natural intermediate step between FRW (which is invariant under rotations about any axis) and the Bianchi IX Universe, in which the rotational invariance is absent due to the presence of three intrinsically different scale factors.

The line element of the Taub space-time reads as

$$ds^2 = N^2(t) dt^2 - e^{2\alpha} (e^{2\beta})_{ab} \omega^a \omega^b, \quad (10.148)$$

where the left-invariant 1-forms  $\omega^a = \omega_\alpha^a dx^\alpha$  satisfy the Maurer-Cartan Eq. (7.28). The variable  $\alpha(t)$  describes the isotropic expansion of the Universe and  $\beta_{ab}(t)$  is the traceless symmetric matrix

$$\beta_{ab} = \text{diag}(\beta_+, \beta_+, -2\beta_+) \quad (10.149)$$

which determines the anisotropy via  $\beta_+$  only. The Taub model then corresponds to the particular case of Bianchi IX as soon as  $\beta_- \equiv 0$ . The determinant of the matrix  $\eta_{ab}$  corresponds to  $\eta = e^{6\alpha}$  and the classical singularity appears for  $\alpha \rightarrow -\infty$ .

Performing the usual Legendre transformation we obtain the Hamiltonian constraint for this model. The complete Hamiltonian framework can be recovered from the Bianchi IX one imposing  $\beta_- \equiv 0$ , and hence  $p_- \equiv 0$ . In particular, we are interested to the analysis of the Taub model in the Poincaré plane (see Sec. 10.9). The obtained dynamics is equivalent to the motion of a particle in a one-dimensional half-closed domain. As we can see from Eqs. (8.53) and (10.144), this particular case arises for

$$\theta = 0 \implies u = -\frac{1}{2}, \quad \xi = \frac{v^2 + 3/4}{\sqrt{3}v}. \quad (10.150)$$

The ADM Hamiltonian of the Taub Universe is obtained from Eq. (10.145b) and reads as

$$\mathcal{H}_{\text{ADM}}^T = vp_v, \quad (10.151)$$

where  $v \in [1/2, \infty)$ . The above Hamiltonian (10.151) can be further simplified defining a new variable

$$x = \ln v \quad (10.152)$$

and becomes

$$\mathcal{H}_{\text{ADM}}^T = p_x \equiv p. \quad (10.153)$$

Within this framework, the Taub model is described by a two-dimensional system in which the variable  $\tau$  is considered as time, while the variable  $x$  describes the single degree of freedom of the Universe, i.e. the change of shape. The classical singularity arises for  $\tau \rightarrow \infty$ . The configuration variable  $x$  is related to the Universe anisotropy  $\beta_+$  via the expression (8.53), for (10.150), as

$$\beta_+ = \frac{e^\tau}{\sqrt{3}v} \left( v^2 - \frac{3}{4} \right) = \frac{e^{\tau-x}}{\sqrt{3}} \left( e^{2x} - \frac{3}{4} \right). \quad (10.154)$$

By this equation, a monotonic relation between  $\beta_+$  and the configuration variable  $x \in [x_0, \infty)$ , where  $x_0 \equiv \ln(1/2)$ , appears, measuring the degree of anisotropy of the Universe. The isotropic shape of the Taub model (which corresponds to  $\beta_+ = 0$ ) comes out for the particular value  $x = \ln(\sqrt{3}/2)$ , leading to the closed FRW Universe.

Let us analyze the corresponding dynamics. The equations of motion follow from Eq. (10.153) and the system describes a free particle (the point-Universe) bouncing against the wall at  $x = x_0$ . The Taub model can be interpreted as a photon in the Lorentzian plane  $(\tau, x)$  and the classical trajectory is on its light-cone. The incoming particle ( $\tau < 0$ ) bounces on the wall ( $x = x_0$ ) and then falls into the classical cosmological singularity ( $\tau \rightarrow \infty$ ) (see Fig. 10.4).

The quantum dynamics of this cosmological model is discussed below in the WDW framework and, in a different quantization scheme, in Sec. 12.6.

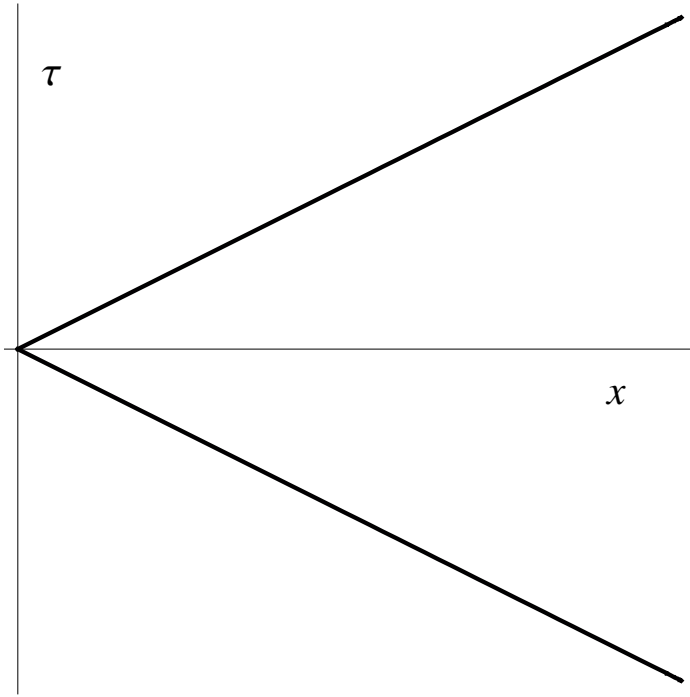
### 10.10.2 Quantum framework

The quantum dynamics of the Taub model is here analyzed in the context of the ADM reduction of the dynamics. For later purposes (see Sec. 12.6) we choose the wave function in the momentum representation. The variable  $\tau$  is then regarded as a time coordinate and a Schrödinger-like equation

$$i\partial_\tau \Psi(\tau, p) = \hat{\mathcal{H}}_{\text{ADM}}^T \Psi(\tau, p) \quad (10.155)$$

holds. Therefore we obtain the eigenvalue problem

$$k^2 \psi_k(p) = p^2 \psi_k(p), \quad (10.156)$$

Figure 10.4 Dynamics of the Taub Universe in the  $(\tau, x)$ -plane

where

$$\Psi(\tau, x) = \int_0^\infty dk A(k) \psi_k(x) e^{-ik\tau}. \quad (10.157)$$

We have to square the eigenvalue problem in order to correctly impose the boundary condition. In order to proceed forward, we assume that the functional form of the eigenfunctions be the same either with or without the square root. Correspondingly, we assume that the eigenvalues are the square of the original problem. The solution to Eq. (10.156) is the Dirac  $\delta$ -distribution

$$\psi_k(p) = \delta(p^2 - k^2). \quad (10.158)$$

The wave functions of the model in the coordinate space are given by

$$\begin{aligned} \psi_k(x) &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \frac{e^{ixp}}{k} (A\delta(p-k) + B\delta(p+k)) \\ &= \frac{1}{2k} [A e^{ixk} + B e^{-ixk}], \end{aligned} \quad (10.159)$$



where  $A$  and  $B$  are integration constants. This way, the boundary condition

$$\psi(x = x_0) = 0 \quad (10.160)$$

fixes one integration constant providing the eigenfunctions

$$\psi_k(x) = \frac{A}{2k} \left( e^{ixk} - e^{i(2x_0-x)k} \right). \quad (10.161)$$

Let us now investigate the fate of the classical singularity at a quantum level. In particular, we will construct and examine the motion of wave packets leading to a precise description of the evolution of the Taub model. The wave packets are superposition of the eigenfunctions (10.161) as in Eq. (10.157). Similarly to the FRW case, we can take  $A(k)$  as a Gaussian-like function

$$A(k) = k e^{-\frac{(k-k_0)^2}{2\sigma^2}} \quad (10.162)$$

peaked at energies much smaller than the Planck one. Let us note that  $k$  at the numerator in Eq. (10.162) simplifies the one in Eq. (10.161). The

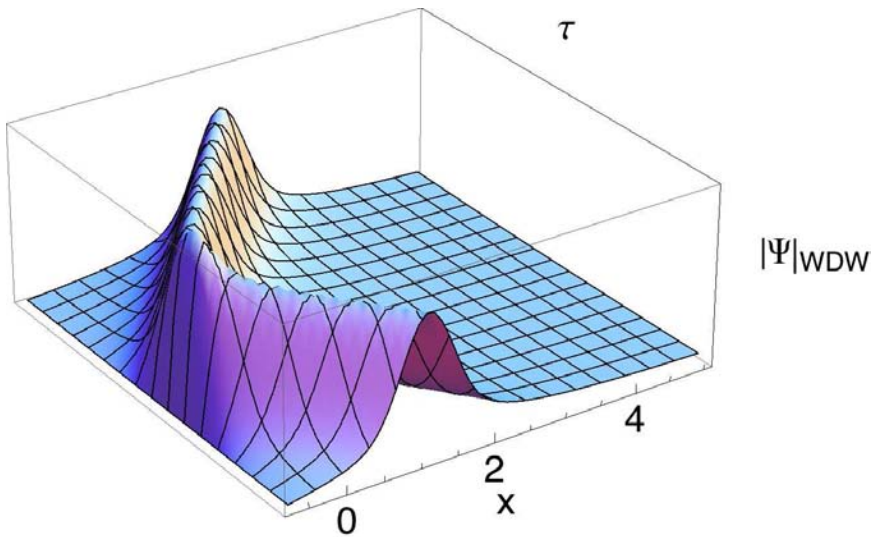


Figure 10.5 The evolution of the probability density of the wave packets  $|\Psi(\tau, x)|$  in the WDW case for the Taub model. The wave packets are peaked along the classical trajectories previously described. The  $x$  variable is in the  $[x_0 \equiv \ln(1/2), 5]$ -interval.

plot resulting from the superposition of the eigenfunctions in Eq. (10.161)

with the Gaussian-like weight function in Eq. (10.162), is given in Fig. 10.5. The wave packets are peaked along the classical trajectories analyzed above. The probability amplitude to find the particle (Universe) is peaked around the whole trajectory, thus no privileged region arises in the  $(\tau, x)$ -plane. As a matter of fact, the “incoming” Universe ( $\tau < 0$ ) bounces to the potential wall at  $x = x_0$  and then falls into the classical singularity ( $\tau \rightarrow \infty$ ).

Also in this case, the WDW formalism is not able to shed light on the necessary quantum resolution of the classical cosmological singularity. We will discuss in Sec. 12.6 how this picture changes in generalized quantization schemes.

### 10.11 Quantization of the Mixmaster in the Misner Picture

In this Section we provide a first insight into the quantum dynamics of the Bianchi IX cosmological model (for the classical description see Sec. 8.2). We discuss the approach relying on an adiabatic approximation ensured by the behavior of the potential term toward the cosmological singularity. In this scheme, which has to go back to the seminal work of Misner in 1969, the potential is modeled as an infinite square *box* with the same measure as in the original triangular picture. We will see that the wave function oscillates with a frequency increasing with the growth of the occupation number. A more complete analysis of the quantum dynamics of the Bianchi IX Universe will be the subject of Sec. 10.12.

As we have seen, by replacing the canonical variables with the corresponding operators and implementing the Hamiltonian constraint as a condition for the physical states, the system is described by the function  $\Psi = \Psi(\alpha, \beta_{\pm})$ . Here we adopt  $\alpha = \ln a$ , where  $a$  is the isotropic scale factor of the Universe, and the classical singularity appears as  $\alpha \rightarrow -\infty$ . Adopting the standard representation in the configuration space we address the WDW equation corresponding to Eq. (8.35) as

$$\hat{\mathcal{H}}_{\text{IX}} \Psi = -\frac{\kappa}{3(8\pi)^2} e^{-3\alpha} (-\partial_{\alpha}^2 + \partial_{+}^2 + \partial_{-}^2 - \mathcal{V}) \Psi = 0. \quad (10.163)$$

A solution to this equation can be searched in the form

$$\Psi(\alpha, \beta_{\pm}) = \sum_n \Gamma_n(\alpha) \psi_n(\alpha, \beta_{\pm}), \quad (10.164)$$

where the coefficients  $\Gamma_n$  are  $\alpha$ -dependent amplitudes. In particular, we require that the evolution in  $\alpha$  is mainly contained in these coefficients

and that the functions  $\psi_n(\alpha, \beta_{\pm})$  depend on  $\alpha$  parametrically only. This condition, known as the adiabatic approximation, reads as

$$|\partial_{\alpha}\Gamma_n| \gg |\partial_{\alpha}\psi_n|. \quad (10.165)$$

Equation (10.163) reduces to the ADM eigenvalue problem

$$(-\partial_+^2 - \partial_-^2 + \mathcal{V}) \psi_n = E_n^2(\alpha) \psi_n. \quad (10.166)$$

Note that the “energy” eigenvalues  $E_n$  are  $\alpha$ -dependent. We now approximate the triangular domain of the potential  $U(\beta_{\pm})$  by a two-dimensional rectangular box centered in  $\beta_{\pm} = 0$ . Such an approximation is valid only asymptotically to the singularity where  $U(\beta_{\pm})$  becomes an infinite potential well on a triangular basis. The two different domains are required to have the same area. The eigenvalues  $E_n$  (and also the eigenfunctions) are assumed to be the same as those in the square box, i.e.  $E_n^2 = (n\pi/L)^2$ . Here  $L^2$  denotes the area of the box which we demand to be equal to

$$L^2 = 3\sqrt{3}\beta_{\text{wall}}^2. \quad (10.167)$$

In fact this is the area of the triangular domain as  $\beta_{\text{wall}} = -\alpha/2$ . The eigenvalues  $E_n$  are thus given by

$$E_n(\alpha) = \left( \frac{4\pi^2}{3^{3/2}} \right)^{1/2} \frac{n}{\alpha} \equiv \frac{a_n}{\alpha}, \quad (10.168)$$

where  $n^2 = n_+^2 + n_-^2$  and  $n_{\pm} \in \mathbb{N}$  are the two independent quantum numbers corresponding to the variables  $\beta_{\pm}$ , respectively. Substituting the expression for the wave function (10.164) in Eq. (10.163) we get the differential equation for  $\Gamma_n$

$$\begin{aligned} \sum_n (\partial_{\alpha}^2 \Gamma_n) \psi_n + \sum_n \Gamma_n (\partial_{\alpha}^2 \psi_n) \\ + 2 \sum_n (\partial_{\alpha} \Gamma_n) (\partial_{\alpha} \psi_n) + \sum_n E_n^2 \Gamma_n \psi_n = 0, \end{aligned} \quad (10.169)$$

which, in the limit of the adiabatic approximation (10.165), simplifies to

$$\frac{d^2 \Gamma_n}{d\alpha^2} + \frac{a_n^2}{\alpha^2} \Gamma_n = 0, \quad a_n^2 = \frac{4\pi^2}{3^{3/2}} n^2. \quad (10.170)$$

The equation above is solved by trigonometric functions in the form

$$\Gamma_n(\alpha) = C_1 \sqrt{\alpha} \sin \left( \frac{1}{2} \sqrt{p_n} \ln \alpha \right) + C_2 \sqrt{\alpha} \cos \left( \frac{1}{2} \sqrt{p_n} \ln \alpha \right), \quad (10.171)$$

where  $p_n = a_n^2 - 1$ . From Eq. (10.171), the self-consistence of the adiabatic approximation is ensured. Figure 10.6 shows the behavior of  $\Gamma_n(\alpha)$  for

various values of the parameter  $a_n$ . Such wave function behaves like an oscillating profile whose frequency increases with occupation number  $n$  and approaching the cosmological singularity, while the amplitude depends on the  $\alpha$  variable only.

By this treatment, Misner himself obtained the interesting result that the occupation number  $n$ , on average, is constant toward the singularity. More precisely, taking an average over many runs and bounces, it is possible to get the relation

$$\langle H_{\text{ADM}} \alpha \rangle = \text{const.} \quad (10.172)$$

where  $H_{\text{ADM}}$  is the Hamiltonian with respect to the time variable  $\alpha$  (see Sec. 8.1). Replacing  $H_{\text{ADM}}$  with the energy eigenvalues (10.168) the result

$$\langle n \rangle = \text{const.} \quad (10.173)$$

is obtained, i.e.  $n$  can be regarded as an adiabatic invariant. Let us consider an initial semiclassical state (in the sense of  $n \gg 1$ ) and extrapolate its backwards evolution toward the cosmological singularity. The semiclassical character of this state is then preserved during the whole dynamics although the Universe reaches a full Planck regime. Such a behavior is in agreement to what obtained in the FRW case (Sec. 10.8) as well as in the Taub model (Sec. 10.10).

## 10.12 The Quantum Mixmaster in the Poincaré Half Plane

The Misner representation (see Sec. 10.11) provided a good insight in some qualitative aspects of the Mixmaster model quantum dynamics and allowed some physical considerations on the evolution toward the singularity. However, in this picture the potential walls move with time providing an obstacle toward a full implementation of a Schrödinger like quantization scheme. In this Section we perform a further description of the quantum properties associated to the Mixmaster dynamics when addressed in the canonical metric approach. Indeed we will consider the MCI variables (characterized by static potential walls) of the Poincaré half plane representation (introduced in Sec. 10.9). The choice of such a parametrization of the Lobačevskij plane allows one to deal with a simple geometry which reduces the differences between the Bianchi I model and the Mixmaster type to a problem of boundary conditions. Such an improvement of the quantization scheme permits to refine the Misner analysis outlining for instance the discreteness of the energy spectrum and the existence of a zero point energy.

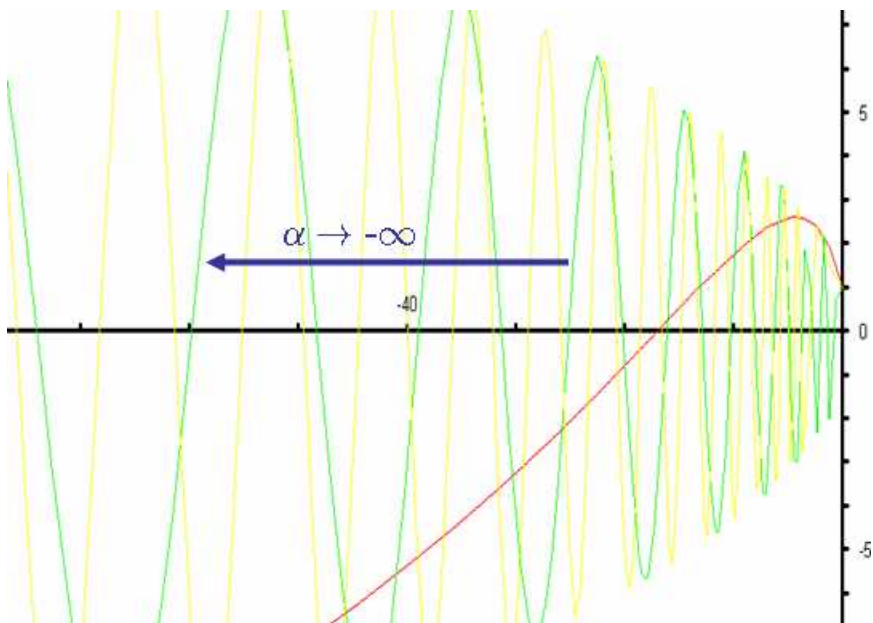


Figure 10.6 Behavior of the solution  $\Gamma_n(\alpha)$  for three different values of the parameter  $k_n = 1, 15, 30$ . The bigger  $k_n$ , the higher the frequency of oscillation is.

### 10.12.1 Continuity equation and the Liouville theorem

Since the Mixmaster provides an energy-like constant of motion toward the singularity, the point Universe randomizes within a closed domain and we can characterize the dynamics as a microcanonical ensemble, as discussed in Sec. 8.3.

The physical properties of a stationary ensemble are described by a distribution function  $\rho = \rho(u, v, p_u, p_v)$ , representing the probability of finding the system within an infinitesimal interval of the phase space  $(u, v, p_u, p_v)$ , and it obeys the continuity equation

$$\frac{\partial(\dot{u}\rho)}{\partial u} + \frac{\partial(\dot{v}\rho)}{\partial v} + \frac{\partial(\dot{p}_u\rho)}{\partial p_u} + \frac{\partial(\dot{p}_v\rho)}{\partial p_v} = 0, \quad (10.174)$$

where the dot denotes the time derivative and the Hamilton equations as-

sociated to Eq. (10.145b) read as

$$\dot{u} = \frac{v^2}{\epsilon} p_u, \quad \dot{p}_u = 0, \quad (10.175a)$$

$$\dot{v} = \frac{v^2}{\epsilon} p_v, \quad \dot{p}_v = -\frac{\epsilon}{v}. \quad (10.175b)$$

From Eq. (10.174) and Eq. (10.175) we obtain

$$\frac{v^2 p_u}{\epsilon} \frac{\partial \rho}{\partial u} + \frac{v^2 p_v}{\epsilon} \frac{\partial \rho}{\partial v} - \frac{\epsilon}{v} \frac{\partial \rho}{\partial p_v} = 0. \quad (10.176)$$

The continuity equation provides an appropriate representation sufficiently close to the initial singularity only, where the infinite potential wall approximation properly works. Such a model corresponds to deal with an energy-like constant of motion  $\epsilon$  and fixes the microcanonical nature of the ensemble. Since we are interested to the distribution function in the  $(u, v)$  space, we will reduce the dependence on the momenta by integrating  $\rho(u, v, p_u, p_v)$  in the momentum space. Assuming  $\rho$  to be a regular, vanishing at infinity in the phase-space, limited function, we can integrate over Eq. (10.176) getting the equation for  $\tilde{w} = \tilde{w}(u, v; k)$  as

$$\frac{\partial \tilde{w}}{\partial u} + \sqrt{\left(\frac{E}{Cv}\right)^2 - 1} \frac{\partial \tilde{w}}{\partial v} + \frac{E^2 - 2C^2 v^2}{Cv^2} \frac{\tilde{w}}{\sqrt{E^2 - (Cv)^2}} = 0, \quad (10.177)$$

where the constant  $C$  appears, due to the analytic expression of the HJ solution to (2.84a) for this model, i.e.

$$S(u, v) = Cu + \sqrt{E^2 - C^2 v^2} - E \ln \left( 2 \frac{E + \sqrt{E^2 - C^2 v^2}}{E^2 v} \right) + D, \quad (10.178)$$

where  $D$  is an integration constant and we have taken  $\epsilon \equiv E$ . In other words, we expressed the time derivative of  $u, v$  in terms of the momenta by the Hamilton equations (10.175) and, in turn, such momenta via the Hamilton function  $S(u, v)$ .

However, the distribution function cannot depend on the initial conditions that fix the constants  $C$  and  $E$ , and they must be ruled out from the final result. We obtain the following solution in terms of a generic function  $g$

$$\tilde{w}(u, v; C) = \frac{g\left(u + v \sqrt{\frac{E^2}{C^2 v^2} - 1}\right)}{v \sqrt{E^2 - C^2 v^2}}. \quad (10.179)$$

The distribution function cannot contain the constant  $C$  and the final result is obtained after the integration over it. We define the reduced distribution  $w(u, v)$  as

$$w(u, v) \equiv \int_A \tilde{w}(u, v; k) dk, \quad (10.180)$$

where the integration is taken over the classical available domain for  $p_u \equiv C$  expressed as  $A = [-E/v, E/v]$ . In Eq. (10.147) we demonstrated that the measure associated to it is the Liouville one; the measure  $w_{\text{mc}}$  (after integration over the admissible values of  $\phi$ ) corresponds to the case  $g = \text{const.}$

$$w_{\text{mc}}(u, v) = \int_{-\frac{E}{v}}^{\frac{E}{v}} \frac{1}{C v^2 \sqrt{\frac{E^2}{C^2 v^2} - 1}} dC = \frac{\pi}{v^2}. \quad (10.181)$$

Summarizing, we have derived the generic expression of the distribution function fixing its form for the microcanonical ensemble. This choice, in view of the energy-like constant of motion  $\mathcal{H}_{\text{ADM}}$ , is appropriate to describe the Mixmaster system restricted to the configuration space. This analysis reproduces in the Poincaré half plane the same result as the stationary invariant measure described in Sec. 8.4.

### 10.12.2 Schrödinger dynamics

The Schrödinger quantum picture is obtained in the standard way, i.e. by promoting the classical variables to operators and imposing Dirichlet boundary conditions onto the wave function as

$$\Psi(\partial\Pi_Q) = 0. \quad (10.182)$$

The quantum dynamics for the state function  $\Psi = \Psi(u, v, \tau)$  is governed by the Schrödinger equation

$$i \frac{\partial \Psi}{\partial \tau} = \hat{\mathcal{H}}_{\text{ADM}} \Psi = \sqrt{-v^2 \frac{\partial^2}{\partial u^2} - v^{2-a} \frac{\partial}{\partial v} \left( v^a \frac{\partial}{\partial v} \right)} \Psi. \quad (10.183)$$

We have to address two main problems: the operator-ordering for the position and momentum (here parametrized by the constant  $a$ ) and the non-locality of the Hamiltonian operator. Indeed, when solving the super-Hamiltonian constraint with respect to  $p_\tau$ , the ADM Hamiltonian contains a square root and consequently it might define a non-local dynamics.

The question of the correct operator-ordering is addressed in the next Section comparing the classic evolution versus the WKB limit of the

quantum-dynamics and requiring a proper matching. On the other hand, we will assume the operators  $\hat{\mathcal{H}}_{\text{ADM}}$  and  $\hat{\mathcal{H}}_{\text{ADM}}^2$  to have the same set of eigenfunctions with eigenvalues  $E$  and  $E^2$ , respectively. It is worth noting that in the domain  $\Pi_Q$ ,  $\mathcal{H}_{\text{ADM}}$  has a positive sign (the potential vanishes asymptotically). Under these assumptions, we will solve the eigenvalue problem for the squared ADM Hamiltonian given by

$$\hat{\mathcal{H}}_{\text{ADM}}^2 \Psi_E = \left[ -v^2 \frac{\partial^2}{\partial u^2} - v^{2-a} \frac{\partial}{\partial v} \left( v^a \frac{\partial}{\partial v} \right) \right] \Psi_E = E^2 \Psi_E, \quad (10.184)$$

where  $\Psi_E = \Psi_E(u, v, E)$ .

In order to study the WKB limit of Eq. (10.184), we separate the wave function into its phase and amplitude

$$\Psi_E(u, v, E) = \sqrt{r(u, v, E)} e^{i\sigma(u, v, E)}. \quad (10.185)$$

In Eq. (10.185) the function  $r(u, v, E)$  represents the probability density, and the quasi-classical regime appears in the limit  $\hbar \rightarrow 0$ ; substituting Eq. (10.185) into Eq. (10.184) and retaining only the lowest order terms in  $\hbar$ , we obtain the system

$$v^2 \left[ \left( \frac{\partial \sigma}{\partial u} \right)^2 + \left( \frac{\partial \sigma}{\partial v} \right)^2 \right] = E^2, \quad (10.186a)$$

$$\frac{\partial r}{\partial u} \frac{\partial \sigma}{\partial u} + \frac{\partial r}{\partial v} \frac{\partial \sigma}{\partial v} + r \left( \frac{a}{v} \frac{\partial \sigma}{\partial v} + \frac{\partial^2 \sigma}{\partial v^2} + \frac{\partial^2 \sigma}{\partial u^2} \right) = 0. \quad (10.186b)$$

In view of the HJ equation and of Hamiltonian (10.145b), we can identify the phase  $\sigma$  as the functional  $S$  defined in Eq. (10.178). Because of this identification, Eq. (10.186b) reduces to

$$C \frac{\partial r}{\partial u} + \sqrt{\left( \frac{E}{v} \right)^2 - C^2} \frac{\partial r}{\partial v} + \frac{a(E^2 - C^2 v^2) - E^2}{v^2 \sqrt{E^2 - C^2 v^2}} r = 0. \quad (10.187)$$

Comparing Eq. (10.187) with Eq. (10.177), we see that they coincide for  $a = 2$  only. This correspondence is expected for a suitable choice of the configurational variables; however, it is remarkable that it arises for the chosen operator-ordering only, allowing to fix a particular quantum dynamics for the system. Summarizing, we have demonstrated that it is possible to get a WKB correspondence between the quasi-classical regime and the ensemble dynamics in the configuration space, and we provided the operator-ordering to quantize the Mixmaster model

$$\hat{v}^2 \hat{p}_v^2 \rightarrow - \frac{\partial}{\partial v} \left( v^2 \frac{\partial}{\partial v} \right). \quad (10.188)$$



### 10.12.3 Eigenfunctions and the vacuum state

Once fixed the operator ordering by  $a = 2$ , the eigenvalue equation (10.184) rewrites as

$$\left[ v^2 \frac{\partial^2}{\partial u^2} + v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v} + E^2 \right] \Psi_E(u, v, E) = 0. \quad (10.189)$$

By redefining  $\Psi_E(u, v, E) = \psi(u, v, E)/v$ , we can reduce (10.189) to the eigenvalue problem for the Laplace-Beltrami operator in the Poincaré plane as

$$\nabla_{\text{LB}} \psi(u, v, E) \equiv v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \psi(u, v, E) = E_s \psi(u, v, E), \quad (10.190)$$

which is central in the harmonic analysis on symmetric spaces and has been widely investigated in terms of its invariance under  $SL(2, \mathbb{C})$ . Its eigenstates and eigenvalues are

$$\psi_s(u, v) = av^s + bv^{1-s} + \sqrt{v} \sum_{n \neq 0} a_n K_{s-1/2}(2\pi|n|v) e^{2\pi i n u} \quad (10.191)$$

$$\nabla_{\text{LB}} \psi_s(u, v) = s(s-1) \psi_s(u, v) \quad (10.192)$$

where  $a, b, a_n \in \mathbb{C}$ ,  $K_{s-1/2}(2\pi n v)$  are the modified Bessel functions of the third kind and  $s$  denotes the index of the eigenfunction. This is a continuous spectrum and the sum runs over every real value of  $n$ . The eigenfunctions for this model read as

$$\Psi_E(u, v, E) = av^{s-1} + bv^{-s} + \sum_{n \neq 0} a_n \frac{K_{s-1/2}(2\pi|n|v)}{\sqrt{v}} e^{2\pi i n u}, \quad (10.193)$$

with eigenvalues

$$E^2 = s(1-s). \quad (10.194)$$

To impose Dirichlet boundary conditions for the wave functions, we will require a vanishing behavior on the edges of the geodesic triangle of Fig. 10.3. Let us approximate the domain with the one in Fig. 10.7; the value of the horizontal line  $v = 1/\pi$  provides the same measure for the exact as well as for the approximate domain

$$\int_{\Pi_Q} \frac{dudv}{v^2} = \int_{\text{Approx domain}} \frac{dudv}{v^2} = \pi. \quad (10.195)$$

The difficulty to deal with the exact boundary conditions relies on the mixing of solutions with different indices  $s$  due to the semicircle that bounds the domain from below. The Laplace-Beltrami operator and the exact

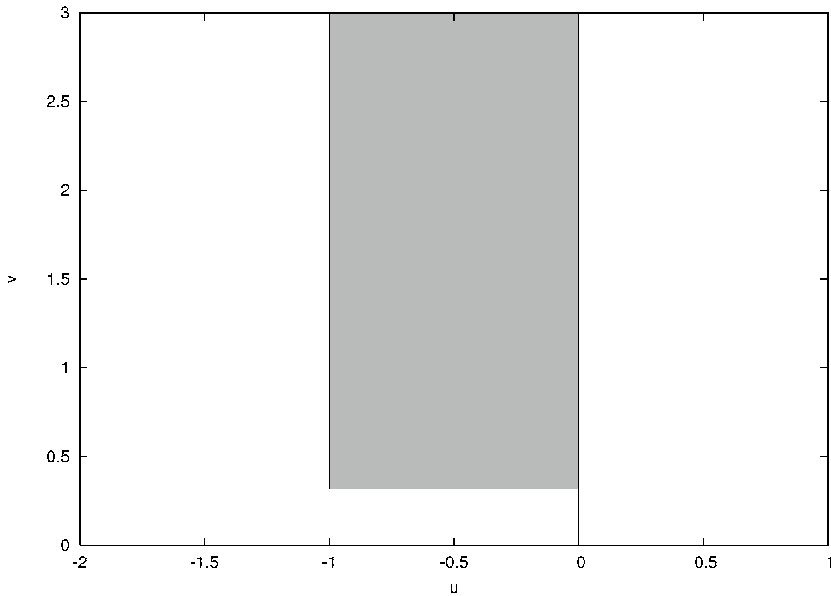


Figure 10.7 The approximate domain where we impose the boundary conditions. The choice  $v = 1/\pi$  for the straight line preserves the measure  $\mu = \pi$ .

boundary conditions are invariant under a parity transformation  $u \rightarrow -u$ ; however, the full symmetry group has two one-dimensional irreducible representations and one two-dimensional representation. The eigenstates transforming accordingly to one of the two-dimensional representations are twofold degenerate, while the remaining are non-degenerate. The latter can be divided into two classes, satisfying either Neumann or Dirichlet boundary conditions. We focus our attention on the second case. The choice of the line  $v = 1/\pi$  approximates the symmetry lines of the original billiard and corresponds to the one-dimensional irreducible representations. The conditions on the vertical lines  $u = 0$ ,  $u = -1$  require to disregard the first two terms in Eq. (10.193); furthermore, we get the condition on the last term as

$$\sum_{n \neq 0} e^{2\pi i n u} \rightarrow \sum_{n=1}^{\infty} \sin(\pi n u) , \quad (10.196)$$

for integer  $n$ . As soon as we restrict to only one of the two one-dimensional

representations, we get

$$\sum_{n \neq 0} e^{2\pi i n u} \rightarrow \sum_{n=1}^{\infty} \sin(2\pi n u), \quad (10.197)$$

while the condition on the horizontal line implies

$$\sum_{n>0} a_n K_{s-1/2}(2n) \sin(2\pi n u) = 0, \quad \forall u \in [-1, 0], \quad (10.198)$$

which in general is satisfied by requiring  $K_{s-1/2}(2n) = 0$  only, for every  $n$ . This last condition, together with the form of the spectrum (10.194), ensures the discreteness of the energy levels, because of the discreteness of the zeros of the Bessel functions. The functions  $K_\nu(x)$  are real and positive for real argument and real index, therefore the index must be imaginary, i.e.  $s = 1/2 + it$ . In this case, these functions have (only) real zeros, and the corresponding eigenvalues turn out to be real and positive.

$$E^2 = t^2 + \frac{1}{4}. \quad (10.199)$$

The eigenfunctions (10.193) exponentially vanish as infinite values of  $v$  are approached. The conditions (10.198) cannot be analytically solved for all the values of  $n$  and  $t$ , and the roots must be numerically worked out for each  $n$ . There are several results on their distribution that allow one to find at least the first levels: a theorem by [373] on the zeros of these functions states that  $K_\nu(\nu x) = 0 \Leftrightarrow 0 < x < 1$ ; furthermore, the energy levels (10.199) monotonically depend on the values of the zeros. Thus, one can search the lowest levels by solving Eq. (10.198) for the first values of  $n$ ; we will discuss below some properties of the spectrum, while now we analyze the ground state only.

A minimum energy exists, as follows from the quadratic structure of the spectrum and from the properties of the Bessel zeros; its value is  $E_0^2 = 19.831$ , and the corresponding eigenfunction is plotted in Fig. 10.8. The eigenstate is normalized through the normalization constant  $N = 739.466$ . The existence of such a ground state has been numerically derived, but it can be inferred on the basis of general considerations about the Hamiltonian structure. The Hamiltonian, indeed, contains a term  $\widehat{v}^2 \widehat{p}_v^2$  which has positive definite spectrum and does not admit vanishing eigenvalues.

#### 10.12.4 Properties of the spectrum

The study of the distribution of the highest energy levels relies on the asymptotic behavior of the zeros for the modified Bessel functions of the

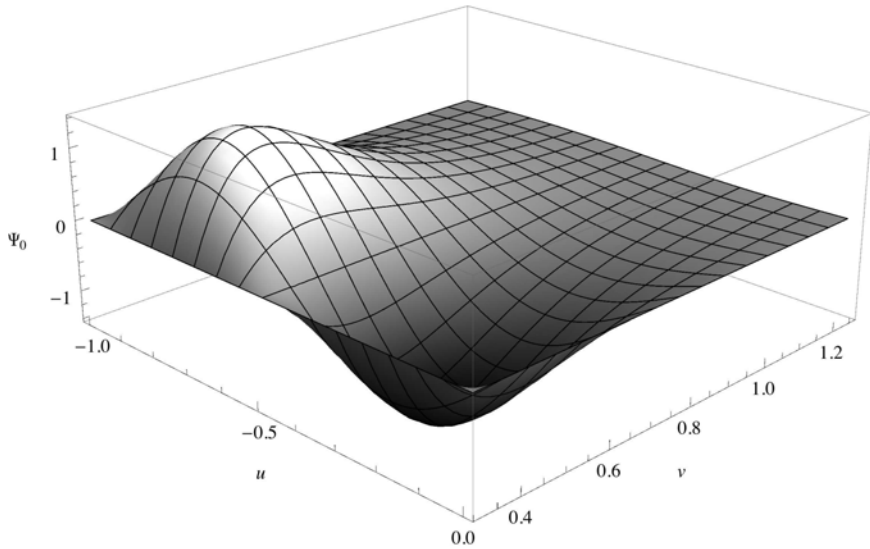


Figure 10.8 The ground state wave function of the Mixmaster model.

third kind. We will discuss the asymptotic regions of the  $(t, n)$  plane in the two cases  $t \gg n$  and  $t \simeq n \gg 1$ .

(i) For  $t \gg n$ , the Bessel functions admit the representation

$$K_{it}(n) = \frac{\sqrt{2\pi}e^{-t\pi/2}}{(t^2 - n^2)^{1/4}} \left[ \sin a \sum_{k=0}^{\infty} \frac{(-1)^k}{t^{2k}} u_{2k} \left( \frac{1}{\sqrt{1-p^2}} \right) + \cos a \sum_{k=0}^{\infty} \frac{(-1)^k}{t^{2k+1}} u_{2k+1} \left( \frac{1}{\sqrt{1-p^2}} \right) \right], \quad (10.200)$$

where  $a = \pi/4 - \sqrt{t^2 - n^2} + \text{tarccosh}(t/n)$ ,  $p \equiv n/t$  and  $u_k$  are the polynomials

$$\begin{cases} u_0(t) = 1, \\ u_{k+1}(t) = \frac{1}{2}t^2(1-t^2)u'_k(t) + \frac{1}{8} \int_0^1 (1-5t^2)u_k(t)dt. \end{cases} \quad (10.201)$$

Retaining in the expression above only terms of order  $\mathcal{O}(n/t)$ , the zeros are fixed by the relation

$$\sin \left[ \frac{\pi}{4} - t + t \log \left( \frac{2}{p} \right) \right] - \frac{1}{12t} \cos \left[ \frac{\pi}{4} - t + t \left( \frac{2}{p} \right) \right] = 0. \quad (10.202)$$

In the limit  $n/t \ll 1$ , Eq. (10.202) can be recast as

$$t \log(t/n) = l\pi \Rightarrow t = \frac{l\pi}{\text{productlog}\left(\frac{l\pi}{n}\right)}, \quad (10.203)$$

where  $\text{productlog}(z)$  is a generalized function giving the solution to the equation  $z = we^w$  and, for a real and positive domain, is a monotonic function of its argument. In Eq. (10.203)  $l$  is an integer number much greater than unity in order to verify  $n/t \ll 1$ .

- (ii) In case the difference between  $2n$  and  $t$  is  $\mathcal{O}(n^{1/3})$  for  $t, n \gg 1$ , we can evaluate the first zeros  $k_{s,\nu}$  by the relations

$$k_{s,\nu} \sim \nu + \sum_{r=0}^{\infty} (-1)^r s_r(a_s) \left(\frac{\nu}{2}\right)^{-(2r-1)/3}, \quad (10.204)$$

where  $a_s$  is the  $s$ -th zero of  $\text{Ai}((2/z)^{1/3})$ ,  $\text{Ai}(x)$  is the Airy function and  $s_i$  are appropriate polynomials. From this expansion it results that, to lowest order

$$t = 2n + 0.030n^{1/3}. \quad (10.205)$$

Equation (10.205) provides the lowest zero (and therefore the energy) for a fixed value of  $n$  and also the relation for the eigenvalues for high occupation numbers as

$$E^2 \sim 4n^2 + 0.12n^{4/3}. \quad (10.206)$$

Let us discuss the completeness of the spectrum and the definition of a scalar product. The problem of completeness can be faced by studying firstly the sine functions and then the Bessel ones. On the interval  $[-1, 0]$ , the set  $\sin(2\pi nu)$  is not a complete basis, but as soon as we request the wave function to satisfy the symmetry of the problem, it becomes complete. Let us take a value  $n > 0$ , thus the functions (10.193) have the form  $\Psi_E(u, v) = \sin(2\pi nu)g(v)$ , which substituted in Eq. (10.189) provides  $v^2 (\partial_v^2 + (2\pi n)^2) g(v) = s(1-s)g(v)$ , whose solutions are exactly the Bessel functions. This property together with the condition on the line  $v = 1/\pi$  forms a Sturm-Liouville problem with a complete set of eigenfunctions. Therefore, such eigenfunctions define a space of functions where we can introduce a scalar product, naturally induced by the metric of the Poincaré plane as

$$(\psi, \phi) = \int \psi(u, v) \phi^\dagger(u, v) \frac{dudv}{v^2}. \quad (10.207)$$

Now we briefly discuss if the presence of a non-local function, like the square-root of a differential operator, can give rise to non-local phenomena. A wavepacket which is non-zero in a finite region of the domain ( $v < M$ ) and far from infinity fails to run to infinity in a finite time, i.e. the probability  $P(v > M)$  to find the packet far away exponentially vanishes. Indeed,

$$\begin{aligned}
 P(v > M) &= \int_{-1}^0 \int_M^\infty |v \sqrt{\partial_u^2 + \partial_v^2} \Psi(u, v)|^2 \frac{dv du}{v^2} \\
 &< 4 M^2 \sqrt{\frac{\pi}{2}} (\sup \Psi)^2 \int_{-1}^0 \int_M^\infty e^{-2v} \frac{dv du}{v^2} \\
 &= 4 M^2 \sqrt{\frac{\pi}{2}} (\sup \Psi)^2 \left( \frac{e^{-2M}}{M} + \text{Ei}(-2M) \right) \\
 &< 4 \sqrt{\frac{\pi}{2}} (\sup \Psi)^2 M e^{-2M}
 \end{aligned} \tag{10.208}$$

where  $\sup(\Psi)$  is the maximum value of the wavepacket in the domain  $v < M$  and  $\text{Ei}(z) = -\int_{-z}^\infty e^{-t}/t dt$  is the exponential integral function. We can conclude that nevertheless the square root is a non-local function, non-local phenomena do not appear (like the case of a wavepacket starting from a localized zone and falling out to infinity).

### 10.13 Guidelines to the Literature

Quantum geometrodynamics, discussed in Sec. 10.1, has been firstly analyzed by DeWitt in [149–151]; for a review see for example [32, 262, 281, 297] while a good textbook is that of Kiefer [279]. The Euclidean approach to quantum gravity (Sec. 10.1.2) has been proposed mainly by Gibbons, Hawking and collaborators in [194, 195, 225, 229]; for a detailed discussion see for example the book edited by Gibbons & Hawking [196]. For the Gibbons-Hawking-York boundary term see also [470]. For a general review on quantum gravity, see also [113].

Discussions about the problem of time (Sec. 10.2) can be found in [262, 442] while for a more recent account see [186, 280, 399, 416]. The role of a quantum perfect clock is discussed in [443]. The Brown-Kuchař mechanism has been proposed in [104, 299]. The evolutionary approach to quantum gravity has been proposed in [353] and developed in [52, 342, 355]. The result of Torre is in [440]. The relational point of view, introduced by Rovelli, has been elaborated in [154, 397, 398].

The first formulation of a minisuperspace theory as described Sec. 10.3

can be found in [344, 346, 465]. Recent reviews on quantum cosmology are [170, 218, 282, 467]. A more detailed list of classical papers is given in [217]. The interpretation of the theory given in Sec. 10.3.2 is discussed in [219, 220, 387, 451], while quantum cosmological singularities (Sec. 10.3.3) are analyzed in [90, 149, 199]. The principle of quantum hyperbolicity can be found in [93].

The path integral quantization of a minisuperspace model discussed in Sec. 10.4, has been formulated in [216] and developed in [221–223, 276]; for some reviews see [218, 360, 417], while for an explicit application to the Kasner Universe see [81].

A clear and complete discussion of matter fields as relational times, is in [262]. In particular, the role of the scalar field, described in Sec. 10.5, is analyzed in [90, 199].

The problem of interpreting the wave function of the Universe (Sec. 10.6) is discussed in [451] and developed in [45, 110, 277, 310] (the analysis of the wave function correlation is in [215]). The application to the quasi-isotropic Mixmaster Universe is formulated in [47].

Boundary conditions, discussed in Sec. 10.7, are reviewed for example [218, 278, 452]. The no-boundary one has been proposed in [225, 230] (a recent development can be found in [179, 226, 227]), while the tunneling one in [322, 449, 450]. The comparison between these two schemes is proposed in [206, 293, 372] (see also [218]).

The quantization of the FRW model presented in Sec. 10.8 has been formulated in [90, 308]. The singularity avoidance conjecture is in [199, 200].

A discussion on the Poincaré half plane presented in Sec. 10.9 is in [286, 354, 435].

The original work that introduces the Taub cosmological model (Sec. 10.10) is [431]; a complete discussion is proposed in [406]; the quantization of the model is in [114, 336].

The scheme presented in Sec. 10.11 is mainly based on the original work [346]. This model has been developed, for example, in [260].

The quantization of the Mixmaster model (Sec. 10.12) in the half plane is presented in [71]; for a different approach see [201], while for a discussion on some problems related to the sign of the Bianchi IX potential see [275]. A clear analysis of the Laplace-Beltrami operator can be found in the book of Terras [435] while an application to quantum gravity is in [389].

The Mixmaster Universe has been investigated in the framework of what is called *quantum chaos*, searching the possible link with the classical

behavior. This topic, not addressed here, has been faced for example in [73, 180, 181, 202]; for a different but related analysis, see [134, 203].



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## Chapter 11

# Generalized Approaches to Quantum Mechanics

This Chapter presents some non-standard approaches to quantum mechanics which will be relevant in modern formulations of quantum cosmology analyzed in Chap. 12. We devote particular attention to the polymer representation of quantum mechanics because of its relation with Loop Quantum Cosmology. Deformed Heisenberg algebras are also analyzed paying attention to their connection with non-commutative geometries as well as with String Theory.

We start with a concise introduction to the algebraic approach to quantum physics. This scheme is relevant in order to investigate singular representations of the canonical commutation relations, being the polymer one of these. Attention is devoted to the uniqueness theorem of quantum mechanics and to the Gelfand-Naimark-Segal construction.

Starting with the relation with the standard representation of quantum mechanics (namely the Schrödinger one), we analyze the structure underlying the polymer quantum mechanics. This framework is the quantum mechanical scheme behind Loop Quantum Gravity (see Sec. 12.1) once a system with a finite numbers of degrees of freedom is taken into account. In this respect, Loop Quantum Cosmology (see Sec. 12.2) can thus be regarded as the implementation of this quantization technique in the minisuperspace dynamics.

We then discuss the notion of the Planck scale as a fundamental minimal length in quantum gravity. A simple derivation of such a scale will be showed in the context of String Theory stressing the differences with respect to the particle framework. The main implication of a minimal length results to be a modification (or deformation) of the Heisenberg uncertainty principle.

The modifications of the Heisenberg algebra in a specific non-

commutative space-time will also be analyzed. We discuss the relation with the String Theory uncertainty principle and we then analyze the formulation of quantum mechanics in the presence of a minimal scale. The implementation of these approaches in quantum cosmology will be given in the second part of Chap. 12.

### 11.1 The Algebraic Approach

In this Section, we will discuss in a pedagogical manner some elementary aspects of the so-called *algebraic approach to quantum physics*. The main results will be used to describe the polymer representation of quantum mechanics.

The main idea of the algebraic approach is to consider the observables as the relevant objects of the theory. This procedure is the opposite to the usual construction where observables are “secondary” objects only. In the standard formulation of quantum mechanics, the first step is the construction of a Hilbert space  $\mathcal{F}$  and the definition of vectors living in it. The vectors in the Hilbert space are the states of the theory and one defines the observables as operators which act upon the states. In this sense, the observables are secondary objects of the theory, while the states are the elementary building blocks. In particular, a self-adjoint operator, defined in  $\mathcal{F}$ , corresponds to each measurable quantity of the classical theory. The spectrum of this operator defines the possible values which may be measured during an experiment. For physical purposes, the description of the Hilbert space and the choice of basis are irrelevant. From a physical point of view, the relevant descriptors are the observables (namely the operators corresponding to measurable quantities). To be more precise, let us take two different formulations of a theory described by two Hilbert spaces  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Furthermore, let  $A_1$  and  $A_2$  be two operators defined in the relative Hilbert spaces  $\mathcal{F}_1$  and  $\mathcal{F}_2$  corresponding to the same observable. These formulations are equivalent if there exists a unitary map  $U : \mathcal{F}_1 \rightarrow \mathcal{F}_2$  such that

$$A_2 = U A_1 U^{-1}. \quad (11.1)$$

It is worth noting that the quantization procedure is far from being unique. As is well known, a classical theory is invariant under canonical transformations, while the quantum approach is invariant under unitary transformations. The group of (classical) canonical transformations is however not isomorphic to the unitary group of the Hilbert space in which  $(q, p)$

are irreducibly represented. This way, *the results of quantization depend on the choice of the classical canonical variables* and the empirical evidence guides in the construction of the quantum theory.

As we mentioned above, the algebraic approach inverts the roles played by observables and states. In such a framework, one begins by constructing an abstract algebra whose elements are the observables. The states are defined in a second moment as the objects which act upon observables by associating a real number to each observable, corresponding to taking the expectation values as in the standard way. The main goal of this approach is that all states, in particular those arising in unitarily inequivalent representations, are treated on an equal footing and one can define a theory without the need to select a preferred construction.

In the first step we will introduce some basic concepts and the uniqueness representation theorem of quantum mechanics, and thereafter we will face the Gelfand-Naimark-Segal (GNS) construction and the Fell theorem.

### 11.1.1 Basic elements

Let us start by introducing the notion of a  $C^*$ -algebra. A vector space over  $\mathbb{C}$  is defined as a set on which are defined the operations of addition and scalar multiplication. An algebra  $\mathcal{A}$  over  $\mathbb{C}$  is a vector space over  $\mathbb{C}$  with an additional multiplication map

$$\times : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, \quad (11.2)$$

which is bilinear (i.e. linear in each variable) and associative. From now on the “ $\times$ ” operator will be dropped. If we add the involution map (also called the  $*$ -operation)  $*$  :  $\mathcal{A} \rightarrow \mathcal{A}$  satisfying

$$\begin{aligned} A^{**} &= A, \\ (AB)^* &= B^* A^* \end{aligned} \quad (11.3)$$

for any  $A, B \in \mathcal{A}$ , then the algebra  $\mathcal{A}$  will be said a  $*$ -algebra. The last element needed to construct a  $C^*$ -algebra is the introduction of a topology, i.e. a definition of a neighborhood of an element of  $\mathcal{A}$ . The most evident topology for the algebra is the norm map  $\|\cdot\| : \mathcal{A} \rightarrow \mathbb{R}^+$  such that:

$$\begin{aligned} \|\alpha A\| &= |\alpha| \|A\|, \\ \|A + B\| &\leq \|A\| + \|B\|, \\ \|AB\| &\leq \|A\| \|B\|, \end{aligned} \quad (11.4)$$

where  $\alpha \in \mathbb{C}$ . Such norm induces the metric  $d(A, B) = \|A - B\|$ . Finally, if we require that

$$\begin{aligned} \|A^*\| &= \|A\|, \\ \|A^*A\| &= \|A\|^2 \end{aligned} \quad (11.5)$$

for all  $A \in \mathcal{A}$ , then the algebra  $\mathcal{A}$  will be a  $C^*$ -algebra.

The other key ingredient in the algebraic approach is the definition of an *algebraic state*, not to be confused with the definition of a state in the ordinary formulation of quantum mechanics. To avoid any confusion, we call the latter a physical state. An algebraic state  $\omega$  of the quantum theory is defined to be a linear map

$$\omega : \mathcal{A} \rightarrow \mathbb{C}, \quad (11.6)$$

from a  $C^*$ -algebra  $\mathcal{A}$  to  $\mathbb{C}$ . This object is positive definite

$$\omega(A^*A) \geq 0 \quad \forall A \in \mathcal{A} \quad (11.7)$$

as well as normalized

$$\omega(\mathbb{I}) = \|\omega\| = 1, \quad (11.8)$$

$\mathbb{I}$  being the identity element of  $\mathcal{A}$ . Let us now assume that:

- (i) the self-adjoint elements of  $\mathcal{A}$  correspond to observables. This means that the  $*$ -operation corresponds to taking the adjoint.
- (ii) The unit element  $\mathbb{I}$  is the trivial observable having the value 1 in any physical state (which are vectors in the Hilbert space).

This way, an algebraic state (a normalized positive linear form  $\omega$ ) can be interpreted as an expectation value over the observables, explicitly

$$\omega(A) = \langle A \rangle. \quad (11.9)$$

This observation clarifies the physical interpretation of the algebraic approach and its precise formulation will be given below in terms of the so-called GNS construction.

Let us illustrate such statement considering the simplest mechanical system, i.e. a particle on the real axis  $\mathbb{R}$ . The phase space of this model is  $\mathbb{R}^2$  with coordinates  $(q, p)$  satisfying the Poisson brackets

$$\{q, p\} = 1. \quad (11.10)$$

The quantum kinematic correspondence to this system is described by operators, represented on a Hilbert space, satisfying the canonical commutation relations

$$[\hat{q}, \hat{p}] = i\hat{1}. \quad (11.11)$$

Consider the algebra generated by the exponentiated versions of the basic operators  $(\hat{q}, \hat{p})$  which are denoted by

$$U(\alpha) = e^{i\alpha\hat{q}}, \quad V(\beta) = e^{i\beta\hat{p}}, \quad (11.12)$$

where  $\alpha$  and  $\beta$  have dimensions of momentum and length, respectively. The canonical commutation relation (11.11) becomes<sup>1</sup>

$$U(\alpha) V(\beta) = e^{-i\alpha\beta} V(\beta) U(\alpha). \quad (11.13)$$

The two quantities in Eq. (11.12) generate the so-called *Weyl algebra*  $\mathcal{W}$  which is obtained by considering the linear combinations of the generators (11.12). A generic element  $W = W(\alpha, \beta)$  of  $\mathcal{W}$  can in general be expressed as

$$W = \sum_i (A_i U(\alpha_i) + B_i V(\beta_i)), \quad (11.14)$$

and the Weyl algebra has the natural structure of a  $C^*$ -algebra.

From this perspective, the quantization of a mechanical system consists of finding a unitary irreducible representation of the Weyl algebra  $\mathcal{W}$  on a Hilbert space  $\mathcal{F}$ . It is natural to ask which is the role of the usual Schrödinger representation of quantum mechanics and to investigate about other possible representations of the canonical commutation relations. The Weyl algebra is exactly the way to answer this question.

The ordinary Schrödinger construction is based on the choice of the Hilbert space

$$\mathcal{F} = L^2(\mathbb{R}, dq), \quad (11.15)$$

which is the space of the square integrable functions with respect to the Lebesgue measure  $dq$  on  $\mathbb{R}$ . The basic operators are then represented as

$$\begin{aligned} \hat{q} \psi(q) &= q \psi(q), \\ \hat{p} \psi(q) &= -i \partial_q \psi(q). \end{aligned} \quad (11.16)$$

The Stone-von Neumann theorem ensures that this is the unique irreducible representation of the Weyl algebra (namely, of the canonical commutation relations) *if* the operators (11.12) are weakly continuous in the parameters  $\alpha$  and  $\beta$ . The Schrödinger representation is unique up to unitarily equivalence. There are however many irreducible representations where

---

<sup>1</sup>Take into account the Campbell-Baker-Hausdorff formula,

$$e^A e^B = e^C \quad \text{with} \quad C = A + B + \frac{1}{2}[A, B] + \dots$$

the continuity condition is not satisfied (such kind of representations are often called *singular representations*) such as, for example, the polymer representation (see Sec. 11.2) belongs to the latter class.

It is worth noting that such a fundamental theorem is valid only for systems with finite degrees of freedom, i.e. for quantum mechanics. For system with infinite degrees of freedom (field theories) there exists a host of inequivalent, irreducible representations of the canonical commutation relations which defies a useful complete classification.

### 11.1.2 GNS construction and Fell theorem

The relation between the algebraic and the ordinary approach to the quantum theory can be formulated through the celebrated construction given by Gelfand, Naimark and Segal (GNS) which can be stated as follows.

**Theorem 11.1 (GNS Construction).** *Let  $\mathcal{A}$  be a  $C^*$ -algebra with unit and let  $\omega : \mathcal{A} \rightarrow \mathbb{C}$  be a state. Then there exist a Hilbert space  $\mathcal{F}$ , a representation<sup>2</sup>  $\pi : \mathcal{A} \rightarrow L(\mathcal{F})$  and a vector  $\Omega \in \mathcal{F}$  such that*

$$\omega(A) = \langle \Omega | \pi(A) | \Omega \rangle_{\mathcal{F}}. \quad (11.17)$$

*These objects satisfy the additional property that  $\Omega$  is cyclic, i.e. the vectors  $\pi(A)\Omega$  for all  $A \in \mathcal{A}$  comprise a dense<sup>3</sup> subspace of  $\mathcal{F}$ . The triplet  $(\mathcal{F}, \pi, \Omega)$  is uniquely determined (up to unitary equivalence) by these properties.*

Each positive linear form  $\omega$  over a  $C^*$ -algebra defines a Hilbert space as well as a representation of the algebra by linear operators acting on the Hilbert space. One key aspect of the GNS construction is that one can have different, but unitarily equivalent, representations of the Weyl algebra which yield equivalent theories.

Let us now sketch some details of this construction. As we have seen, the algebra  $\mathcal{A}$  is a linear space (over  $\mathbb{C}$ ) and the state  $\omega$  defines a Hermitian scalar product on  $\mathcal{A}$  by

$$\langle A | B \rangle = \omega(A^* B) \quad (11.18)$$

for  $A, B \in \mathcal{A}$ . However, due to the positivity condition (11.7), this scalar product is semi-definite positive. In fact, it can occur that, for some  $X \in \mathcal{A}$ , we have  $\omega(X, X) = 0$ . In order to have a (properly) positive definite

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<sup>2</sup>We denote by  $L(\mathcal{F})$  the collection of all bounded linear maps on  $\mathcal{F}$ . It has the structure of a  $C^*$  algebra.

<sup>3</sup>A subspace  $Y$  of a topological space  $T$  is said to be dense in  $T$  if the closure of  $Y$  is equal to  $T$ .

inner product, we have to factor out the contributions given by the set  $\mathcal{J}$  of elements  $X$ . The inner product will be taken in the corresponding factor space  $\mathcal{A}/\mathcal{J}$ . An element of the factor space is denoted by  $[A]$  and corresponds to the equivalence class

$$[A] = \{A + X\} \quad \text{with} \quad A \in \mathcal{A}, X \in \mathcal{J}. \quad (11.19)$$

The Hilbert space  $\mathcal{F}$  is thus defined by the completion of  $\mathcal{A}/\mathcal{J}$  with respect to the norm (11.7). The product in  $\mathcal{A}$  then defines the representation  $\pi : \mathcal{A} \rightarrow L(\mathcal{F})$  by

$$\pi(A)[B] = [AB], \quad (11.20)$$

for all  $A \in \mathcal{A}$ . Finally, the cyclic vector  $\Omega$  corresponds to the identity element of the algebra  $\mathcal{A}$ .

This construction can be inverted. In general, any vector  $\Psi \in \mathcal{F}$  defines an algebraic state

$$\omega_\Psi(A) = \langle \Psi | \pi(A) | \Psi \rangle_{\mathcal{F}}, \quad (11.21)$$

and furthermore, since  $\Omega$  is a cyclic vector, the vector  $\Psi$  can be approximated by  $\pi(B)\Omega$ . The state (11.21) can be given as

$$\omega_\Psi(A) \simeq \omega(B^*AB) = \langle B | AB \rangle, \quad (11.22)$$

with  $B \in \mathcal{A}$ .

The GNS construction shows that states over a  $C^*$ -algebra come in families. In fact, a single (algebraic) state  $\omega$  determines a family of (physical) states by means of Eq. (11.21). More generally, it is possible to consider the states

$$\omega_\rho(A) = \text{Tr}[\rho \pi(A)], \quad (11.23)$$

where  $\rho$  is a density matrix. The collection of all the states (11.23) is the so-called *folium of the representation*  $\pi$ . The notion of a folium is fundamental in order to enunciate the Fell theorem, one of the most important ones in the algebraic approach to the quantum theory.

**Theorem 11.2 (Fell Theorem).** *The folium of a faithful representation<sup>4</sup> of a  $C^*$ -algebra is weakly dense in the collection of all states.*

For a better understanding, let us reformulate this theorem. Let  $(\mathcal{F}_1, \pi_1)$  and  $(\mathcal{F}_2, \pi_2)$  be (possibly unitarily inequivalent) representations of the Weyl algebra  $\mathcal{W}$  in the sense of the GNS construction. Let  $A_1, \dots, A_n \in \mathcal{W}$  and

<sup>4</sup>A representation  $\pi$  is said to be faithful if  $\pi(A) \neq 0$  for  $A \neq 0$ .



$\epsilon_1, \dots, \epsilon_n > 0$ . Let  $\omega_1$  be an algebraic state corresponding to a density matrix on the Hilbert space  $\mathcal{F}_1$ . The Fell theorem ensures that there exists a state  $\omega_2$ , corresponding to a density matrix on  $\mathcal{F}_2$ , such that

$$|\omega_1(A_i) - \omega_2(A_i)| < \epsilon_i, \quad (11.24)$$

for all  $i = 1, \dots, n$ . The theorem shows that, although two representations of  $\mathcal{W}$  can be inequivalent, the determination of a *finite* number of expectation values of observables in  $\mathcal{W}$ , made with *finite* accuracy, cannot distinguish between different representations. In physics it is not possible to perform infinitely many experiments and furthermore each experiment has a finite accuracy. This way, by monitoring a state, we can at most determine a weak neighborhood in the space of all states. The Fell theorem states that we cannot find out in which folium the state lies.

Let us put forward this consideration by assuming that the observables in an algebra  $\mathcal{A}$  are the only measurable quantities of a quantum field theory. Thus, because of the physical realistic limitation of finitely many measurements with finite accuracy, different (namely inequivalent) representations of the algebra are “physically equivalent” and the choice of the representation is physically irrelevant. Such a (fascinating) statement is however not valid in general as, in fact, there are additional observables in the theory which cannot be represented in  $\mathcal{A}$ , as for example the energy-momentum tensor. Two representations should not be “physically equivalent” with respect to these additional observables and, in those cases, not treated in this book, the so-called Hadamard condition has to be invoked.

## 11.2 Polymer Quantum Mechanics

The polymer representation of quantum mechanics is based on a singular (non-standard) representation of the canonical commutation relations. In particular, in a two-dimensional phase space, it is possible to choose a discretized operator, whose conjugate variable cannot be directly promoted as an operator. From a physical point of view, this scheme can be interpreted as the quantum-mechanical framework for the introduction of a cutoff. Its continuum limit, which corresponds to the removal of such a cutoff, has to be understood as the equivalence class of theories modified at different microscopical scales.

This framework is relevant to make a bridge with the Planck scale physics. In particular, it is interesting when treating the quantum-mechanical properties of a background-independent canonical quantum the-

ory of gravity. More precisely, the holonomy-flux algebra used in Loop Quantum Gravity (LQG, see Sec. 12.1) reduces to a polymer-like algebra, when a system with a finite number of degrees of freedom is considered. Loop Quantum Cosmology (LQC, see Sec. 12.2) can be regarded as the implementation of this quantization technique in the minisuperspace dynamics. Finally, from a quantum-field theoretical point of view, the polymer representation is substantially equivalent to introducing a lattice structure on the space.

In this Section we analyze the polymer quantum mechanics starting from its relation with the Schrödinger representation. The kinematics and the dynamics of the polymer particle will be discussed later.

### 11.2.1 *From Schrödinger to polymer representation*

As we have seen in Sec. 11.1.1, the Stone-Von Neumann uniqueness theorem ensures that the Schrödinger representation is (up to unitary equivalence) the only irreducible representation of the Weyl algebra  $\mathcal{W}$  in which the operators (11.12) are continuous functions of  $\alpha$  and  $\beta$ . The polymer case is a particular representation in which this condition is not satisfied, providing a unitarily inequivalent representation of the canonical commutation relations and then the physical predictions of the two frameworks will differ. The link between the Schrödinger and the polymer representations is implicitly given by the Fell theorem (11.2) since, as we have seen, it ensures that it is possible to approximate states in the standard representation by states in a singular representation of the Weyl algebra. However, the Fell theorem is not constructive because it does indicate how to recover a non-standard representation. We will show how it is possible to obtain an explicit singular representation of the Weyl algebra and its manifest link to the Schrödinger one.

A classical system is described, in the Hamiltonian formalism, in terms of a symplectic manifold  $(\Gamma, \varpi)$  where  $\Gamma$  is the phase space and  $\varpi$  is the symplectic 2-form which defines the Poisson brackets as

$$\{f, g\} = \varpi^{ab} \nabla_a f \nabla_b g. \quad (11.25)$$

The quantization aims to find a representation of the canonical commutation relations (11.13) in a Hilbert space. To analyze the representations of the Weyl algebra it is useful to introduce the complex structure  $J : \Gamma \rightarrow \Gamma$  such that  $J^2 = -1$ . We focus on one-dimensional mechanical systems and

thus  $J$  can be defined by a length scale  $d$  only, so that explicitly we have

$$J = \begin{pmatrix} 0 & -d^2 \\ 1/d^2 & 0 \end{pmatrix}, \quad (11.26)$$

and thus  $J : (q, p) \rightarrow (-d^2 p, q/d^2)$ . This has to be compatible with the symplectic structure and thus it induces a positive definite, real, inner product on  $\Gamma$  by

$$g(v, v') = \varpi(v, Jv'), \quad (11.27)$$

where  $v$  denotes a vector in the phase space  $\Gamma = \mathbb{R}^2$ , namely  $v = (q, p)$ . By means of the complex structure  $J$ , the Hilbert space can be viewed as a real vector space. In particular, the hermitian (complex) inner product is given by

$$\langle v|v' \rangle = \frac{1}{2}g(v, v') + \frac{i}{2}\varpi(v, v') \quad (11.28)$$

and then it explicitly decomposes into a real and an imaginary part. Notably, the triple  $(J, g, \varpi)$  equips the Hilbert space with the structure of a Kähler space, providing the starting point of the so-called “geometric formulation of quantum mechanics”.

A relation with the Schrödinger representation (namely with the Hilbert space (11.15)) is recovered by the GNS construction. From Eq. (11.9), the complex structure (11.26) uniquely defines the algebraic state  $\omega$  that yields

$$\langle U(\alpha) \rangle = e^{-\frac{1}{4}d^2\alpha^2} \quad (11.29a)$$

$$\langle V(\beta) \rangle = e^{-\frac{1}{4}\beta^2/d^2}. \quad (11.29b)$$

The Hilbert space underlying this framework, i.e. the one equipped with the extra structure  $J$  (or  $d$ ), is given by

$$\mathcal{F}_d = L^2(\mathbb{R}, dq_d) \quad (11.30)$$

where the measure  $dq_d$  is no longer trivial and reads as

$$dq_d = \frac{1}{d\sqrt{\pi}} e^{-q^2/d^2} dq. \quad (11.31)$$

The relation with the standard representation is given in terms of a map between the two frameworks. More precisely, the Hilbert space  $\mathcal{F}_d$  can be mapped into the Schrödinger one (11.15) by means of an isometric isomorphism  $\mathcal{K} : \mathcal{F}_d \rightarrow \mathcal{F}$  which is explicitly given by

$$\mathcal{K} = \frac{e^{-\frac{q^2}{2d^2}}}{\sqrt{d\sqrt{\pi}}}. \quad (11.32)$$

All the  $d$ -representations are unitarily equivalent and this is nothing but an explicit manifestation of the Stone-Von Neumann uniqueness theorem. However, in the limiting cases  $d \rightarrow 0$  and  $d \rightarrow \infty$ , the map (11.32) is ill defined and indeed the polymer representation arises in these “regimes”.

### 11.2.2 Kinematics

The polymer representation of quantum mechanics is constructed so far in an abstract way, i.e. without addressing any relation with the Schrödinger one. We start by considering abstract kets  $|\mu\rangle$ , where  $\mu \in \mathbb{R}$ , and a suitable finite subset defined by  $\mu_i \in \mathbb{R}$  with  $i = 1, 2, \dots, N$ . The polymer inner product between these kets and bra is assumed to be

$$\langle \mu | \nu \rangle = \delta_{\mu\nu}, \quad (11.33)$$

which is a Kronecker-delta rather than the usual Dirac-delta distribution. The kets are then assumed to be an orthonormal basis along which any state  $|\psi\rangle$  can be projected. Given two states  $|\psi\rangle = \sum_i a_i |\mu_i\rangle$  and  $|\phi\rangle = \sum_j b_j |\nu_j\rangle$ , the inner product between them is given by

$$\langle \phi | \psi \rangle = \sum_i \sum_j b_j^\dagger a_i \langle \nu_j | \mu_i \rangle = \sum_k b_k^\dagger a_k, \quad (11.34)$$

where  $k$  labels the available intersection points. This defines a Hilbert space  $\mathcal{F}_{\text{pol}}$ .

In quantum mechanics there are two basic operators, the multiplication and the displacement: let us investigate how they act on the polymer Hilbert space. The symmetric “label” operator  $\hat{\epsilon}$  is such that

$$\hat{\epsilon}|\mu\rangle = \mu|\mu\rangle. \quad (11.35)$$

The second group of operators is given by a one-parameter family of unitary operators,  $\hat{s}(\lambda)$ , such that

$$\hat{s}(\lambda)|\mu\rangle = |\mu + \lambda\rangle. \quad (11.36)$$

Because all kets are orthonormal by means of Eq. (11.33),  $\hat{s}(\lambda)$  is (weakly) discontinuous. As a result, it cannot be obtained from any hermitian operator by exponentiation. In this sense, the continuity hypothesis of the Stone-Von Neumann theorem has been relaxed and therefore the polymer quantum dynamics turns out to be a non-standard representation of the Weyl algebra. It is worth noting that the polymer Hilbert space is not separable.<sup>5</sup>

For the toy model of a one-dimensional system, whose phase space is spanned by the variables  $p$  and  $q$ , the polymer representation techniques find interesting applications when one of the two variables is supposed to be discrete. This discreteness will affect both wave functions, obtained by projecting the physical states on the  $p$  or  $q$  basis (polarization), together

<sup>5</sup>A Hilbert space is separable if and only if it admits a countable orthonormal basis.

with the operators associated to the canonical variables, acting on them. We will discuss only the case of a “discrete” position variable  $q$ , and the corresponding momentum polarization. In this case, the wave functions are given by

$$\psi_\mu(p) = \langle p | \mu \rangle = e^{ip\mu}. \quad (11.37)$$

Accordingly to the previous discussion, the label operator  $\hat{e}$  is easily identified with the position operator  $\hat{q}$ , i.e.

$$\hat{q} \psi_\mu = -i \partial_p \psi_\mu = \mu \psi_\mu. \quad (11.38)$$

On the other hand, the  $V(\lambda)$  multiplicative operator in (11.12) corresponds exactly to the “shift” operator  $\hat{s}(\lambda)$ , and in fact one has

$$V(\lambda) \psi_\mu = e^{i\lambda p} e^{ip\mu} = \psi_{(\mu+\lambda)}. \quad (11.39)$$

This way, since such operator is discontinuous in  $\lambda$ , the variable  $p$  cannot be directly implemented as the operator  $\hat{p}$  in the Hilbert space and only the operator  $V(\lambda)$  is well defined.

The position operator  $\hat{q}$  is discrete, but in a weaker sense with respect to have a discrete spectrum. More precisely, although its eigenvalues are continuous ( $\mu \in \mathbb{R}$ ), all the eigenvectors are normalizable. Hence, this Hilbert space can be expanded out as a direct sum, rather than as a direct integral, of the one-dimensional eigenspaces of  $\hat{q}$ . This clarifies the difference with respect to the ordinary representation.

It can be shown that, from generic representation theory arguments, the corresponding Hilbert space is given by

$$\mathcal{F}_{\text{pol}} = L^2(\mathbb{R}_B, d\mu), \quad (11.40)$$

which is the set of square-integrable functions defined on the Bohr compactification of the real line  $\mathbb{R}_B$ , with a Haar measure  $d\mu$ . Since the kets  $|\mu\rangle$  are arbitrary but finite, the wave functions can be interpreted as quasi-periodic functions, with the inner product

$$\begin{aligned} \langle \psi_\mu | \psi_\lambda \rangle &= \int_{\mathbb{R}_B} d\mu \psi_\mu^\dagger(p) \psi_\lambda(p) \\ &= \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L dp \psi_\mu^\dagger(p) \psi_\lambda(p) = \delta_{\mu\lambda}. \end{aligned} \quad (11.41)$$

Such construction corresponds to the limit  $d \rightarrow \infty$  case of the previous discussion (in which  $p$  and  $q$  are interchanged). It is worth noting that the Hilbert space of polymer quantum mechanics  $\mathcal{F}_{\text{pol}}$  is exactly the same of LQC (see Sec. 12.2). This concludes our analysis on the kinematical aspects of the polymer quantization procedure.

### 11.2.3 Dynamics

The Hamiltonian  $\mathcal{H}$  describing a quantum mechanical system is usually a function of both position and momentum, i.e. of the form  $\mathcal{H} = p^2 + V(q)$ . As we have seen, in the particular case of a discrete position variable in the momentum polarization,  $p$  cannot be implemented as an operator, so that some restrictions on the model are still required.

First of all, a suitable approximation of the kinetic term is needed. For this purpose, it is useful to restrict the arbitrary kets  $|\mu_i\rangle$ , with  $i \in \mathbb{R}$  to  $|\mu_i\rangle$  with  $i \in \mathbb{Z}$ . In other words, we introduce the notion of regular graph  $\gamma_{\mu_0}$ , defined as a numerable set of equidistant points, whose separation is given by the parameter  $\mu_0$  expressed as

$$\gamma_{\mu_0} = \{q \in \mathbb{R} \mid q = n\mu_0, \forall n \in \mathbb{Z}\}. \quad (11.42)$$

The associated Hilbert space  $\mathcal{F}_{\gamma_{\mu_0}}$  is now separable. Because of the regular graph  $\mu_0$ , the eigenfunctions of  $\hat{p}_{\mu_0}$  must be of the form

$$\exp(im\mu_0 p), \quad (11.43)$$

with  $m \in \mathbb{Z}$ , which are Fourier modes of period  $2\pi/\mu_0$ . The inner product (11.41) is equivalent to the inner product on a circle  $S^1$  with uniform measure, i.e.

$$\langle \phi(p) | \psi(p) \rangle_{\mu_0} = \frac{\mu_0}{2\pi} \int_{-\pi/\mu_0}^{\pi/\mu_0} dp \phi^\dagger(p) \psi(p). \quad (11.44)$$

The “dynamical” Hilbert space reads as

$$\mathcal{F}_{\gamma_{\mu_0}} = L^2(S^1, dp) \quad (11.45)$$

and there it is possible to construct an approximation for the displacement operator  $V(\lambda)$  whose action is the shift of a ket  $|\mu_n\rangle$  to the next one  $|\mu_{n+1}\rangle$ . Thus, the parameter  $\lambda$  has to be fixed to the lattice scale  $\mu_0$  leading to the desired result

$$V(\mu_0)|\mu_n\rangle = |\mu_n + \mu_0\rangle = |\mu_{n+1}\rangle. \quad (11.46)$$

Thereafter, we can build a regulated operator  $\hat{p}_{\mu_0}$  to implement the usual incremental ratio in a discrete manner and is defined as

$$\begin{aligned} \hat{p}_{\mu_0}|\mu_n\rangle &= \frac{1}{2i\mu_0} (V(\mu_0) - V(-\mu_0)) |\mu_n\rangle \\ &= \frac{1}{2i\mu_0} (|\mu_{n+1}\rangle - |\mu_{n-1}\rangle). \end{aligned} \quad (11.47)$$

This basic shift operator will be of fundamental importance when constructing (approximating) any function of  $p$ , e.g. the Hamiltonian  $\mathcal{H}$  itself. From

the relation (11.47), the polymer paradigm can be recovered by the formal substitution

$$p \rightarrow \frac{1}{\mu_0} \sin(\mu_0 p), \quad (11.48)$$

where the incremental ratio (11.47) has been evaluated for exponentiated operators.

The Hamiltonian operator in the polymer Hilbert space includes the  $\hat{p}_{\mu_0}^2$  operator which can be defined in (at least) two ways. The first one is to apply the operator (11.47) twice, i.e.  $\hat{p}_{\mu_0}^2 = \hat{p}_{\mu_0} \cdot \hat{p}_{\mu_0}$ , leading to two-step shifts in the graph. The second possibility is to define  $\hat{p}_{\mu_0}^2$  from its approximation in terms of a cosine function, i.e.

$$\begin{aligned} \hat{p}_{\mu_0}^2 |\mu_n\rangle &\simeq \frac{2}{\mu_0^2} [1 - \cos(p\mu_0)] |\mu_n\rangle \\ &= \frac{1}{\mu_0^2} (2 - V(\mu_0) - V(-\mu_0)) |\mu_n\rangle, \end{aligned} \quad (11.49)$$

leading to a one-step shift and, for such reason, the second possibility appears more suitable. The Hamiltonian operator  $\mathcal{H}_{\mu_0}$ , which lives in  $\mathcal{F}_{\gamma_{\mu_0}}$ , reads as

$$\mathcal{H}_{\mu_0} = \frac{\hat{p}_{\mu_0}^2}{2m} + V(\hat{q}), \quad (11.50)$$

where  $\hat{p}_{\mu_0}^2$  is given by Eq. (11.49) and the differential operator  $\hat{q}$  is well defined in the Hilbert space.

To conclude, we will discuss how to remove the regulator  $\mu_0$  which was introduced as an intermediate step when constructing the dynamics. The physical Hilbert space can be defined as the continuum limit of effective theories at different scales and can be shown to be unitarily isomorphic to the ordinary one  $\mathcal{F}_S = L^2(\mathbb{R}, dp)$ . Indeed, it is impossible to obtain  $\mathcal{F}_S$  starting from a given graph  $\gamma_0 = \{q_k \in \mathbb{R} | q_k = ka_0, \forall k \in \mathbb{Z}\}$  by dividing each interval  $a_0$  into  $2^n$  new intervals of length  $a_n = a_0/2^n$ , because  $\mathcal{F}_S$  cannot be embedded into  $\mathcal{F}_{\text{pol}}$ . It is however possible to go the other way round and to look for a continuous wave function that is approximated by a wave function over a graph, in the limit of the graph becoming finer. In fact, if one defines a scale  $C_n$ , i.e. a decomposition of  $\mathbb{R}$  in terms of the union of closed-open intervals that have lattice points as end points and cover  $\mathbb{R}$  without intersecting, then can approximate continuous functions with functions that are constant on these intervals. As a result, at any given scale  $C_n$ , the kinetic term of the Hamiltonian operator can be approximated

as in Eq. (11.49), and effective theories at given scales are related by coarse-graining maps. In particular, it is necessary to regularize the Hamiltonian, treated as a quadratic form, as a self-adjoint operator at each scale by introducing a normalization factor in the inner product. The convergence of microscopically corrected Hamiltonians is based on the convergence of the energy levels and on the existence of completely normalized eigenvectors compatible with the coarse-graining operation.

### 11.3 On the Existence of a Fundamental Scale

The combination of quantum mechanics and GR leads to a fundamental minimal length, naturally related to the Planck scale. The argumentations for this intuition are very general and any approach to quantum gravity<sup>6</sup> predicts such a scale. In this Section we describe in a very simple manner how such minimal scale appears as soon as we deal with energy regimes where both quantum effects and gravity are relevant. This argument has to be considered as an attempt to loosely explain why the idea that quantum gravity implies an absolute limit on the localization of events is commonly accepted in the scientific community. A rigorous description of quantum space time is however not possible at the present stage of the theories, because there is not yet a complete quantum theory of gravity. Nevertheless, a minimal scale appears from a very general overlap between the existing physical theories.

A maximal localization scale is already present in relativistic quantum mechanics and historically lead to the birth of quantum field theories. The Heisenberg uncertainty relation implies that the position uncertainty is proportional to the inverse of the momentum uncertainty, i.e.  $\Delta q \Delta p \gtrsim 1$  (numerical factors have been ignored). For a relativistic particle ( $E \sim p$ ) this relation stands as  $\Delta q \Delta E \gtrsim 1$  and, if we consider a position uncertainty smaller than its Compton length ( $\Delta q \lesssim 1/E$ ), we get the key relation

$$\Delta E \gtrsim E. \quad (11.51)$$

The energy uncertainty of a relativistic particle is larger than its rest mass, i.e. the concept of a single particle is rather unclear. We have thus to turn

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<sup>6</sup>The two main approaches for a quantum theory of the gravitational field are Loop Quantum Gravity and String Theory. The existence of a minimal length is a prediction of both these (very different) theories. In the former, the physical space appears to be granular (see Sec. 12.1), while in the latter a minimal length is a direct consequence of considering strings instead of particles (see Sec. 11.4).



on a multi-particle theory which can be formulated in terms of a quantum field theory.

Let us now introduce GR effects. The space time is here dynamical and the metric evolves in tandem with matter according to the Einstein equations. This way, a quantum uncertainty in the position of a particle affects (by Heisenberg) the uncertainty in momentum, which leads (by Einstein) to an uncertainty in geometry which, in turn (again by Einstein), induces an additional uncertainty in the position of the particle. Furthermore, to get a high resolution we need high energy photons which strongly curve the space-time which in turn increases the disturbance on the measure. The geometry itself is now subjected to quantum fluctuations and, as already argued by Wheeler in 1963 the quantum space time should appear as a foam.

These considerations can be better explained by the following simple example. Suppose that we want to resolve a spherical region of radius  $l$  and then a photon of wavelength less than  $l$  (its energy will be greater than  $1/l$ ). Such photon carries out an energy density  $\rho$  greater than  $1/l^4$  and the corresponding Einstein equations for this system can be roughly written as

$$\partial^2 g \sim \kappa \rho \gtrsim \frac{\kappa}{l^4}, \quad (11.52)$$

where  $g$  is the amplitude of the space-time metric. Therefore, the photon generates a gravitational potential  $g > \kappa/l^2$  and the length that is being measured will have an uncertainty equal to

$$\sqrt{gl^2} > l_P. \quad (11.53)$$

A minimal uncertainty of order of the Planck length is thus predicted in any quantum gravity framework. Such a basic example illustrates a result which can be obtained in a large variety of more sophisticated and rigorous schemes.

## 11.4 String Theory and Generalized Uncertainty Principle

A general prediction of String Theory is the existence of a minimal observable physical length, leading to the Generalized Uncertainty Principle (GUP), i.e. a modification of the Heisenberg uncertainty relation. Such result is expected to be quite general since it can be derived from model-independent arguments, like the analysis of the transition amplitudes. This result was also obtained in high-energy Gedanken string scattering experiments.

In what follows we briefly review how a minimal length naturally arises in string theory using the path integral formulation.

String theory assumes as fundamental objects one dimensional entities, i.e. strings. The world line of a string (which is called the worldsheet) is a two-dimensional object rather than one-dimensional as in the standard particle framework. The starting point of the theory is the Polyakov action for a bosonic string. It is assumed that strings propagate in the Minkowski space-time with the worldsheet given by  $X^i(\sigma^\alpha)$ , where  $\sigma^0 = \tau$  and  $\sigma^1 = \sigma$  are the (time-like and space-like, respectively) coordinates on the worldsheet. The Polyakov action reads as

$$S_{\text{string}} \propto -\frac{1}{l_s^2} \int d\tau d\sigma \sqrt{G} G^{\alpha\beta} \partial_\alpha X^i \partial_\beta X_i, \quad (11.54)$$

where  $G_{\alpha\beta} = G_{\alpha\beta}(\sigma, \tau)$  is the intrinsic metric of the worldsheet,  $G = |\det G_{\alpha\beta}|$  and

$$l_s \propto l_P \quad (11.55)$$

is the fundamental length characterizing the string theory (the proportionality constant is a parameter model-dependent).

Let us consider the quantum mechanics of a non-relativistic particle. The usual Feynman path integral is obtained by a regularization of the world line of a particle  $x(\tau)$  with parameter  $\epsilon$  and the generating functional is given by

$$Z_{\text{particle}} \sim \int \mathcal{D}x \exp \left[ \frac{i}{\lambda^2} \sum \epsilon \left( \frac{\delta_\tau x}{\epsilon} \right)^2 \right], \quad (11.56)$$

where  $\lambda$  is a dimensional constant. From this expression, the particle travels, at each step, a distance

$$\langle (\delta x)^2 \rangle \sim \epsilon \lambda^2, \quad (11.57)$$

which means that taking  $\epsilon \rightarrow 0$  the resolution can be arbitrarily high, i.e. also infinite. Such a diverging resolution allows us to introduce the concept of particle state in quantum mechanics and the local-operator formalism, as well as equal-time commutators, in quantum field theory. On the other hand, as is well known, the infinite resolution is at the basis of ambiguities in non-linear systems like the operator-ordering ambiguity, the UV divergences and so on.

This picture drastically changes in string theory. Consider a discretization of the worldsheet coordinates as regularization. The path integral for the action (11.54) reads as

$$Z_{\text{string}} \sim \int \mathcal{D}X \exp \left\{ \frac{i}{l_s^2} \sum \epsilon^2 \left[ \left( \frac{\delta_\tau X}{\epsilon} \right)^2 + \left( \frac{\delta_\sigma X}{\epsilon} \right)^2 \right] \right\}, \quad (11.58)$$

in which the  $\epsilon^2$  factor in front of the square bracket arises since the surface we are regularizing (i.e. the worldsheet) is two-dimensional. This way, analogously to Eq. (11.57), the resolution in string theory reads as

$$\langle(\delta X)^2\rangle \sim l_s^2. \quad (11.59)$$

The  $\epsilon$  factor is ruled out and, no matter how finely the worldsheet is discretized ( $\epsilon \rightarrow 0$ ), the geodesic distance between two adjacent points remains  $l_s$ . The appearance of the minimal length  $l_s$  is the direct consequence of dealing with two-dimensional objects.

The presence of such a scale modifies the Heisenberg uncertainty relations. In particular, the ordinary uncertainty relation is replaced by the GUP one given by (see Fig. 11.1)

$$\Delta q \gtrsim \frac{1}{2} \left( \frac{1}{\Delta p} + l_s^2 \Delta p \right). \quad (11.60)$$

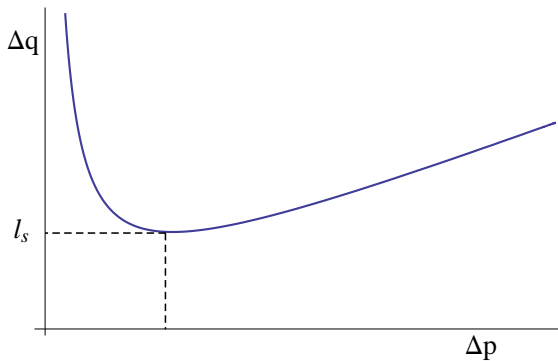


Figure 11.1 Plot of the uncertainty  $\Delta q$  vs.  $\Delta p$  (Eq. (11.60)), showing the existence of a minimum uncertainty for the  $q$  variable.

We conclude by stressing that an ultra-violet/infra-red divergence mixing is explicitly manifest in the relation (11.60). When the uncertainty in momentum  $\Delta p$  is large (UV), the uncertainty in the position  $\Delta x$  is proportional to  $\Delta p$  and therefore is also large (IR). This is an evidence that, in string theory, the physics at short distances (in contrast to local quantum field theory) is not clearly separated from the physics at large scales.

### 11.5 Heisenberg Algebras in Non-Commutative Snyder Space-Time

The analysis of non-commutative space-time geometries has recently attracted a significant attention by the theoretical physics community. Such theories are in fact believed to be candidates for an effective limit of quantum gravity as soon as the degrees of freedom of gravity are integrated out. This way, the only remaining trace of (quantum) gravity should be the presence of an energy scale (related to the Planck one). Such scale is the responsible for the non-commutativity of the space-time coordinates. This intuition has been corroborated by different results. In particular, a non-commutative space-time exactly arises as the flat limit of a three-dimensional quantum gravity model; furthermore, non-commutative geometries have a natural connection with string theory and another motivation comes from the so-called *doubly special relativity*, which corresponds to the framework of special relativity where two observer-independent scales are taken into account. The first quantity is obviously the speed of light, while the second one is an energy scale coming out from a (quantum) gravity reminiscence and the presence of two scales justifies the definition “doubly”. Interestingly, the space-time underlying such an extended (often also called deformed) special relativity is non-commutative.

In this Section we will consider the Snyder space-time, a particular type of non-commutative space-time geometry, briefly describing the modifications induced on the Heisenberg uncertainty relation. The non-commutativity of the Snyder space-time is encoded in the commutators between the coordinates. These are proportional to the (undeformed) Lorentz generators and the (algebraic) Poincaré symmetry underlying this space-time is undeformed, mentioning only that the symmetry deformations appear at the co-algebraic sector level only. Such a geometry will be implemented in quantum cosmology in Sec. 12.5, showing a quantum Big Bounce for the Universe dynamics.

Let us consider an  $n$ -dimensional non-commutative (deformed) Minkowski space-time such that the commutators between the coordinates have the non-trivial structure ( $\{i, j, \dots\} \in \{0, \dots, n\}$ )

$$[\tilde{q}_i, \tilde{q}_j] = s M_{ij}, \quad (11.61)$$

where  $\tilde{q}_i$  denote the non-commutative coordinates and  $s \in \mathbb{R}$  is the deformation parameter with dimensions of a squared length. For  $s = 0$  we recover the ordinary Minkowski framework. Let us demand that the symmetries of such a space are described by an undeformed Poincaré algebra

and therefore the Lorentz generators

$$M_{ij} = -M_{ji} = i(q_i p_j - q_j p_i) \quad (11.62)$$

satisfy the ordinary  $SO(n, 1)$  algebra

$$[M_{ij}, M_{kl}] = \eta_{jk} M_{il} - \eta_{ik} M_{jl} - \eta_{jl} M_{ik} + \eta_{il} M_{jk} \quad (11.63)$$

and the translation group is not deformed, i.e.

$$[p_i, p_j] = 0. \quad (11.64)$$

We also assume that the momenta  $p_i$  and the non-commutative coordinates  $\tilde{q}_i$  transform as undeformed vectors under the Lorentz algebra, i.e. the commutators

$$[M_{ij}, \tilde{q}_k] = \eta_{jk} \tilde{q}_i - \eta_{ik} \tilde{q}_j, \quad (11.65)$$

$$[M_{ij}, p_k] = \eta_{jk} p_i - \eta_{ik} p_j \quad (11.66)$$

hold. The quantity  $p^2 = \eta^{ij} p_i p_j$  is then a Lorentz invariant.

The relations (11.61)-(11.66) define the non-commutative Snyder space-time geometry but, however, they do not uniquely fix the commutators between  $\tilde{q}_i$  and  $p_j$ . In fact, there are infinitely many commutators which are all compatible (in the sense that the algebra closes by virtue of the Jacobi identities) with the above natural requirements. This feature can be understood by means of the concept of *realization*.

A realization on a non-commutative space is defined as the rescaling of the deformed coordinates  $\tilde{q}_i$  in terms of the ordinary phase space variables  $(q_i, p_j)$  as

$$\tilde{q}_i = \Phi_{ij}(p) q_j. \quad (11.67)$$

The most general  $SO(n, 1)$  covariant realization for  $\tilde{q}_i$  is given by

$$\tilde{q}_i = q_i \varphi_1(A) + s (q_j p_j) p_i \varphi_2(A), \quad (11.68)$$

in which  $\varphi_1$  and  $\varphi_2$  are two functions of the dimensionless quantity  $A = sp^2$ . The boundary conditions one has to impose in order to recover the commutative framework read as

$$\varphi_1(s=0) = 1. \quad (11.69)$$

The rescaling (11.68) depends on the adopted algebraic structure, but the two functions  $\varphi_1$  and  $\varphi_2$  are not uniquely fixed. Indeed, given any function  $\varphi_1$  satisfying (11.69), the function  $\varphi_2$  is determined by inserting (11.68) into the commutator (11.61), that is

$$\varphi_2 = \frac{1 + 2\dot{\varphi}_1 \varphi_1}{\varphi_1 - 2A\dot{\varphi}_1}, \quad (11.70)$$

where the dot denotes differentiation with respect to  $A$ . On the other hand, the realization (11.68) inserted into Eq. (11.65) provides an identity and therefore only a single condition on  $\varphi_1$  and  $\varphi_2$  is required. The generic realization (11.68) is completely specified by the function  $\varphi_1$  and there are infinitely many ways to express, via  $\varphi_1$ , the non-commutative coordinates (11.61) in terms of the ordinary ones without deforming the original symmetry.

The commutator between  $\tilde{q}_i$  and  $p_j$  arises from the realization (11.68) and can be expressed as

$$[\tilde{q}_i, p_j] = i(\delta_{ij}\varphi_1 + s p_i p_j \varphi_2). \quad (11.71)$$

This relation describes a deformed Heisenberg algebra and from it we obtain the (generalized) uncertainty principle underlying the Snyder geometry as

$$\Delta\tilde{q}_i \Delta p_j \geq \frac{1}{2} |\delta_{ij}\langle\varphi_1\rangle + s\langle p_i p_j \varphi_2\rangle|. \quad (11.72)$$

The ordinary framework is recovered in the  $s \rightarrow 0$  limit. In other words, the deformation of the unique commutator between the spatial coordinates defined in Eq. (11.61) leads to infinitely many realizations of the algebra, and thus of generalized uncertainty relations (11.72), all of them consistent with the assumptions underlying the model. Notice that, unless  $\varphi_2 = 0$ , compatible observables no longer exist but they are coupled to each other and an exactly simultaneous measurable couple  $(\tilde{q}_i, p_j)$  is no longer allowed. A measure of the  $i$ -component of the (non-commutative) position will always affect a measure of the  $j(\neq i)$ -component of the momentum by an amount  $\Delta p_j \sim |s\langle p_i p_j \varphi_2\rangle|/\Delta\tilde{q}_i$ .

Let us now consider the Euclidean subspace of the Snyder geometry, i.e. a Snyder space. This case can be recovered from the above framework by considering  $SO(n)$  generators (instead of the Lorentz ones), deformed coordinates and momenta invariant under rotations and the Euclidean metric instead of the Lorentzian one. Also in this case infinitely many realizations of such non-commutative geometry exist. However, for one-dimensional mechanical systems, this picture is (almost) uniquely fixed. In this case the symmetry group is trivial ( $SO(1) = \mathbb{I}$ ) and the most general realization is given by

$$\tilde{q} = q\varphi(A) = q\sqrt{1 - sp^2}. \quad (11.73)$$

The commutation relation (11.71) then reduces to

$$[\tilde{q}, p] = i\sqrt{1 - sp^2} \quad (11.74)$$

and the only freedom relies on the sign of the deformation parameter  $s$ . In such a case, the uncertainty relation (11.72) rewrites as

$$\Delta\tilde{q}\Delta p \geq \frac{1}{2}|\langle\sqrt{1-sp^2}\rangle| \quad (11.75)$$

and, if  $s < 0$ , the minimal observable length

$$\Delta\tilde{q}_{\min} = \frac{\sqrt{-s}}{2} \quad (11.76)$$

is predicted. On the other hand, if  $s > 0$ , a natural cut-off on the momentum arises given by  $|p| < \sqrt{1/s}$ .

Expanding inequality (11.75) to first order in  $s$ , the generalized uncertainty relation predicted by String Theory (11.60) holds. In this case, the string length  $l_s$  is related to the deformation parameter  $s$  by the relation  $l_s = \sqrt{-s/2}$  (thus  $s < 0$ ). On the other hand, if  $s > 0$ , a zero uncertainty in the non-commutative coordinate appears as soon as  $\Delta p = \sqrt{(1-s\langle p \rangle)/s}$ . We can conclude that a maximum momentum or a minimal length are predicted by the Snyder-deformed commutator (11.74) if  $s > 0$  or  $s < 0$ , respectively.

## 11.6 Quantum Mechanics in the GUP Framework

Let us now discuss some aspects and results of a non-relativistic quantum mechanics with non-zero minimal uncertainty in position. In one dimension, we consider the Heisenberg algebra generated by  $q$  and  $p$  obeying the commutation relations

$$[q, p] = i(1 + sp^2), \quad (11.77)$$

where  $s > 0$  is a deformation parameter. This commutator can be seen as the first order term of the Snyder relation<sup>7</sup> (11.74). The commutation relations (11.77) lead to the uncertainty relation

$$\Delta q \Delta p \geq \frac{1}{2} (1 + s(\Delta p)^2 + s\langle p \rangle^2), \quad (11.78)$$

i.e. the same form as for String Theory (11.60). The canonical Heisenberg algebra can be recovered in the limit  $s = 0$  and the generalization to higher dimensions will be discussed below. We start our analysis from the modified algebra (11.77) and thus we do not take the deformation parameter  $s$  to be directly related to the string scale  $l_s$ .

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<sup>7</sup>The deformation parameter used in this Section is minus one half of the one of Eq. (11.74). For convenience, we denote it again with  $s$ .

The generalized uncertainty principle (11.78) implies a finite minimal uncertainty in position as

$$\Delta q_{\min} = \sqrt{s} \quad (11.79)$$

which is considered as a fundamental minimal scale of the theory.

Let us now consider the quantization of this system. The existence of a non-zero uncertainty in position implies some relevant differences with respect to the ordinary quantum theory. In particular, the physical states that correspond to position eigenstates are no longer allowed. In fact, an eigenstate of an observable necessarily has vanishing uncertainty. To be more precise, let us assume the commutation relations to be represented on some dense domain  $D \subset \mathcal{F}$ ,  $\mathcal{F}$  being a Hilbert space. In the ordinary case, it is always possible to find a sequence of physical states  $|\psi_n\rangle \in D$  with position uncertainties decreasing to zero. Therefore, the position eigenstates can usually be approximated by arbitrary precision by  $|\psi_n\rangle$ . On the other hand, in the presence of a minimal uncertainty  $\Delta q_{\min} > 0$ , it is not possible any more to find some  $|\psi_n\rangle \in D$  such that

$$\lim_{n \rightarrow \infty} (\Delta q_{\min})_{|\psi_n\rangle} = \lim_{n \rightarrow \infty} \langle \psi | (q - \langle q \rangle)^2 | \psi \rangle = 0. \quad (11.80)$$

Thus, although it is possible to construct position eigenvectors, they are only formal eigenvectors and not physical states. This feature comes out from the corrections to the canonical commutation relation (11.77). In the GUP framework we have lost direct information on the position itself and, in particular, we cannot directly work in the configuration space but a notion of *quasiposition* has to be introduced. Notice that, in general, a non-commutativity of the coordinates does not imply a finite minimal uncertainty in position (see discussion in Sec. 11.5).

The algebra (11.77) can be represented in the momentum space, where the  $\hat{q}$ ,  $\hat{p}$  operators act as

$$\begin{aligned} \hat{p} \psi(p) &= p \psi(p), \\ \hat{q} \psi(p) &= i(1 + sp^2) \partial_p \psi(p), \end{aligned} \quad (11.81)$$

on a dense domain  $S$  of smooth functions. The measure in the scalar product, with respect to which the operators  $q$  and  $p$  are symmetric (namely  $(\langle \psi | q) | \phi \rangle = \langle \psi | (q | \phi \rangle)$  and  $(\langle \psi | p) | \phi \rangle = \langle \psi | (p | \phi \rangle)$ ), is deformed and, in particular, the scalar product reads as

$$\langle \psi | \phi \rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + sp^2} \psi^\dagger(p) \phi(p). \quad (11.82)$$



Information on the position can be recovered by studying the states which realize the maximally-allowed localization. The states of maximal localization  $|\psi_\zeta^{ml}\rangle$ , which are proper physical states around a position  $\zeta$ , satisfy the properties

$$\langle \psi_\zeta^{ml} | q | \psi_\zeta^{ml} \rangle = \zeta, \quad (\Delta q)_{|\psi_\zeta^{ml}\rangle} = \Delta q_{\min} \quad (11.83)$$

and are called *maximal localization states* because they obey the minimal uncertainty condition

$$\Delta q \Delta p = \frac{|\langle [q, p] \rangle|}{2}. \quad (11.84)$$

Therefore the following equation holds

$$\left( q - \langle q \rangle + \frac{\langle [q, p] \rangle}{2(\Delta p)^2} (p - \langle p \rangle) \right) |\psi_\zeta^{ml}\rangle = 0, \quad (11.85)$$

whose solution, in the momentum space, is given by<sup>8</sup>

$$\psi_\zeta^{ml}(p) \propto \frac{1}{(1 + sp^2)^{1/2}} \exp \left( -i \frac{\zeta}{\sqrt{s}} \arctan(\sqrt{s} p) \right), \quad (11.86)$$

where the proportionality factor is given by a normalization constant. As we can see, these states reduce, in the  $s = 0$  case, to the ordinary plane waves. However, differently from the canonical case, the states (11.86) are normalizable and their scalar product is a function rather than the Dirac distribution.

Furthermore, we can project an arbitrary state  $|\psi\rangle$  on the maximally localized states  $|\psi_\zeta^{ml}\rangle$ , giving the probability amplitude for a particle to be maximally localized around the position  $\zeta$ , i.e. with standard deviation  $\Delta q_{\min}$ . We call these projections the *quasiposition wave functions*  $\psi(\zeta) \equiv \langle \psi_\zeta^{ml} | \psi \rangle$ , explicitly given by

$$\psi(\zeta) \propto \int_{-\infty}^{+\infty} \frac{dp}{(1 + sp^2)^{3/2}} \exp \left( i \frac{\zeta}{\sqrt{s}} \arctan(\sqrt{s} p) \right) \psi(p). \quad (11.87)$$

This is nothing but a generalized Fourier transformation and in the  $s = 0$  limit the ordinary position wave function  $\psi(\zeta) = \langle \zeta | \psi \rangle$  is recovered.

As last point we will analyze the generalization of relation (11.77) to  $n$  spatial dimensions. Let us consider the following generalization

$$[q_i, p_j] = i\delta_{ij}(1 + sp^2) + is'p_i p_j, \quad p^2 = p_i p^i, \quad (11.88)$$

$s' > 0$  being a new parameter. Furthermore, assuming that the translation group is not deformed ( $[p_i, p_j] = 0$ ) the commutation relations among the

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<sup>8</sup>The absolutely minimal uncertainty in position reads as  $\Delta q_{\min} = \sqrt{s}$ , and then the maximal localization states are obtained for  $\langle p \rangle = 0$ .

coordinates are almost uniquely determined by the Jacobi identity. The deformed classical dynamics is summarized in the modified symplectic geometry arising from the classical limit of the quantum-mechanical commutators, as soon as the parameters  $s$  and  $s'$  are regarded as independent constants with respect to  $\hbar$ . It is possible to replace the quantum-mechanical commutators  $-i[\cdot, \cdot]$  via the Poisson bracket  $\{\cdot, \cdot\}$ , thus dealing with a phase space algebra given by the commutators

$$\begin{aligned}\{q_i, p_j\} &= \delta_{ij}(1 + sp^2) + s'p_i p_j, \\ \{p_i, p_j\} &= 0, \\ \{q_i, q_j\} &= \frac{(2s - s') + (2s + s')sp^2}{1 + sp^2}(p_i q_j - p_j q_i).\end{aligned}\tag{11.89}$$

From a String Theory point of view, keeping the parameters  $s$  and  $s'$  fixed as  $\hbar \rightarrow 0$  corresponds to maintaining the string momentum scale fixed as the string length scale shrinks to zero.

The GUP Poisson brackets are immediately obtained from Eq. (11.89) and, for any phase space function, they read as

$$\{F, G\} = \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_j} \right) \{q_i, p_j\} + \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial q_j} \{q_i, q_j\}.\tag{11.90}$$

It is worth noting that, for  $s' = 2s$ , the coordinates  $q_i$  become commutative up to higher order corrections, that is

$$\{q_i, q_j\} = 0 + \mathcal{O}(s^2).\tag{11.91}$$

This can be considered a preferred choice of the parameters. However, although terms like  $\mathcal{O}(s^2)$  can be neglected, the case in which  $sp^2 \gg 1$  is allowed due to the absence of restrictions on the  $p$ -domain, i.e.  $p \in \mathbb{R}$ .

Let us conclude by discussing the application of the deformed Heisenberg algebras in quantum cosmology (see Sec. 12.5, Sec. 12.6 and Sec. 12.7). As we have seen in Sec. 10.3, in the minisuperspace theory only a finite number of the gravitational degrees of freedom are invoked at quantum level (the remaining are set to zero imposing some symmetry on the spatial metric). In this respect, the implementation of deformed Heisenberg algebras in quantum cosmology seems to be physically grounded, also in the light of the analogy existing between the point Universe in the minisuperspace and a relativistic particle moving on a curved background. Indeed, the GUP scheme relies on a modification to the canonical quantization prescriptions and thus can be reliably applied to any dynamical system.

## 11.7 Guidelines to the Literature

The algebraic approach to quantum physics presented in Sec. 11.1 is developed in the textbook of Haag [214]; for an introduction to the basic aspects, see Wald [457]. A good textbook on  $C^*$ -algebra theory, is for example, that of Brattelli & Robinson [102], while a pedagogical introduction is given in the review [302]. The uniqueness theorem of quantum mechanics, the GNS construction and the Fell theorem are described in the textbooks [214, 457].

The polymer quantum mechanics analyzed in Sec. 11.2 has been proposed for example in [56, 115]; recent developments can be found in [26, 128, 129, 174].

A clear review on a minimal length in quantum gravity (Sec. 11.3) is [190].

A simple exposition about the prediction of a minimal length in String theory, as discussed in Sec. 11.4, can be found in [8, 268, 292, 447].

The Snyder non-commutative space-time (Sec. 11.5) has been proposed in [424] and developed in [35, 50, 51, 209]. A relation between quantum gravity and non-commutative geometry is in [175]. The connection with String theory has been formulated in [411]. Doubly special relativity has been proposed by Amelino-Camelia in [9, 10]; see also [333]. For a recent review, see [11].

The quantum mechanics in a GUP framework, discussed in Sec. 11.6, has been studied in [36, 103, 269–271]. Generalizations to more spatial dimensions are in [68, 117].

## Chapter 12

# Modern Quantum Cosmology

This Chapter is devoted to the analysis of modern approaches to quantum cosmology developed in the last ten years. Motivated by the Ashtekar discovery of a gauge-like formulation of GR, a great improvement of the Wheeler-de Witt quantum theory of gravity has been pursued. This work culminated in the Loop Quantum Gravity theory of which Loop Quantum Cosmology is the main application. In such framework the classical Big Bang singularity is replaced by a quantum Big Bounce opening the way for a *Planck scale physics*. Notably, the chaotic dynamics of the Mixmaster Universe is tamed by loop quantum effects and a relation with non-commutative geometries can be outlined as well.

In the first part of the Chapter, we address the loop approach to quantum cosmology, while in the second one we discuss non-commutative inspired quantum cosmological models.

As a starting point, we analyze Loop Quantum Gravity posing our attention on the novel tools, developing in some details the basic steps in the construction of a kinematic Hilbert space based on the holonomy-flux algebra and the prediction of a granular structure of space. A particular attention is also devoted to the way the new constraints are imposed at a quantum level, as well as to the closure of the quantum constraints algebra in view of an anomaly free construction.

The implementation of the loop techniques in the minisuperspace arena leads to a new quantum cosmology theory: Loop Quantum Cosmology. While its predictions are very close to those of the old quantum geometrodynamics theory in the low curvature regime, there is a drastic difference once the energy density of the Universe approaches the Planck scale: a Big Bounce is predicted by means of repulsive quantum geometrical effects. We analyze in details this theory in the isotropic (FRW models) and

anisotropic (Bianchi IX) regimes. The problems regarding the relation of Loop Quantum Cosmology with the full theory are also investigated.

We then discuss a very recent approach to quantum cosmology whose aim is to relate Loop Quantum Gravity with its minisuperspace theory. This approach, known as Triangulated Loop Quantum Cosmology, is based on a truncation of Loop Quantum Gravity down to a graph with a finite number of links. The simplest choice is based on a “dipole” graph formed by two nodes connected by four links. This framework determines a Hilbert space which describes the Bianchi IX Universe plus few inhomogeneous degrees of freedom. The dynamics of Loop Quantum Cosmology is there recovered without heuristic arguments.

The recent interest in non-commutative space-time geometries as a “low energy limit” of quantum gravity has suggested their implementation in primordial cosmology. We analyze a quantum cosmological model based on a particular type of non-commutative space-time in which the commutators between the four-coordinates are proportional to the Lorentz generators. In this specific framework, a quantum cosmological bounce *à la* Loop Quantum Cosmology takes place, setting a bridge between these two different theories.

Non-commutative geometries are naturally related to generalizations of the uncertainty principle. In fact, the underlying Heisenberg algebra is deformed and results in a modification of the uncertainty relations. We then discuss the implementation of this scheme towards the Taub Universe as well as towards the Bianchi IX model. In particular, the Taub Universe offers a nice framework to contrast the polymer loop quantization scheme with the one behind deformed algebras. The removal of the cosmological singularity appears to be dependent on the adopted quantization scheme and also the choice of the variables to be quantized plays a crucial role.

## 12.1 Loop Quantum Gravity

In this Section we will analyze some of the basic aspects of Loop Quantum Gravity (LQG) in a pedagogical manner for a non-expert reader.

LQG is a conceptually clear and mathematically rigorous attempt to quantize GR in a background independent scheme. It aims to define a quantum field theory just on a differential manifold  $\mathcal{M}$  and not on a background space-time  $(\mathcal{M}, g_0)$ , i.e. independently of the choice of a fixed background metric  $g_0$ . In this respect, LQG follows an approach opposite to that of String Theory. In fact, in String Theory a target space background metric

(mainly Minkowski or Anti de Sitter) is fixed and strings propagate on it.

The construction of a quantum field theory in the absence of a fixed background (namely a diffeomorphism invariant QFT) is a highly non-trivial task. We remind that the construction of the ordinary QFT is entirely based on the Wightman axioms, which are in turn based on a *fixed* background metric  $g_0$ , usually the Minkowski one. Such fixed background structure implies a preferred notion of causality (locality) and the Poincaré symmetry group. The only quantum field theories in four dimensions that are fully understood are the ones related to the behavior of free or perturbatively interacting fields. On the other hand, in GR the metric itself is a dynamical entity which has to be quantized and there is not anymore a background on which (quantum) physics happens.

LQG is the most advanced implementation of canonical quantum gravity and it is able to overcome some of the problems of the previous geometrodynamical approach (described in Chap. 10) by adopting new conceptual and technical ingredients. The main result of LQG is the discreteness of the spectrum of geometric operators like as the area and volume. The quantum three-geometry can be viewed as composed by quanta of space.

The starting point of LQG is the Hamiltonian formulation of GR in terms of connection variables (see Sec. 2.6). The Einstein theory, in this framework, has the form of a background independent  $SU(2)$  Yang-Mills theory. The new phase space is endowed with an  $SU(2)$  connection  $A_\alpha^a$  and its conjugate (gravitational electric) field  $E_a^\alpha$ . The symplectic geometry is determined by the only non-trivial Poisson brackets

$$\{A_\alpha^a(x, t), E_b^\beta(x', t)\} = \kappa \delta_b^a \delta_\alpha^\beta \delta^3(x - x'). \quad (12.1)$$

The diffeomorphism (vector) and scalar constraints of the theory (2.77) are rewritten in terms of these new variables respectively as in Eq. (2.139)

$$\mathcal{H}_\alpha = E_a^\beta F_{\alpha\beta}^a = 0 \quad (12.2)$$

$$\mathcal{H} = \frac{1}{2\sqrt{|h|}} E_a^\alpha E_b^\beta \left( \epsilon^{ab}{}_c F_{\alpha\beta}^c - 2(1 + \gamma^2) K_{[\alpha}^a K_{\beta]}^b \right) = 0, \quad (12.3)$$

where the parameter  $\gamma$  is the Immirzi parameter. In comparison with the metric approach, the additional Gauss constraint (2.138) appearing in the connection formalism

$$G_a = \mathcal{D}_\alpha E_a^\alpha = \partial_\alpha E_a^\alpha - \gamma \epsilon_{ab}{}^c A_\alpha^b E_c^\alpha = 0 \quad (12.4)$$

holds in addition to Eqs. (12.2). It gets rid of the  $SU(2)$  degrees of freedom. These seven constraints arise from requiring a theory with a whole gauge of

freedom and that any observable (a gauge invariant phase space function) has to commute with all these quantities. The theory is now described (in the phase space) by 18 variables ( $A_\alpha^a, E_b^\beta$ ) subjected to seven first-class constraints, each of them eliminating two degrees of freedom. The four phase space degrees of freedom of the gravitational field are then recovered.

Let us now summarize the conceptual path of the canonical quantization of gravity *à la* Dirac, consisting basically of four steps:

- a) Find a representation of a set of phase space functions, generating a Poisson algebra, as operators in a kinematic Hilbert space  $\mathcal{F}_{\text{kin}}$ .
- b) Implement the constraints, here the seven ones in Eqs. (12.2)-(12.4), as (self-adjoint) operators in  $\mathcal{F}_{\text{kin}}$ .
- c) Describe the space of solutions of the constraints and define the corresponding inner product. This determines the physical Hilbert space  $\mathcal{F}_{\text{phys}}$  allowing for a probabilistic interpretation.
- d) Find a complete set of gauge invariant observables.

The WDW theory (see Sec. 10.1) does not satisfy the first assessment (a) of the previous list, since no kinematic Hilbert space can be obtained there. The step (b) is solved only formally, and the remaining two points (c and d) are even not addressed. The main problem in the geometrodynamics approach can be drawn back to the choice of  $h_{\alpha\beta}$  and  $\Pi^{\gamma\delta}$  as basic variables of the theory. In fact, it is not possible to find a meaningful representation of these functionals which satisfies the constraints. LQG is able to overcome these limitations basically by using a more suitable phase space algebra.

### 12.1.1 Kinematics

A crucial step to construct a suitable Hilbert space for LQG is to appropriately smear the fields  $A_\alpha^a$  and  $E_b^\beta$ . This procedure is inspired by what is usually done in gauge theories. A natural quantity associated to the connections consists of the *holonomies* which will be regarded as the basic variables for the quantum theory. Given a curve on the three-dimensional surface  $\Sigma$ , i.e. an edge  $\ell$ , a holonomy  $h_\ell[A]$  is defined as

$$h_\ell[A] = \mathcal{P} \exp \left( \int_\ell A_\alpha^a \tau_a dx^\alpha \right). \quad (12.5)$$

Here  $\mathcal{P}$  denotes the path ordering and the (anti-hermitian) matrices  $\tau_a$  are generators of  $SU(2)$ , i.e. they satisfy the commutator relation

$$[\tau_a, \tau_b] = \epsilon_{abc} \tau^c. \quad (12.6)$$

This basis is related to the Pauli matrices  $\sigma_a$  by the equality

$$\tau_a = \frac{\sigma_a}{2i}. \quad (12.7)$$

Notice that  $\tau_a$  are proportional to the  $T_a$  used in Sec. 2.2.4 being  $\tau_a = -iT_a$ . The holonomies  $h_\ell$  are elements of  $SU(2)$  and define the parallel transport of the connection  $A_\alpha^a$  along the edge  $\ell$ . They contain  $SU(2)$  gauge invariant informations (their trace is gauge invariant) of the connection in the restriction to a graph  $\Gamma$  (see below). It is worth noting that they have a one-dimensional support rather than being smeared over all  $\Sigma$ . Taking the trace of the holonomy (12.5) for a closed edge leads to the so-called Wilson loop which is at the ground of LQG and its name.

On the other hand, the densitized triad  $E_a^\alpha$  induces an  $SU(2)$ -valued two form  $E_a^\alpha \epsilon_{\alpha\beta\gamma}$ . It is then natural to smear the electric field  $E_a^\alpha$  on a two-dimensional surface  $S \subset \Sigma$ . This defines the *flux* electric vector  $(P_S[E])_a$  as

$$P_S[E] = (P_S[E])_a \tau^a = \int_S E_a^\alpha \tau^a \epsilon_{\alpha\beta\gamma} dx^\beta \wedge dx^\gamma. \quad (12.8)$$

In order to compute the Poisson brackets between the functionals  $h_\ell[A]$  and  $(P_S[E])_a$ , let us consider a surface  $S$  and an edge  $\ell$  intersecting  $S$  in one point  $p$ . Then we divide the edge into the two sub-edges  $\ell_1$  and  $\ell_2$  such that  $p$  is the source of  $\ell_1$  and  $p$  is target of  $\ell_2$ . The associated holonomies are respectively denoted as  $h_{\ell_1}$  and  $h_{\ell_2}$  and lead to

$$\{h_\ell[A], (P_S[E])_a\} = \kappa\gamma \alpha(\ell, S) h_{\ell_1}[A] \tau_a h_{\ell_2}[A]. \quad (12.9)$$

In Eq. (12.9),  $\alpha = 0$  refers to the case of the edge not intersecting the surface and  $\alpha = \pm 1$  when the edge and surface orientation are the same or the opposite, respectively. We stress that the commutation relation (12.9) is non-canonical for the presence of  $h_\ell[A]$  on the right-hand side.

The quantum kinematics can be constructed by promoting such variables to quantum operators obeying appropriate commutation relations. The essential feature of LQG is to consider the holonomy (12.5) as the configuration variable of the theory. The holonomies  $h_\ell$  are thus promoted to operators rather than the connections  $A_\alpha^a$  themselves. The Poisson algebra of holonomies and fluxes is well defined and the resulting Hilbert space is *unique*. More precisely, requiring the three-diffeomorphism invariance (there must be a unitary action of such diffeomorphism group on the representation by moving edges and surfaces in space), there is a unique representation of the *holonomy-flux algebra* that defines the kinematic Hilbert



space  $\mathcal{F}_{\text{kin}}$ . This is an important theorem ensuring the self-consistency of the theory.

Let us investigate the kinematical Hilbert space  $\mathcal{F}_{\text{kin}}$ . This is known as the spin networks Hilbert space. Spin network states  $|S\rangle$ , expressed as

$$|S\rangle = |\Gamma, j_\ell, i_n\rangle, \quad (12.10)$$

are defined by three ingredients: (i) a graph  $\Gamma \subset \Sigma$  consisting of a finite number of edges  $\ell$  and nodes  $n$ ; (ii) a collection of spin quantum numbers  $j_\ell = 1/2, 1, 3/2, \dots$ , one for each edge; (iii) other quantum numbers  $i_n$ , the intertwiners, one for each node  $n$ . Spin networks are *colored graphs*. Notice that although spin networks are defined on a three-dimensional manifold (the edges and the nodes are in fact defined on  $\Sigma$ ), no physical metric is carried out. The holonomies defined in Eq. (12.5) are taken to transform in an  $SU(2)$  representation of arbitrary spin. Such spin  $j_\ell$ -valued holonomy is denoted as  $\rho_j(h_\ell[A])$ .

The wave functional on the spin network is thus given by

$$\Psi_{\Gamma, \psi}[A] = \psi(\rho_{j_1}(h_{\ell_1}[A]), \dots, \rho_{j_n}(h_{\ell_n}[A])) \quad (12.11)$$

where the wave function  $\psi$  is  $SU(2)$  gauge invariant and satisfies the Gauss constraint. More precisely, the function  $\psi$  joints the collection of holonomies (in the arbitrary spin representation) into an  $SU(2)$  invariant complex number by contracting all the gauge indices with the intertwiners, the latter being invariant tensors localized at each node. The states (12.11) are called cylindrical functionals since they have a one-dimensional support, i.e. they probe the  $SU(2)$  gauge connection on one-dimensional networks only. The space of these functions is called Cyl.

Let us now promote the fundamental variables of the theory ( $h_\ell[A], P_S[E]$ ) to quantum operators. Their action on the wave functions (12.11) is given by

$$\hat{h}_\ell[A] \Psi_{\Gamma, \psi}[A] = h_\ell[A] \Psi_{\Gamma, \psi}[A] \quad (12.12a)$$

$$(\widehat{P_S[E]})_a \Psi_{\Gamma, \psi}[A] = i\{(P_S[E])_a, \Psi_{\Gamma, \psi}[A]\}, \quad (12.12b)$$

where the relation (12.12) is defined by means of Eq. (12.9).

A key result in LQG is the construction of the kinematic scalar product between two cylindrical functions. Indeed, the discreteness of area and volume operators spectra, mainly based on the compactness of the  $SU(2)$  group, can be obtained from it. The kinematic scalar product is defined as

$$\langle \Psi_\Gamma | \Psi_{\Gamma'} \rangle = \begin{cases} 0 & \text{if } \Gamma \neq \Gamma' \\ \int \prod_{\ell \in \Gamma} dh_\ell \psi_\Gamma^\dagger(h_{\ell_1}, \dots) \psi_{\Gamma'}(h_{\ell_1}, \dots) & \text{if } \Gamma = \Gamma', \end{cases} \quad (12.13)$$

where the integrals  $\int h_\ell$  are performed with the  $SU(2)$  Haar measure. Such definition is based on the strong uniqueness theorem previously discussed. The inner product vanishes if the graphs  $\Gamma$  and  $\Gamma'$  do not coincide and it is invariant under spatial diffeomorphisms, even if the states  $\Psi_1$  and  $\Psi_2$  themselves are not. This happens because the coincidence between two graphs is a diffeomorphism invariant notion. The information about the position of the graphs, carried by the wave function, disappears in the scalar product (12.13). The Hilbert space  $\mathcal{F}_{\text{kin}}$  is the Cauchy completion of the space of the cylindric functions  $\text{Cyl}$  with respect to the above inner product.

Two remarks are appropriate:

- i) The Hilbert space obtained is not-separable as it does not admit a countable basis. The set of all spin networks is not numerable and two non-coincident spin networks are orthogonal with respect to the scalar product (12.13).
- ii) States with negative norm are absent without imposing the constraints (12.2) and (12.4). This is in contrast with the usual gauge theories where the negative norm states can be eliminated only after imposing the constraints.

Let us now briefly analyze the area operator in LQG (the construction of the volume operator is conceptually the same and it will not be discussed here). The idea is to construct the areas as functions of basic variables (holonomies and fluxes) and then promote them into operators. The area element  $dA_a$  of a surface  $S \subset \Sigma$  is expressed in terms of tetradic vectors  $e^a = e^a_\alpha dx^\alpha$  as

$$dA_a = \frac{1}{2} \epsilon_{abc} e^b \wedge e^c = \frac{1}{2} \epsilon_{abc} e^b_\beta e^c_\gamma dx^\beta \wedge dx^\gamma = \epsilon_{\alpha\beta\gamma} E^\alpha_a dx^\beta \wedge dx^\gamma, \quad (12.14)$$

where in the last step we have used the definition (2.133). The area of  $S$  is then given by

$$A_S = \int_S \sqrt{dA_a dA^a}, \quad (12.15)$$

and the surface can be divided in  $N$  cells  $S_I$  such that  $S = \cup_I S_I$ . We can then express the area as the limit of a sum, explicitly as

$$A_S = \lim_{N \rightarrow \infty} A_S^N, \quad A_S^N = \sum_I \sqrt{(P_{S_I}[E])_a (P_{S_I}[E])^a}, \quad (12.16)$$

where  $(P_{S_I}[E])_a$  is the flux (12.8) of  $E^\alpha_a$  through the  $I$ -th cell. Such expression can be meaningfully quantized and the area operator can be written

as

$$\hat{A}_S = \lim_{N \rightarrow \infty} \hat{A}_S^N. \quad (12.17)$$

It is well defined because the operator associated to the flux exists in the Hilbert space  $\mathcal{F}_{\text{kin}}$ . It is not difficult to show that, using the relations (12.12) and (12.9), the action of the operator  $\widehat{(P_{S_I})_a} \widehat{(P_{S_I})^a}$  on  $\rho_j(h_\ell[A])$  is given by

$$\widehat{(P_{S_I})_a} \widehat{(P_{S_I})^a} \rho_j(h_\ell) = i^2 (\kappa\gamma)^2 (h_{\ell_1} \tau_a \tau^a h_{\ell_2})_j = (\kappa\gamma)^2 j(j+1) \rho_j(h_\ell). \quad (12.18)$$

In the first step we have assumed that the edge of the holonomy crosses  $S_I$  only once ( $\ell$  is decomposed into  $\ell_1$  and  $\ell_2$ ), while the unitary irreducible representation of  $SU(2)$  has been used in the second one<sup>1</sup>. The spin network states are then eigenstates of the quantum area operator and we have

$$\hat{A}_S \Psi = \kappa\gamma \sum_p \sqrt{j_p(j_p+1)} \Psi, \quad (12.19)$$

where  $p$  punctures in the  $I$ -th cell are considered. The minimal accessible length appears to be of the Planck order and the spectrum is clearly discrete. Notice that the spectrum depends on the Immirzi parameter  $\gamma$ .

### 12.1.2 Implementation of the constraints

Let us now discuss the implementation of the seven constraints of the theory as operators. To deal with the constraints at a quantum level, one needs to follow two steps: (i) to express them in terms of holonomies and fluxes, and (ii) to investigate their properties.

The Gauss constraint (12.4) is the most simple one and it is already implemented by the construction of the cylindrical functionals (12.11), requiring the  $SU(2)$  gauge invariance of the spin network states. This invariance is achieved by contracting the representation indices of a given node in an  $SU(2)$  invariant manner. This way, the total (quantum) flux of the gravitational electric field vanishes. The Hilbert space of  $SU(2)$  gauge invariant states is given by  $\mathcal{F}_{\text{kin}}^G \subset \mathcal{F}_{\text{kin}}$ .

The diffeomorphism constraint  $\mathcal{H}_\alpha = 0$  is more delicate to deal with, because it cannot be treated in operatorial terms. The main difference with respect to the WDW approach is that spin networks are supported on one-dimensional space and not on all  $\Sigma$ . Diffeomorphism generators do not exist as operators and diffeomorphism invariant states (with the exception

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<sup>1</sup>Notice that  $\tau_a$  are anti-hermitian matrices so the Casimir reads  $\tau_a \tau^a = -j(j+1)$ .

of the empty spin network state  $\Psi = 1$ ) do not exist in  $\mathcal{F}_{\text{kin}}$ . This feature is due to the non-compactness of the gauge orbits of the diffeomorphism group. This constraint is imposed implementing a *group averaging method* in a way properly adapted to the scalar product (12.13). A key step arises from solving the constraint on a larger space, i.e. the dual  $\text{Cyl}^*$  of the cylindric functions  $\text{Cyl}$ , such that

$$\text{Cyl} \subset \mathcal{F}_{\text{kin}} \subset \text{Cyl}^*. \quad (12.20)$$

This is the so-called Gelfand triplet and the dual space is the space of the distribution.

$\text{Cyl}^*$  can be characterized as follows. Given any operator  $O \in \text{Cyl}$  and its adjoint  $O^\dagger \in \text{Cyl}$ , the action of its dual  $O^* \in \text{Cyl}^*$  on any element  $\mathcal{X} \in \text{Cyl}^*$  is given by

$$\langle O^* \mathcal{X} | \Psi \rangle = \langle \mathcal{X} | O^\dagger \Psi \rangle, \quad \forall \Psi \in \text{Cyl}. \quad (12.21)$$

A diffeomorphism invariant state is then obtained by averaging  $\Psi_\Gamma \in \text{Cyl}$  over the diffeomorphism group as

$$\eta(\Psi_\Gamma) = \sum_{\phi \in \text{Diff}(\Sigma/\Gamma)} \phi^* \circ \Psi_\Gamma, \quad (12.22)$$

where the diffeomorphisms  $\phi$  are in  $\text{Diff}(\Sigma)$  modulo those leaving invariant the graph  $\Gamma$ . From the inner product (12.13), it turns out that only a finite number of terms contributes to the kinematical scalar product between a spin network state and a diffeomorphisms invariant one. This way, a state  $\eta(\Psi_\Gamma)$  is well-defined making sense as a distribution living outside of  $\mathcal{F}_{\text{kin}}$ . We denote the space of the diffeomorphisms invariant distributions as  $\text{Cyl}_{\text{diff}}^*$ . The scalar product between two diffeomorphism invariant states is thus given by

$$\langle \eta(\Psi_{\Gamma'}) | \eta(\Psi_\Gamma) \rangle_{\text{diff}} = \langle \eta(\Psi_{\Gamma'}) | \Psi_\Gamma \rangle. \quad (12.23)$$

The diffeomorphism invariant Hilbert space  $\mathcal{F}_{\text{diff}}$  is obtained by completion with respect to this norm, i.e. it is an averaged version of  $\mathcal{F}_{\text{kin}}$ .

Let us point out two remarks.

- i) The new Hilbert space  $\mathcal{F}_{\text{diff}}$  is separable (differently from  $\mathcal{F}_{\text{kin}}$ ) and it is a subspace of  $\text{Cyl}^*$ .
- ii) The elements of  $\mathcal{F}_{\text{kin}}$  related by a diffeomorphism are mapped to the same state of  $\text{Cyl}^*$ .

The final challenge is to find a space annihilated by all the constraints and defining a physical inner product which yields the final physical Hilbert space  $\mathcal{F}_{\text{phys}}$ . This step is accomplished by quantizing the scalar constraint (12.3) which encodes the dynamics of the theory. As mentioned in the WDW formalism, the main problem of all (canonical) quantum gravity approaches is to impose this constraint at a quantum level. Let us briefly sketch its construction which is mainly due to Thiemann.

Let us firstly rewrite the function (12.3) in terms of basic variables corresponding to well-defined quantum operators. This can be done using classical identities to express the triads, the extrinsic curvature and the field strength in terms of holonomies and well-defined operators like the volume. In particular, introducing the integral trace of the extrinsic curvature of  $\Sigma$  as

$$\bar{K} = \int_{\Sigma} d^3x \sqrt{|h|} K_{\alpha\beta} h^{\alpha\beta} = \int_{\Sigma} d^3x K_{\alpha}^a E_a^{\alpha}, \quad (12.24)$$

the smeared version of the (classical) scalar constraint turns out to be

$$\mathcal{H}(N) = \int_{\Sigma} d^3x N \epsilon^{\alpha\beta\gamma} \left( \delta_{ab} F_{\alpha\beta}^a \{A_{\gamma}^b, V\} - 2(1 + \gamma^2) \frac{\epsilon_{abc}}{\kappa^2 \gamma^3} \{A_{\alpha}^a, \{\mathcal{H}^E(1), V\}\} \{A_{\beta}^b, \{\mathcal{H}^E(1), V\}\} \{A_{\gamma}^c, V\} \right). \quad (12.25)$$

Here

$$V = \int_{\Sigma} d^3x \sqrt{|h|} = \int_{\Sigma} d^3x \sqrt{\frac{1}{3!} |\epsilon_{\alpha\beta\gamma} \epsilon^{abc} E_a^{\alpha} E_b^{\beta} E_c^{\gamma}|} \quad (12.26)$$

denotes the volume of  $\Sigma$  and  $\mathcal{H}^E(1)$  is the smeared Euclidean part of the scalar constraint (12.3) as  $N = 1$  defined as

$$\mathcal{H}^E(1) = \int_{\Sigma} d^3x \frac{E_a^{\alpha} E_b^{\beta}}{\sqrt{|h|}} \epsilon^{ab}{}_c F_{\alpha\beta}^c = \int_{\Sigma} d^3x \epsilon^{\alpha\beta\gamma} \delta_{ab} F_{\alpha\beta}^a \{A_{\gamma}^b, V\}. \quad (12.27)$$

Such formulation of the scalar constraint, containing only the connection and the volume, is the starting point for the quantization and, similarly to ordinary QFT, this is achieved in three steps:

- i) The constraint (12.25) has to be regularized (by a parameter  $\epsilon$ ).
- ii) The classical objects have to be promoted to operators and the Poisson brackets to commutators.
- iii) The regularization has to be removed at the end of the computations. This last step is highly non-trivial and poses serious challenges.

Given an infinitesimal loop  $P_{\alpha\beta}(\epsilon)$  with coordinate area  $\epsilon^2$ , both curvature  $F$  and Poisson brackets  $\{A, V\}$  can be regularized in terms of holonomies and volumes. This procedure leads to the regularized quantum super-Hamiltonian  $\hat{\mathcal{H}}(N, \epsilon)$  which is a sum over cells of volume  $\epsilon^3$  (these cells are centered on the nodes of the spin networks) and is given by

$$\hat{\mathcal{H}}(N, \epsilon) = \sum_I N_I \epsilon^{\alpha\beta\gamma} \text{Tr} \left\{ \left( \hat{h}_{P_{\alpha\beta}(\epsilon)} - \hat{h}_{P_{\alpha\beta}(\epsilon)}^{-1} \right) \hat{h}_\gamma^{-1} [\hat{h}_\gamma, \hat{V}] \right. \\ \left. - 2(1 + \gamma^2) \hat{h}_\alpha^{-1} [\hat{h}_\alpha, \bar{K}] \hat{h}_\beta^{-1} [\hat{h}_\beta, \bar{K}] \hat{h}_\gamma^{-1} [\hat{h}_\gamma, \hat{V}] \right\}, \quad (12.28)$$

where  $h_\alpha = h_{e_\alpha}^I$ . The essential piece in the expression above is the operator

$$\left( \hat{h}_{P_{\alpha\beta}(\epsilon)} - \hat{h}_{P_{\alpha\beta}(\epsilon)}^{-1} \right) \quad (12.29)$$

which defines the action of the scalar constraint on a spin network state. The quantum scalar constraint modifies the spin networks by *creating a plaquette*  $P(\epsilon)$  attached to a node.

The underlying graph is thus changed but two states supported on different networks are orthogonal by virtue of the scalar product (12.13). This feature is reflected as soon as the regulator  $\epsilon$  has to be removed. In particular, for any cylindrical function  $\Psi$ , the limit

$$\lim_{\epsilon \rightarrow 0} \hat{\mathcal{H}}(N, \epsilon) \Psi \quad (12.30)$$

does not exist in Cyl. Usually one transfers the action of the scalar constraint to the dual space, i.e. adopting a weaker notion of limit. More specifically, the limit  $\epsilon \rightarrow 0$  is defined by

$$\langle \hat{\mathcal{H}}^*(N) \mathcal{X} | \Psi \rangle = \lim_{\epsilon \rightarrow 0} \langle \mathcal{X} | \hat{\mathcal{H}}^\dagger(N, \epsilon) \Psi \rangle \quad (12.31)$$

for all  $\Psi \in \text{Cyl}$  and  $\mathcal{X} \in \text{Cyl}_{\text{diff}}^*$ . Such notion of convergence is mainly due to the spatial diffeomorphism invariance. In fact, the limit  $\epsilon \rightarrow 0$  is just a diffeomorphism and therefore, in the diffeomorphism invariant Hilbert space, it corresponds to a trivial operation. This can be considered as the *key point of LQG*, i.e. *a gauge theory is quantized in a diffeomorphism invariant way*.

Finally, we express three remarks on the quantum super-Hamiltonian constraint:

- i) it does not suffer from UV singularities;
- ii) the new nodes created carry zero volume and are invisible to its action;
- iii) the modifications induced do not propagate over the whole graph since they are localized in the neighborhood of a node.

### 12.1.3 Quantum constraints algebra

Space-time covariance is ensured at classical level by the closure of the Dirac algebra (2.80), see Sec. 2.3. The most striking feature is its non-Lie algebraic structure due to the field dependent structure constants in the  $\{\mathcal{H}(N), \mathcal{H}(N')\}$  Poisson brackets. A crucial issue is to prove the closure of this algebra at quantum level, i.e. the closure of the Dirac algebra generated by the quantum operators corresponding to the constraints. This should lead to the so-called *quantum space-time covariance* which can be regarded as a fundamental requirement of any quantum theory of gravity.

The first two relations in Eqs. (2.80) do not pose any problem since they can be reformulated in a finite manner and can be directly quantized. As we have seen, the diffeomorphism invariance is implemented by an averaging procedure that makes the states invariant under finite diffeomorphisms. The main challenge is thus to check the fate of the third relation in Eqs. (2.80) at quantum level. The necessary use of the weaker limit (12.31) is reflected on the quantum constraints algebra. In fact it turns out that the quantum algebra closes since

$$\langle [\hat{\mathcal{H}}^*(N), \hat{\mathcal{H}}^*(N')] \mathcal{X} | \Psi \rangle = 0, \quad (12.32)$$

for all  $\Psi \in \text{Cyl}$  and  $\mathcal{X} \in \text{Cyl}_{\text{diff}}^*$ . This relation is defined in  $\mathcal{F}_{\text{diff}}$  and ensures that the construction is *anomaly-free*. It is worth noting that the closure of the algebra is required only after the imposition of the constraints, i.e. it is partly on-shell. In this sense, it can be regarded as weaker than the off-shell closure which manifestly addresses the quantum space-time covariance. However, since the only states of interest are those invariant under diffeomorphisms, such notion of closure works properly.

Let us conclude this paragraph by some considerations. LQG has consistently improved, with respect to the geometrodynamics (WDW) approach, the program of the canonical quantization of the Einstein GR and its main results provide a rigorous kinematic construction of the quantum theory and the derivation of a quantized space (often called as the clear realization of the old Wheeler intuition about the foam of space). LQG is however not (yet) the final theory of quantum gravity and its main limit is the implementation of the dynamics. The Thiemann scalar quantum constraint is an outstanding proposal which suffers of ambiguities mainly due to the strong dependence on the regularization. Furthermore, no physical scalar product has been constructed and no correct semiclassical limit is addressed. These are two fundamental (related) open problems of LQG. Recently, it has been to overcome these shortcomings by the *Master Constraint Program* or by

the *Spin Foam Models*. The first approach is an implementation of the LQG (canonical) formalism in which the Dirac algebra (2.80) is transformed in a true Lie form. On the other hand, Spin Foam Models can be regarded as an attempt to construct a space-time version of spin networks, i.e. a rigorous path integral formulation of LQG. Both approaches are matter of current active research.

## 12.2 Loop Quantum Cosmology

A new quantum cosmology theory, motivated by LQG, has been recently formulated and it is known as Loop Quantum Cosmology (LQC). LQC is a minisuperspace model which is quantized according to the methods of LQG. LQC however is not the cosmological sector of LQG since the inhomogeneous fluctuations are switched off by hand ab initio rather than being quantum-mechanically suppressed. Nonetheless, LQC is an important arena to test the full theory leading to several relevant results such as:

- (i) the absence of the classical Big Bang singularity, replaced by a Big Bounce, in the isotropic setting
- (ii) a geometrical inflation as well as the suppression of the Mixmaster chaotic behavior, in the homogeneous one.

From a technical point of view with respect to the WDW scheme, these results are recovered here since the two quantum theories are (unitarily) inequivalent. In fact, LQC can be considered as the implementation of polymer quantum mechanics (see Sec. 11.2) to cosmological models written in the Ashtekar-Barbero-Immirzi connection formalism.

This Section is devoted to analyze this approach in the isotropic framework. The fate of the cosmological singularity and the differences with respect to the WDW quantum cosmology are discussed in details. In the next section, the theory will be implemented to the Bianchi IX dynamics.

### 12.2.1 Kinematics

Following what is done in the full theory, the first step in obtaining the quantum cosmological scheme is to select the basic variables of the model. As we have seen, the phase space of GR in its isotropic sector is two-dimensional and the scalar factor  $a = a(t)$  is the only degree of freedom of



the system. Due to such symmetry of the manifold, the connection  $A_\alpha^a$  and the electric field  $E_b^\beta$  are respectively reduced to

$$A = c(t) ({}^o e^a \tau_a), \quad E = p(t) \left( \sqrt{h^{\text{RW}}} {}^o e_a \tau^a \right). \quad (12.33)$$

Here, a fiducial metric on the Cauchy surface  $\Sigma$  is fixed by the triad  ${}^o e_a$  and co-triad  ${}^o e^a$  as

$$h_{\alpha\beta}^{\text{RW}} = k_{ab} {}^o e_\alpha^a {}^o e_\beta^b, \quad (12.34)$$

$k_{ab}$  being the Killing-Cartan metric on symmetry group of the spatial surface while the fiducial triad and co-triad satisfy the relation  ${}^o e_a ({}^o e^b) = \delta_a^b$ . The physical three-metric  $h_{\alpha\beta}$  is determined by the scale factor  $a(t)$  as

$$h_{\alpha\beta} = a^2(t) h_{\alpha\beta}^{\text{RW}}. \quad (12.35)$$

The phase space of the model has coordinates  $(c, p)$ , which are conjugate variables satisfying

$$\{c, p\} = \frac{\kappa\gamma}{3}. \quad (12.36)$$

The connection formalism is related to the metric one via the relations

$$|p| = a^2, \quad c = \frac{1}{2}(K + \gamma\dot{a}), \quad (12.37)$$

where  $K = 0, \pm 1$  is the usual curvature parameter of the FRW model. The Gauss and the diffeomorphism constraints of the full theory are here already satisfied by using Eq. (12.33). The scalar constraint rewrites as<sup>2</sup>

$$\mathcal{H}^{(c,p)} = -\frac{3}{\kappa} \sqrt{|p|} \left( \frac{1}{\gamma^2} (c - \Gamma)^2 + \Gamma^2 \right) = 0, \quad (12.38)$$

where the spin connection  $\Gamma$  is given by  $\Gamma = K/2$ . In the flat case ( $K = 0$ ), the relation (12.38) reduces to the simpler form

$$\mathcal{H}^{(c,p)} = -\frac{3}{\kappa\gamma^2} c^2 \sqrt{|p|} = 0. \quad (12.39)$$

We stress that in general  $p \in (-\infty, +\infty)$  and the classical singularity appears for  $p = 0$ . Differently from the WDW framework, it is not a boundary of the configuration space. Changes in the sign of  $p$  are here allowed and correspond to the changes of the orientation of the physical triad  $e_a^\alpha$  and co-triad  $e_\alpha^a$ . These are related to  ${}^o e_\alpha^a$  and  ${}^o e_a^\alpha$  via the equations

$$e_\alpha^a = {}^o e_\alpha^a \text{sign}(p) / \sqrt{|p|}, \quad e_a^\alpha = {}^o e_a^\alpha \text{sign}(p) \sqrt{|p|}. \quad (12.40)$$

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<sup>2</sup>Here we adopt the conventions adopted in literature.

The construction of quantum kinematic of this model follows the lines of LQG. As before, we need to construct  $SU(2)$  holonomies and electric fluxes which can be meaningfully implemented as (quantum) operators. The holonomy is formulated along the straight edges  $\ell = \mu^o e_a^\alpha$  and is given by

$$h_a(c) = \exp(\mu\tau_a c) = \cos\left(\frac{\mu c}{2}\right) \mathbb{I} + 2\tau_a \sin\left(\frac{\mu c}{2}\right). \quad (12.41)$$

Here  $\mathbb{I}$  is the identity  $2 \times 2$  matrix and  $\mu \in (-\infty, +\infty)$  is a real continuous parameter along which the holonomies are computed. Due to isotropy, the flux  $P_S(E)$  of the densitized triads across a surface  $S$  is proportional to the momentum  $p$  itself

$$P_S(p) = A_S p, \quad (12.42)$$

where  $A_S$  is a factor determined by the fiducial metric. The elements of the holonomies can be recovered from the functions

$$N_\mu(c) = e^{i\mu c/2}, \quad (12.43)$$

which are almost periodic functions because  $\mu$  is an arbitrary (non-integer) real number. The cylindrical functions of this reduced model are thus given by

$$\Psi(c) = \sum_j \xi_j e^{i\mu_j c/2}, \quad (12.44)$$

where  $\xi_j \in \mathbb{C}$ . Such states are defined in the space of the symmetric cylindric functions, denoted as  $\text{Cyl}_S$ . The holonomy-flux algebra is generated by  $e^{i\mu c/2}$  and by  $p$ . We deal with a hybrid representation between the Heisenberg ( $p$ ) and Weyl ( $e^{i\alpha x}$ ) ones.

The kinematical Hilbert space of LQC is obtained requiring that the  $N_\mu(c)$  form an orthonormal basis, i.e.

$$\langle N_\mu | N_{\mu'} \rangle = \delta_{\mu, \mu'}, \quad (12.45)$$

in analogy with the scalar product (12.13) of the full theory. Notice that on the right-hand side there is a Kronecker-delta rather than the usual Dirac distribution. From general theoretical considerations, the Hilbert space is necessarily  $\mathcal{F}_S = L^2(\mathbb{R}_B, d\mu)$ , where  $\mathbb{R}_B$  is a compact Abelian group (the so-called Bohr compactification of the real line) and  $d\mu$  is an appropriate measure on it. This is the space of the almost periodic functions and can be characterized as the Cauchy completion of the space  $\text{Cyl}_S$  with respect to the norm

$$\|\Psi\|^2 = \lim_{d \rightarrow \infty} \frac{1}{2d} \int_{-d}^d dc \Psi^\dagger(c) \Psi(c). \quad (12.46)$$

Two important remarks are in order:

- (i) assuming the scalar product as in Eq. (12.45), a new representation of the Weyl algebra has been introduced. Such a representation turns out to be inequivalent to the standard Schrödinger one since *the connection  $c$  itself does not exist as an operator in the Hilbert space*. This feature determines the failure of the Stone-Von Neumann uniqueness theorem (see Sec. 11.2). New features are thus allowed in the theory.
- (ii) The Hilbert space  $\mathcal{F}_S$  is not separable.

Let us now investigate the action of some fundamental operators in  $\mathcal{F}_S$ . The operators  $\hat{N}_\mu$  and  $\hat{p}$  act as multiplication and differentiation respectively as

$$\hat{N}_\mu \Psi = e^{i\mu c/2} \Psi, \quad \hat{p} \Psi = -i \frac{\kappa\gamma}{3} \frac{d\Psi}{dc}. \quad (12.47)$$

Introducing the Dirac bra-ket notation as  $N_\mu(c) = \langle c|\mu\rangle$ , the action of  $\hat{p}$  on the eigenstates  $|\mu\rangle$  is

$$\hat{p}|\mu\rangle = \frac{\kappa\gamma}{6} \mu |\mu\rangle \quad (12.48)$$

and the operator corresponding to the volume  $V = |p|^{3/2}$  has a spectrum given by

$$\hat{V}|\mu\rangle = V_\mu |\mu\rangle = \left( \frac{\kappa\gamma}{6} |\mu| \right)^{3/2} |\mu\rangle. \quad (12.49)$$

The volume operator in LQC has a *continuous* spectrum. This feature is in contrast with respect to LQG and can be attributed to the high degree of symmetry. In fact, in LQG the spin networks are characterized by a pair  $(\ell, j)$  consisting of a continuous edge  $\ell$  and a discrete spin  $j$ . Due to the symmetry, such pair now collapses to a single continuous label  $\mu$ . In the reduced theory, the spectrum is discrete in a weaker sense: all the eigenvectors are normalizable. Hence the Hilbert space can be expanded as a direct sum, rather than as a direct integral, over the one-dimensional eigenspaces of  $\hat{p}$ .

In view of the analysis of the Big Bang singularity, we have to address the role of the inverse scale factor fundamental operator. At a classical level, the inverse scale factor  $\text{sign}(p)/\sqrt{|p|}$  diverges towards the singularity ( $p \rightarrow 0$ ). Let us express it in terms of holonomies and fluxes and then proceed to the quantization. Let us note that the following classical identity

$$\frac{\text{sign}(p)}{\sqrt{|p|}} = \frac{4}{\kappa\gamma} \text{Tr} \left( \sum_a \tau^a h_a \left\{ h_a^{-1}, V^{1/3} \right\} \right) \quad (12.50)$$

holds, where the holonomy  $h_a$  is evaluated along any given edge. Since of the term  $h_a h_a^{-1}$ , the choice of the edge is not important, i.e. we do not introduce a regulator and the expression (12.50) is exact. When dealing with the scalar constraint, the situation will be different. The quantization of the inverse scale factor (12.50) is now well-defined in  $\mathcal{F}_S$ . The eigenvalues are given by

$$\left( \frac{\widehat{\text{sign}(p)}}{\sqrt{|p|}} \right) |\mu\rangle = \frac{6}{\kappa\gamma} \left( V_{\mu+1}^{1/3} - V_{\mu-1}^{1/3} \right) |\mu\rangle, \quad (12.51)$$

where  $V_\mu$  is the volume operator eigenvalue defined in Eq. (12.49), whose main properties are of being bounded from above and to coincide with the operator  $1/\sqrt{|\hat{p}|}$  as  $|\mu| \gg 1$ . The upper bound is obtained for the value  $\mu = 1$  and reads as

$$|p|_{\text{max}}^{-1/2} = \sqrt{\frac{12}{\kappa\gamma}} \sim l_{\text{P}}^{-1}. \quad (12.52)$$

We recall that the classical Ricci scalar of curvature is given by  $R \sim 1/a^2$  and therefore, considering such bound, it assumes its maximum value at  $R \sim 1/l_{\text{P}}^2$ . According to the local characterization of a space-time singularity (given by the divergence of the scalars built on the Riemann tensor), the LQC model is singularity-free.

The physical picture emerging is intriguing. Although the volume operator admits a continuous spectrum and a zero volume eigenstate (the  $|\mu = 0\rangle$  state), the inverse scalar factor is non-diverging at the classical singularity, but is bounded from above. This can indicate that, at a kinematic level, the classical singularity is avoided in the quantum framework. The semiclassical picture, i.e. the WDW behavior of the inverse scalar factor, is recovered for  $|\mu| \gg 1$  and therefore far from the fully quantum regime. Such behavior contrasts with the WDW formalism where the inverse scale factor is unbounded from above. As we mentioned, the differences are due to the non-standard Hilbert space adopted, or equivalently to the holonomy-flux algebra. As a matter of fact, differently from the WDW theory, all eigenvectors of  $p$  are normalizable in LQC including the one with zero eigenvalue. However, the boundedness of the inverse scale factor is by itself (as a local criterion) neither necessary nor sufficient for the cosmological singularity avoidance.

Let us investigate the quantum dynamics of the model.

### 12.2.2 Quantum dynamics and Big Bounce

To describe the quantum dynamics we have to impose the scalar constraint at quantum level, so as to discuss the fate of the Big Bang singularity from a dynamical point of view. In order to follow the lines of the full LQG theory, the starting point will be the constraint (12.3) and not the reduced one (12.38). In fact, the connection  $c$  itself is present in Eq. (12.38) rather than the holonomies and therefore the operator  $\hat{\mathcal{H}}^{(c,p)}$  is not well defined at a quantum level, since  $\hat{c}$  is not.

Let us investigate the flat model. Because of the spatial flatness, the Euclidean (first) and Lorentzian (second) terms in the scalar constraint (12.3) are proportional to each other. Mimicking the procedure followed in the full theory (see the previous Section), we rewrite the constraint in terms of holonomies and fluxes. The term involving the triad can be rewritten using a classical identity as

$$\epsilon_{abc} \frac{E^{\alpha b} E^{\beta c}}{\sqrt{|h|}} = 4 \sum_c \frac{\text{sign}(p)}{\kappa \gamma \mu_0} {}^\circ \epsilon^{\alpha \beta \gamma} {}^\circ e_\gamma^c \text{Tr} (h_c \{h_c^{-1}, V\} \tau_a), \quad (12.53)$$

where the holonomy is computed along an edge with length  $\mu_0$ .

Let us consider the field strength part. As in the lattice gauge theories, the curvature  $F_{\alpha\beta}^a$  can be (classically) written in terms of a trace of holonomies over a square loop  $\square_{ab}$  (each side having length  $\mu_0$ ) with its area shrinking to zero as

$$F_{\alpha\beta}^c \tau_c = -2 \lim_{\mu_0 \rightarrow 0} \text{Tr} \left( \frac{h_{ab} - 1}{\mu_0^2} \right) {}^\circ e_\alpha^a {}^\circ e_\beta^b. \quad (12.54)$$

Here  $h_{ab}$  denotes the holonomy computed around the square  $\square_{ab}$ , explicitly given by

$$h_{ab} \equiv h_a h_b h_a^{-1} h_b^{-1}. \quad (12.55)$$

This way, the (regulated) scalar constraint (12.3), for the flat FRW cosmological model, rewrites as

$$\mathcal{H}_{\mu_0}^{(h,p)} = -\frac{4 \text{sign}(p)}{\kappa \gamma^3 \mu_0^3} \sum_{abc} \epsilon^{abc} \text{Tr} (h_{ab}^{\mu_0} h_c^{\mu_0} \{(h_c^{\mu_0})^{-1}, V\}) = 0. \quad (12.56)$$

The dependence of the holonomies on  $\mu_0$  is explicitly written and the lapse function is fixed as  $N = 1$ . Differently from expression (12.50), the dependence on  $\mu_0$  does not drop out and now plays the role of a regulator. At a classical level, we can take the limit  $\mu_0 \rightarrow 0$  and verify that the resulting expression coincides with the classical Hamiltonian (12.38), i.e.  $\mathcal{H}_{\mu_0}^{(h,p)} \rightarrow \mathcal{H}^{(c,p)}$ .

This limit does *not* exist at quantum level, essentially because the operator  $\hat{c}$  itself does not exist in the Hilbert space  $\mathcal{F}_S$ . *In the reduced theory there is no way to remove the regulator.* This feature is solved in comparison with the full LQG theory. If we assume that the predictions of LQG are true, then the regulator  $\mu_0$  can be shrunk until the minimal admissible length given by the area operator spectrum. In this sense, the  $\mu_0 \rightarrow 0$  limit is physically meaningless and the failure of the limit to exist is a reminder that an underlying quantum geometry exists. This way the field strength operator (12.54) is non-local since  $\mu_0$  approaches a minimal non-zero value related to the minimum of the area operator (12.19). Such a criterion is the so-called *minimal area gap* argument.

However, how the reduced theory (LQC) can see a minimal length coming out from LQG is not fully understood. In fact, LQC is not the cosmological sector of LQG, but a usual cosmological minisuperspace model quantized through the LQG methods. A way for merging LQC in LQG will be discussed in Sec. 12.4.

Let us now investigate the physical states. As in the full theory, the physical states are those annihilated by all the constraints and live in some larger space  $\text{Cyl}_S^*$  (the algebraic dual of  $\text{Cyl}_S$ ), i.e. they do not need to be normalizable. Promoting the expression (12.56) to a quantum operator  $\hat{\mathcal{H}}_{\mu_0}^{(h,p)}$ , its action on the eigenstates  $|\mu\rangle$  is given by

$$\hat{\mathcal{H}}_{\mu_0}^{(h,p)}|\mu\rangle = \frac{3}{\kappa\gamma^3\mu_0^3}(V_{\mu+\mu_0} - V_{\mu-\mu_0})(|\mu+4\mu_0\rangle - 2|\mu\rangle + |\mu-4\mu_0\rangle). \quad (12.57)$$

A generic state (the notation  $|\Psi\rangle$  is here adopted to the eventuality of non-renormalizable states) can be expanded as

$$|\Psi\rangle = \sum_{\mu} \psi(\mu, \phi) |\mu\rangle, \quad (12.58)$$

where  $\phi$  represents a generic matter field included into the dynamics. Such states satisfy the constraint equation

$$(\Psi| \left( \hat{\mathcal{H}}_{\mu_0}^{(h,p)} + \hat{\mathcal{H}}_{\mu_0}^{\phi} \right)^{\dagger} = 0, \quad (12.59)$$

$\mathcal{H}_{\mu_0}^{\phi}$  representing the matter term appropriately regularized with  $\mu_0$ . The function  $\psi(\mu, \phi)$  has to satisfy the equation

$$\begin{aligned} \frac{3}{\gamma^3\mu_0^3\kappa} [(V_{\mu+5\mu_0} - V_{\mu+3\mu_0})\psi(\mu+4\mu_0, \phi) - 2(V_{\mu+\mu_0} - V_{\mu-\mu_0})\psi(\mu, \phi) \\ + (V_{\mu-3\mu_0} - V_{\mu-5\mu_0})\psi(\mu-4\mu_0, \phi)] = -\hat{\mathcal{H}}_{\mu_0}^{\phi} \psi(\mu, \phi). \end{aligned} \quad (12.60)$$

This is nothing but a recurrence relation for the coefficients  $\psi(\mu, \phi)$  which ensures that  $(\Psi|$  is indeed a physical state. Even though  $\mu$  is a continuous variable, the quantum constraint (12.60) is an algebraic (*difference*) equation rather than a differential one. This is in evident contrast with respect to the WDW theory.

We now discuss the implications of LQC on the fate of the classical Big Bang singularity at a dynamical level. The singularity corresponds to the state  $|\mu = 0\rangle$  and we have to analyze whether the quantum dynamics breaks down at  $\mu = 0$ . If this is not the case, the principle of quantum hyperbolicity (see Sec. 10.3) is satisfied and the classical singularity can be considered as tamed by quantum effects. As we can see from Eq. (12.60), starting at  $\mu = -4N\mu_0$  we can compute all the coefficients  $\psi(4\mu_0(n - N), \phi)$  for  $n > 1$ . However, the coefficient  $\psi(\mu = 0, \phi)$  remains undetermined because the generic coefficient vanishes if and only if  $n = N$ . The quantum evolution seems to break down just at the classical singularity since it is not possible to evolve the states beyond it, but this is anyhow not the case. In fact, the coefficient  $\psi(\mu = 0, \phi)$  is decoupled from the others thanks to  $V_{\mu_0} = V_{-\mu_0}$  and the condition  $\hat{H}_{\mu_0}^\phi \psi(\mu = 0, \phi) = 0$  realizes. Therefore the coefficients in (12.60) are such that one can unambiguously evolve the states through the singularity even though  $\psi(\mu = 0, \phi)$  is not determined. We can thus conclude that *the Big Bang singularity is solved in the LQC framework*.

We will now briefly discuss the outstanding recent results obtained by Ashtekar and collaborators on the *quantum Big Bounce*. Such works have significantly extended the previous discussions displaying in detail a clear picture of the Universe evolution during the Planck era. The considered model is the flat FRW Universe filled with a massless scalar field  $\phi$  whose energy density is

$$\rho = \frac{p_\phi^2}{|p|^3}. \quad (12.61)$$

For the analysis of this model in the WDW framework see Sec. 10.8. The scalar constraint (10.128) rewrites in the connection formalism as

$$-\frac{3}{\kappa\gamma^2}c^2\sqrt{|p|} + \frac{p_\phi^2}{|p|^{3/2}} = 0, \quad (12.62)$$

where, as usual,  $p_\phi$  is the momentum canonically conjugate to  $\phi$ , i.e.  $\{\phi, p_\phi\} = 1$ . As before, each trajectory can be specified in the  $(p, \phi)$ -plane since  $p_\phi$  is a constant of motion and they are given by Eq. (10.131) where here  $|p| = a^2$ . The idea that a massless scalar field  $\phi$  plays the role of a

relational time (see Sec. 10.5) is implemented in LQC by demanding that the whole Hilbert space is given by

$$\mathcal{F}_{\text{tot}} = L^2(\mathbb{R}_B, B(\mu)d\mu) \otimes L^2(\mathbb{R}, d\phi), \quad (12.63)$$

where  $B(\mu)$  denotes the eigenvalue of the operator  $\widehat{1/|p|^{3/2}}$ . The time  $\phi$  is thus treated in the standard (Schrödinger) representation, while only the effective degree of freedom  $\mu$  is analyzed in the polymer representation (see Sec. 11.2). In complete analogy to the WDW framework, the quantum dynamics of this model is summarized by the Klein-Gordon-type equation

$$(\partial_\phi^2 + \Theta_{\text{LQC}})\Psi = 0, \quad (12.64)$$

where  $\Psi = \Psi(\mu, \phi)$  is the wave function of the Universe and  $\Theta_{\text{LQC}}$  denotes a *difference* operator which depends on  $\mu_0$ . As before, it is possible to construct wave packets peaked at late time and evolve them according to (12.64) towards the singularity. Such analysis sheds light on the singularity resolution in LQC. As a result, the semiclassical states remain sharply peaked around modified classical trajectories (see the next subsection) by which the classical Big Bang is replaced by a quantum Big Bounce. In particular, the Universe experiences the bounce for a density of the Planck order. At curvatures much smaller than the Planck one the states are peaked on the classical trajectories (10.131). Loosely speaking, LQC leads to a “quantum bridge” between the expanding and contracting classical Universes.

### 12.2.3 *Effective classical dynamics*

It is useful to obtain an effective Hamiltonian constraint which captures the essential features of the discrete evolution equation of LQC.

As we have seen, the connection  $c$  itself cannot be implemented as an operator in the Hilbert space essentially because of the form of the inner product (12.45). From the polymer perspective, only its exponentiated version can be defined as an operator. This way, let us consider the polymer substitution formula (11.48) as

$$c \rightarrow \sin(\mu_0 c) / \mu_0. \quad (12.65)$$

The scalar constraint (12.62) thus rewrites as

$$\mathcal{H}_{\text{eff}} = -\frac{3}{\kappa\gamma^2\mu_0^2}\sqrt{|p|}\sin^2(\mu_0 c) + \frac{p_\phi^2}{|p|^{3/2}} = 0, \quad (12.66)$$



which summarizes the LQC quantum corrections to the standard Friedmann dynamics. The ordinary evolution is recovered as  $\mu_0 c \ll 1$ . It is worth noting that such expression can be analytically obtained by applying the methods of geometric quantum mechanics directly to LQC. Thus, one can obtain the effective Friedmann equation for the model. The equation of motion for  $p$  is

$$\dot{p} = -N \frac{\kappa\gamma}{3} \frac{\partial \mathcal{H}_{\text{eff}}}{\partial c} = \frac{2N}{\gamma\mu_0} \sqrt{|p|} \sin(\mu_0 c) \cos(\mu_0 c), \quad (12.67)$$

and the constraint (12.66) implies that

$$\sin^2(\mu_0 c) = \frac{\kappa\gamma^2\mu_0^2}{3} \frac{p_\phi^2}{|p|^2} = \frac{\kappa\gamma^2\mu_0^2}{3} |p|\rho. \quad (12.68)$$

Combining Eqs. (12.67) and (12.68) the *effective Friedmann equation* of LQC, in the synchronous reference frame ( $N = 1$ ), is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{p}}{2p}\right)^2 = \frac{\kappa}{3}\rho \left(1 - \frac{\rho}{\rho_c}\right), \quad (12.69)$$

where

$$\rho_c = \frac{3}{\kappa\gamma^2\mu_0^2|p|}, \quad (12.70)$$

denotes the critical energy density. As  $\mu_0 \rightarrow 0$ , this quantity diverges and the standard Friedmann dynamics is recovered. The modifications arising from the LQC quantum effects in Eq. (12.69) are manifested in the form of a  $\rho^2$  term. Such factor is relevant in the high energy regime and, as  $\rho$  reaches the critical value  $\rho_c$ , the Hubble function vanishes and the Universe experiences a bounce (or, more generally, a turn-around) in terms of the scale factor. In the ordinary dynamics, the Hubble function cannot vanish unless for the trivial case  $\rho = 0$ . For energy densities much smaller than the critical one, the Friedmann dynamics appears.

The value of the critical density (12.70) is usually fixed by demanding a relation with the minimum area operator eigenvalue of LQG. The physical area spanned by the elementary loop edge  $\mu_0\sqrt{|p|}$  is given by  $\mu_0^2|p|$  and this way, from Eq. (12.19), the relation

$$\mu_0^2|p| \sim l_{\text{P}}^2, \quad (12.71)$$

is obtained and the critical energy density  $\rho_c$  is of the Planck order since  $\rho_{\text{P}} = 1/l_{\text{P}}^4$ . It is worth noting that in this framework the regulator  $\mu_0$  has been treated as a constant. However, for a consistent quantization it must behave as  $\mu_0 \sim 1/\sqrt{|p|}$ . In order to include this varying parameter, one

must change variables from the triad basis ( $\mu$ ) to the volume one, leading to new basic variables to be quantized in the same way as above. In the literature, this framework is known as “improved quantization” for which we recommend the original works.

As a last point, let us consider the scalar field  $\phi$  as the internal time for the dynamics. As we have seen in Sec. 10.5, this fixes the gauge requiring the lapse function to be

$$N = \frac{|p|^{3/2}}{2p_\phi} = \frac{1}{2\sqrt{\rho}}. \quad (12.72)$$

The effective Friedmann Eq. (12.69) in the  $(p, \phi)$  plane reads as

$$\left(\frac{1}{p} \frac{dp}{d\phi}\right)^2 = \frac{\kappa}{3} \left(1 - \frac{\kappa\gamma^2\mu_0^2}{3} \frac{p_\phi^2}{|p|^2}\right) \quad (12.73)$$

and, as  $\rho_c \rightarrow \infty$ , Eq. (10.130) is recovered. The solution to Eq. (12.73) is given by

$$p(\phi) \sim e^{-\sqrt{\frac{\kappa}{3}}(\phi-\phi_0)} \left(\mu_0^2 \tilde{p}_\phi^2 + e^{2\sqrt{\frac{\kappa}{3}}(\phi-\phi_0)}\right), \quad (12.74)$$

where  $\tilde{p}_\phi^2 = 2\gamma^2 p_\phi^2 / \pi^2$ . The ordinary solutions (10.131) are recovered at late times, i.e. for  $|\phi| \rightarrow \infty$  (see Fig. 12.1). The effective trajectory (12.74) approximates quite well the loop quantum dynamics previously described and clearly represents a bouncing solution (i.e. a singularity-free Universe) over which the wave packets are sharply peaked during the whole evolution. The initial semiclassical state follows the classical trajectory (10.131) until it reaches a purely quantum era where the effects of quantum geometry become dominant. The resulting dynamics is that of a quantum Big Bounce replacing the classical Big Bang.

### 12.3 Mixmaster Universe in LQC

In this Section we describe the dynamics of the Mixmaster Universe in the LQC framework. The discussion here presented covers the basic aspects, referring for notation and details to the original literature.

The Bianchi IX evolution towards the singularity sees infinite sequences of Kasner epochs characterized by a series of permutations as well as by possible rotations of the expanding and contracting spatial directions (for details, see Sec. 7.4.1). However, this infinite number of bounces within the potential, at the ground of the stochastic properties, is a consequence of an

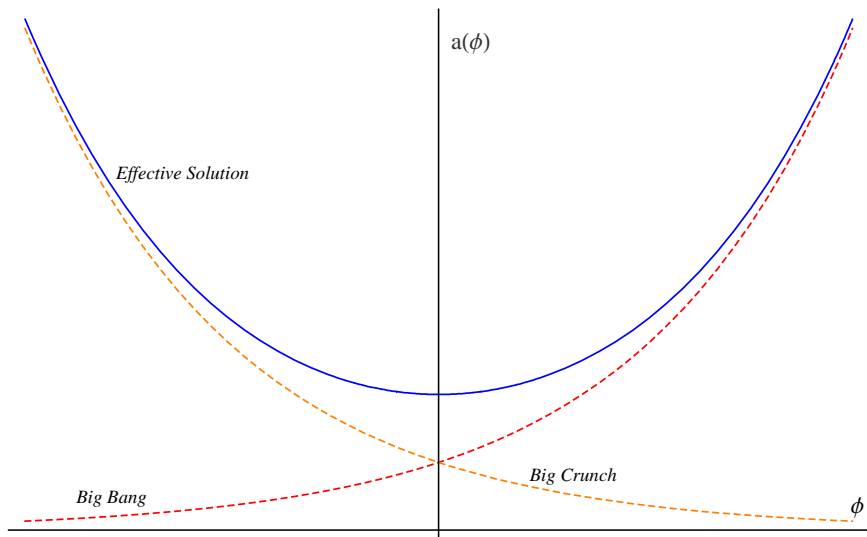


Figure 12.1 The effective LQC trajectory is compared to the ordinary ones for an isotropic model. A Big Bounce clearly replaces the classical singularity at the Planck scale.

unbounded growth of the spatial curvature. When the theory offers a cut-off length and the curvature is bounded, the Bianchi IX model naturally shows a finite number of oscillations and, in LQC, a quantum suppression of the chaotic behavior takes place close to the singularity.

### 12.3.1 Loop quantum Bianchi IX

Let us formulate the vacuum Bianchi IX model in the connection formalism. The spatial metric

$$dl^2 = \sum_c a_c^2 (\omega^c)^2 \quad (12.75)$$

can be taken diagonal, leaving three degrees of freedom<sup>3</sup> only. The basic variables for a homogeneous model are

$$A_\alpha^a = c_{(a)}(t) O_b^a \omega_\alpha^b, \quad E_a^\alpha = p^{(a)}(t) O_a^b e_b^\alpha, \quad (12.76)$$

where the indices in the brackets are not summed over. Here  $\omega^a$  are the left-invariant 1-forms satisfying the Maurer-Cartan equation (7.28) and  $e_a$  are the vector fields dual to  $\omega^a$  ( $\omega^a(e_b) = \delta_b^a$ ). These fields thus form an

<sup>3</sup>Differently from Sec. 7.3, we denote the three scale factors  $a, b, c$  by  $a_1, a_2, a_3$ .

invariant basis. We remember (see Sec. 7.1) that an invariant basis  $\{e_a\}$  is defined as Lie-invariant under the action of the Killing vectors field  $\xi_a$ , that is

$$\mathcal{L}_\xi e = [\xi_a, e_b] = 0, \quad (12.77)$$

which in the case of Bianchi IX carry the  $SO(3)$  isometry. The isomorphism between the Cauchy surfaces  $\Sigma$  and the isometry group  $SO(3) \simeq SU(2)$  appears and the  $SO(3)$ -matrix  $O_b^a$  contains the pure gauge degrees of freedom.

The physical information of the model is provided by the gauge invariant functions  $c_a$  and  $p^b$  which satisfy the Poisson brackets

$$\{c_a, p^b\} = \kappa \gamma \delta_a^b. \quad (12.78)$$

The connection variables are related to the metric formalism (scale factors  $a_c$ , spin connections  $\Gamma_c$  and the extrinsic curvature  $K_c = -\dot{a}_c/2$ ) as

$$c_a = \Gamma_a - \gamma K_a, \quad p^a = |a_b a_c| \text{sign}(a_a), \quad (12.79)$$

where

$$\Gamma_a = \frac{1}{2} \left( \frac{a_b}{a_c} + \frac{a_c}{a_b} - \frac{a_a^2}{a_b a_c} \right) = \frac{1}{2} \left( \frac{p^c}{p^b} + \frac{p^b}{p^c} - \frac{p^b p^c}{(p^a)^2} \right). \quad (12.80)$$

The classical dynamics is governed by the scalar constraint expressed as

$$\mathcal{H}_{\text{IX}} = \frac{2}{\kappa} \left[ (\Gamma_b \Gamma_c - \Gamma_a) a_a - \frac{1}{4} a_a \dot{a}_b \dot{a}_c + \text{cyclic} \right] = 0. \quad (12.81)$$

The potential term from (12.81) is given by

$$W(p) = 2 (p^a p^b (\Gamma_a \Gamma_b - \Gamma_c) + \text{cyclic}) \quad (12.82)$$

which has infinite walls at small  $p^a$  due to the divergence of the spin connection components. Thus the cosmological singularity in Bianchi IX appears whenever  $a_b = 0$  for some  $b$ .

The closed ( $K = 1$ ) FRW model is recovered by setting  $a_1 = a_2 = a_3 = a$  in the Bianchi IX phase space. In this case the triadic projection of the Christoffel symbols (12.80) becomes the constant  $\Gamma = 1/2$  and the isotropic connection and momentum are respectively given by

$$c = \frac{1}{2} (1 + \gamma \dot{a}), \quad |p| = a^2. \quad (12.83)$$

The loop quantization of the Bianchi IX model is performed straightforwardly following the isotropic case. In fact, an orthonormal basis is given by the  $\hat{p}^a$ -eigenstates

$$|\mu_1, \mu_2, \mu_3\rangle = |\mu_1\rangle \otimes |\mu_2\rangle \otimes |\mu_3\rangle \quad (12.84)$$

and the Hilbert space is taken as the direct product of the isotropic ones. It is separable and is a subspace of the kinematical non-separable Hilbert space. The cylindrical functions are given by a superposition of the basis state  $\langle c|\mu\rangle \sim \exp(i\mu c/2)$ . The basic quantum operators are the gauge invariant triad operators  $\hat{p}^a$  (fluxes) and the holonomies

$$h_a(c) = \cos\left(\frac{c_a}{2}\right) + 2O_a^b \tau_b \sin\left(\frac{c_a}{2}\right). \quad (12.85)$$

When implemented as operators in the Hilbert space,  $\hat{p}^a$  and  $\hat{h}_a$  act as differentiation and multiplication, respectively. In particular, the eigenvalues  $p^a$  of the triad operator  $\hat{p}^a$  are given by

$$\hat{p}^a |\mu_1, \mu_2, \mu_3\rangle = \frac{\kappa\gamma}{2} \mu_a |\mu_1, \mu_2, \mu_3\rangle. \quad (12.86)$$

The volume operator is defined from  $\hat{p}^a$  as  $\hat{V} = \sqrt{|\hat{p}^1 \hat{p}^2 \hat{p}^3|}$  and its action on the eigenstates  $|\mu_1, \mu_2, \mu_3\rangle$  reads as

$$\hat{V} |\mu_1, \mu_2, \mu_3\rangle = \left(\frac{\kappa\gamma}{2}\right)^{3/2} \sqrt{|\mu_1 \mu_2 \mu_3|} |\mu_1, \mu_2, \mu_3\rangle \quad (12.87)$$

having a continuous spectrum also in the anisotropic case.

From these basic operators we can obtain, similarly to the isotropic case, the inverse triad operator. Since it is diverging towards the classical singularity, we are interested in its behavior at a quantum level. Conceptually, its construction in the anisotropic cosmological sector is the same as in the isotropic one. The only difference resides in the computations and in the appearance of some quantization ambiguities, as the half-integer values of  $j$  and the continuous parameter  $l \in (0, 1)$ , although all the results are independent of them. This behavior mimics the quantization ambiguities present either in LQG or in the isotropic sector of LQC. While in the full theory we find an ambiguous choice of the spin number  $j$  associated to a given edge of the spin network, in the isotropic LQC such choice is reflected on the Hamiltonian regulator  $\mu_0$ . Nevertheless, this is but a parameter that cannot be shrunk to zero, neither fixed in some way in the context of LQC theory itself, but it comes out from extrapolating a prediction of another theory (namely LQG). In the inverse scale factor term of isotropic LQC, none of quantization ambiguity appears (see Sec. 12.2).

Let us construct the inverse triad operator. The technique is the same as before: we express  $1/|p^a|$  in terms of holonomies and volumes via a classical identity and it can be meaningfully quantized as

$$\widehat{|p^a|^{-1}} |\mu_1, \mu_2, \mu_3\rangle = A_{j,l} f_{j,l}(\mu_a) |\mu_1, \mu_2, \mu_3\rangle, \quad (12.88)$$

where

$$f_{j,l}(\mu_a) = \left( \sum_{k=-j}^j k |\mu_a + 2k|^l \right)^{1/1-l}, \quad (12.89)$$

and  $A_{j,l}$  denotes a function of the quantization ambiguities  $j$  and  $l$ . The values  $f_{j,l}(\mu)$  decrease for  $\mu < 2j$  and the relation

$$\widehat{|p^a|^{-1}}|\mu_a = 0\rangle = 0, \quad (12.90)$$

holds because of  $f_{j,l}(0) = 0$ . Thus, the inverse triad operator annihilates the state corresponding to the classical singularity  $|\mu_a = 0\rangle$ .

The fundamental properties of the eigenvalues of the operator (12.88) can be extracted from the asymptotic expansions of  $F_{j,l}(\mu_a) \equiv A_{j,l}f_{j,l}(\mu_a)$ . For large  $j$  one has  $F_{j,l}(\mu_a) = F_l(\nu_a)$ , with  $\nu_a = \mu_a/2j$ , with no explicit dependence on  $j$  and, in particular,

$$F_l(\nu_a) = \begin{cases} 1/\nu_a & \text{if } \nu_a \gg 1 \quad (\mu_a \gg j) \\ \left( \frac{\nu_a}{l+1} \right)^{1/1-l} & \text{if } \nu_a \ll 1 \quad (\mu_a \ll j). \end{cases} \quad (12.91)$$

The classical behavior of the inverse triad components is obtained for  $\nu_a \gg 1$ , while the loop quantum modifications arise for  $\nu_a \ll 1$ . Rewriting the spin connection in the triad representation, the potential (12.82) is given by

$$W_{j,l}(\nu) = 2(\kappa\gamma j)^2 (\nu_a \nu_b (\Gamma_a \Gamma_b - \Gamma_c) + \text{cyclic}), \quad (12.92)$$

where

$$\Gamma_a(\nu_a) = \frac{1}{2} [\nu_c \text{sign}(\nu_a) F_l(\nu_b) + \nu_b \text{sign}(\nu_c) F_l(\nu_c) - \nu_b \nu_c F_l^2(\nu_a)]. \quad (12.93)$$

We are ready to quantize the scalar constraint of the model to extract the dynamics. The loop quantization of the scalar constraint is however different in the homogeneous case. In fact, unlike the full theory, the Christoffel symbols are tensors in homogeneous space-times and cannot vanish as in LQG. As far as they are chosen to be different from zero, it is not possible to perform a diffeomorphism (maintaining the homogeneity) which makes them vanish. Also the holonomies will depend also on the spin connections and the quantum scalar operator leads to a partial difference equation, like in the isotropic LQC. Such expression is anyway extremely complex and an *effective* description turns out to be more suitable to extract the effects of quantum geometry on the classical model.

### 12.3.2 *Effective dynamics*

We will discuss the Mixmaster Universe in the LQC framework, analyzing its behavior at a semiclassical level, i.e. considering the modifications induced to the dynamics by the loop quantization. To obtain an *effective* Hamiltonian from the underlying quantum evolution, we will proceed in two steps. At the first pace one develops a sort of continuum approximation, while at the second one a WKB expansion of the wave function is performed, more explicitly

- i) for a slow varying solution, the differences equation is specialized for the continuum regime  $\mu_a \sim p_a/\kappa\gamma \gg 1$ , thus obtaining the WDW-like equation

$$\left( \kappa^2 p^a p^b \frac{\partial^2 \sqrt{|p^1 p^2 p^3|} S(p)}{\partial p^a \partial p^b} + \text{cyclic} \right) + W_{j,l}(p) \sqrt{|p^1 p^2 p^3|} S(p) = -\kappa |p^1 p^2 p^3|^{3/2} \hat{\rho}_\phi S(p), \quad (12.94)$$

where  $\phi$  denotes a generic matter field;

- ii) the WKB limit of the wave function

$$T(p) = \sqrt{|p^1 p^2 p^3|} S(p) \quad (12.95)$$

is considered, i.e.  $T \sim e^{iA/\hbar}$ . This approximation leads to the Hamilton-Jacobi equation for the phase  $A$  to zeroth order in  $\hbar$ .

The classical dynamics plus the quantum loop corrections are therefore obtained. The key point for the classical analysis of the effective dynamics is that the classical region  $\mu_a \gg 1$  (taking  $j \gg 1$ ) can be separated into two subregions. In fact, remembering the  $p$ -dependence in the WDW equation (12.94) given by  $\mu_a/2j \sim p_a/j\kappa\gamma$ , we obtain the condition  $\mu_a \gg 1$  for

$$p^a \ll j\kappa\gamma, \quad j \gg \mu_a \gg 1 \quad (12.96)$$

as well as

$$p^a \gg j\kappa\gamma, \quad \mu_a \gg j \gg 1. \quad (12.97)$$

The second subregion ( $\mu_a \gg j \gg 1$ ) is the purely classical one, i.e. where the Misner picture is still valid. From the expansion (12.91), the eigenvalues of the inverse triad operator correspond to the classical values. On the other hand, the (classical) region where  $j \gg \mu_a \gg 1$  is characterized by loop quantum modifications since there the inverse triad operator eigenvalues have a power law dependence. The quantum modifications to the

classical dynamics are controlled by the parameter  $j$ : if it is large enough, one can move the quantum effects within the effective potential into the semiclassical domain. However, the WKB limit ( $\hbar \rightarrow 0$ ) is strictly valid only in the Misner region ( $\mu_a \gg j \gg 1$ ), because in the first region a dependence on  $(\kappa\gamma)^{-1}$  appears in the potential term. The validity of such approximation in the first region ( $j \gg \mu_a \gg 1$ ) is a reasonable requirement since the inverse triads vanish as  $p^a \rightarrow 0$ .

A qualitative study of the modified classical evolution arises from analyzing the potential term, whose explicit expression is more complex than the original one, making the dynamics more tricky. The volume variable is regarded as a time variable (as in the ordinary approach) but in general the dependence on it does not factorize out. Nonetheless, in the second region the Misner potential is restored.

Considering the particular case  $\beta_- \equiv 0$  (namely the Taub Universe), we can qualitatively study the effective potential of LQC. This case corresponds to taking  $\nu_2 = \nu_3 \equiv \nu$  and

$$\nu_1 \equiv \sigma = V^2 / ((2j)^3 \nu), \quad (12.98)$$

and therefore  $\Gamma_2 = \Gamma_3$ . The wall appears for  $\nu \gg 1$  so that  $F_l(\nu) \simeq \nu^{-1}$ , but  $\sigma$  is not negligible, i.e. the relation  $F_l(\sigma) \simeq \sigma^2$  holds. The potential wall (12.92) becomes

$$W_{j,l} \simeq \frac{V^4}{j^4 \sigma^2} F_l^2 \sigma (3 - 2\sigma F_l(\sigma)), \quad (12.99)$$

where  $V \propto e^{3\alpha}$  denotes the volume of the Universe.

For  $\sigma \gg 1$ ,  $F_l(\sigma) \simeq \sigma^{-1}$  and the classical wall  $e^{4\alpha-8\beta_+}$  is restored. The key difference with the standard case relies on the *finite height* of the wall. As the volume decreases, the wall moves inwards and its height decreases as well. In the subsequent evolution, the wall completely disappears as it reaches its maximum for  $\beta_+ = -\alpha$  and vanishes as  $e^{12\alpha} \propto V^4$  towards the classical singularity ( $\alpha \rightarrow -\infty$ ).

This peculiar behavior shows that the Mixmaster-like evolution breaks at a given time and therefore *the chaotic features of the model disappear*. In fact, when the volume is so small that the quantum modifications arise, the point-Universe will never bounce against the potential wall and the Kasner epochs will continue forever. It is worth noting that this behavior predicted by the LQC framework produces (qualitatively) the same results as those induced by a massless scalar field on the Universe dynamics: also in that case, at a given time, the point-Universe performs the last bounce and then it evolves freely (see Sec. 8.7.1).



## 12.4 Triangulated Loop Quantum Cosmology

As we have seen, LQC is the most remarkable application of LQG and its results are explicitly grounded on the physical discreteness of quantum geometry (see Sec. 12.1). However, LQC is still a minisuperspace theory in which the symmetries are imposed at a classical level and the quantization is performed by means of LQG techniques (essentially based on a singular representation of the Weyl algebra). Thus LQC is not the cosmological sector of LQG and the inhomogeneous fluctuations are switched off by hand *ab initio* rather than being quantum-mechanically suppressed: there is not a symmetry reduction of the quantum theory.

In this Section we will show how LQC can be merged into LQG and, in particular, how the LQC dynamics naturally arises from a truncated LQG model (namely with a finite number of degrees of freedom). This model, based on a triangulation of a topological three-sphere, is known as triangulated loop quantum cosmology and is based on a constrained  $SU(2)$  lattice gauge theory describing the Bianchi IX Universe plus some inhomogeneous degrees of freedom. We will first discuss the model and then identify the isotropic as well as the homogeneous sectors of LQC, devoting some space to the inclusion of inhomogeneity. In what follows, we will assume the Immirzi parameter  $\gamma = 1$ .

### 12.4.1 The triangulated model

The basic idea is to triangulate the spatial surfaces slicing the Bianchi IX Universe which are (topologically) three-spheres  $S^3$ . Let us consider an oriented triangulation  $\Delta_n$  of

$$S^3 \simeq SO(3) \simeq SU(2) \quad (12.100)$$

formed by  $n$  tetrahedra  $t$  glued by their triangles<sup>4</sup>  $f$ . We consider a triangulation in which the tetrahedra (and thus the triangles) are curved. This kind of discretization is called a cellular complex decomposition as it differs from the simplicial triangulation adopted for example in the Regge calculus, where the tetrahedra are flat. We associate a group and an algebra element to each triangle given by

$$U_f \in SU(2), \quad E_f = E_f^a \tau_a \in su(2), \quad (12.101)$$

---

<sup>4</sup>For a triangulation making use of  $n$  tetrahedra  $t$  there are  $2n$  triangles  $f$ .

respectively, where  $\tau_a$  are generators of  $SU(2)$  defined in (12.6). The phase space of this model is described by the Poisson brackets

$$\{U_f, U_{f'}\} = 0, \quad (12.102)$$

$$\{E_f^a, U_{f'}\} = \delta_{ff'} \tau^a U_f, \quad (12.103)$$

$$\{E_f^a, E_{f'}^b\} = -\delta_{ff'} \epsilon^{abc} E_f^c. \quad (12.104)$$

The phase space is that of a canonical lattice  $SU(2)$  Yang-Mills theory being the cotangent bundle of  $SU(2)^{2n}$  with its natural symplectic structure.

The dynamics is encoded in two sets of constraints: the Gauss constraint (providing three constraints per tetrahedron) given by

$$G_t^a = \sum_{f \in t} E_f^a = 0, \quad (12.105)$$

where the sum is over the four faces of the tetrahedron, and the Hamiltonian (scalar) constraint expressed as

$$\mathcal{H}_t = V_t^{-1} \sum_{ff' \in t} \text{Tr}[U_f U_{f'}^{-1} E_{f'} E_f] = 0, \quad (12.106a)$$

$$V_t = \sqrt{\text{Tr}[E_f E_{f'} E_{f''}]}, \quad (12.106b)$$

where the operator “Tr” denotes the trace in the  $su(2)$  Lie algebra,  $V_t$  is the volume of the tetrahedron  $t$  and  $U_f^{-1} = U_{f^{-1}}$ . In what follow we will consider  $\tilde{\mathcal{H}}_t = V_t \mathcal{H}_t$ . The scalar constraint (12.106) provides the time evolution of the system.

The interpretation of this model can be sketched considering the dual triangulation  $\Delta_n^*$ , defined as follows: for each  $t \in \Delta_n$  it is associated a node  $n \in \Delta_n^*$ , while for each  $f \in \Delta_n$  we associate a link  $l \in \Delta_n^*$ . Consider now real Ashtekar-Barbero variables  $A_\alpha^a$  and  $E_a^\alpha$  on  $S^3$ . Then,  $U_f$  is the parallel transport of the connection along the link  $l$  and  $E_f$  is the flux of the conjugate electric field across the triangle  $f$  (parallelly transported to the center of the tetrahedron).

The constraint in Eq. (12.106) is the discrete (triangulated) version of the Euclidean part of the Ashtekar-Barbero Hamiltonian constraint<sup>5</sup> (12.3). Firstly let us note that the Hamiltonian (12.3) has the correct continuum limit as soon as considering the Gauss constraint. A holonomy  $U_\alpha$  along a loop  $\alpha$  can be approximated by

$$U_\alpha \sim \exp \int_\alpha A \sim \mathbb{1} - |\alpha|^2 F_\alpha + \mathcal{O}(|\alpha|^4 A^2), \quad (12.107)$$

---

<sup>5</sup>In the Euclidean case, for  $\gamma = 1$ , the constraint (12.3) reduces to the first part only. In the Lorentzian case, this happens when considering the original complex Ashtekar variables.

where  $F$  is the field strength of the connection  $A$ . By means of this holonomy expansion, Eq. (12.106) can be formally recast as

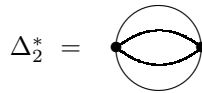
$$\tilde{\mathcal{H}}_t = V_t \mathcal{H}_t = \sum_{ff' \in t} \text{Tr}[E_f E_{f'}] - |\alpha|^2 \sum_{ff' \in t} \text{Tr}[F_{ff'} E_{f'} E_f] = 0. \quad (12.108)$$

The former term in the relation (12.108) vanishes because of the Gauss constraint (12.105) and thus the second term undergoes the expected continuum limit

$$\mathcal{H}\sqrt{h} = \text{Tr}[F_{\alpha\beta} E^\alpha E^\beta] = 0. \quad (12.109)$$

Notice that this happens not only for small values of the length of the loop  $|\alpha|$ , i.e. for a fine triangulation, but also for a coarse triangulation (large  $|\alpha|$ ) provided that  $|\alpha|^2 F$  is small. This model can be regarded as a lattice approximation of the geometrodynamics of a closed Universe.

Let us now consider the simplest triangulation of  $S^3$ , i.e. the one formed by two tetrahedra glued together along all their faces. This model is called *dipole cosmology* and is defined by the dual graph formed by two nodes joined by four links as



This in turns specifies the cellular complex triangulation of the three-sphere. The unconstrained phase space of the theory defined by this triangulation has 24 dimensions and is coordinatized by  $(U_f, E_f^a)$ . At each node there are one Hamiltonian  $\mathcal{H} = 0$  and three Gauss constraints  $G^a = 0$ , but it is easy to verify that the constraints of the two nodes are indeed the same, giving a total of four constraints only, bringing the number of degrees of freedom down to eight (or nine up to the dynamics generated by  $\mathcal{H} = 0$ ).

The dipole model is related to a homogeneous Universe with the topology of a three-sphere, i.e. to Bianchi IX. As we have seen in Sec. 7.1, the building blocks of homogeneous spaces are the left invariant 1-forms  $\omega^a$  satisfying the Maurer-Cartan structure equation

$$d\omega^a = \frac{1}{2} C_{bc}^a \omega^b \wedge \omega^c. \quad (12.110)$$

If the symmetry group is  $SO(3)$ , the Cauchy surfaces are  $S^3$  and the structure constants are  $C_{bc}^a = \epsilon_{bc}^a$ . In order to translate this language in the triangulated framework, let us consider the Plebanski 2-form of the connection  $\omega^a$

$$\Sigma^a(\omega) = \frac{1}{2} \epsilon_{bc}^a \omega^b \wedge \omega^c. \quad (12.111)$$

Let  $\omega_f^a$  be the surface integral of this 2-form on the triangle  $f$  of the triangulation. Using the Maurer-Cartan equation (12.110), we have

$$\omega_f^a = \int_f \Sigma^a = \frac{1}{2} \int_f \epsilon^a{}_{bc} \omega^b \wedge \omega^c = \int_f d\omega^a = \oint_{\partial f} \omega^a. \quad (12.112)$$

The flux of the Plebanski 2-form across a triangle is then equal to the line integral of  $\omega^a$  along the boundary of the triangle. Let us note two properties of  $\omega_f^a$ : firstly, for each tetrahedron  $t$ , the relation

$$\sum_{f \in t} \omega_f^a = \sum_{f \in t} \oint_{\partial f} \omega^a = 0 \quad (12.113)$$

holds. Secondly, because of the  $SU(2)$  symmetry, the vectors  $\omega_f^a$  are proportional to the normals of a regular tetrahedron in  $\mathbb{R}^3$ . This way, the angle between two of them reads as  $\cos \theta_{ff'} = \omega_f^a \omega_{f'}^a = 1/3$ . In conclusion, the set of  $su(2)$  vectors  $\omega_f^a$  forms a natural background fiducial structure for the discrete theory, analogous to the  $\omega^a$  fiducial connection in the continuous theory.

Let us provide a physical meaning to the dipole cosmology model by identifying its degrees of freedom with the ones of Bianchi IX plus inhomogeneous perturbations.

#### 12.4.2 Isotropic sector: FRW

The phase space of the isotropic sector of Bianchi IX is coordinatized by the connection  $c = c(t)$  and the conjugate variable  $p = p(t)$  (see Eq. (12.83)). An embedding of the isotropic geometry in the phase space of the dipole cosmology model is defined by

$$U_f = \exp((c + \delta) \omega_f^a \tau_a), \quad E_f = p \omega_f^a \tau_a, \quad (12.114)$$

where  $\delta$  is a constant taking into account the curvature of the spatial manifold (see the discussion about the Christoffel symbols in Sec. 12.3) and  $\omega_f^a$  has been defined in Eq. (12.112).

Let us discuss the dynamics of this model. The Gauss constraint (12.105) is automatically satisfied because of (12.113), while the Hamiltonian constraint (12.106) becomes

$$\tilde{\mathcal{H}} = \frac{17}{6} p^2 (\cos(c - \delta) - 1) = 0. \quad (12.115)$$

The value of  $\delta$  is fixed by requiring the matching with ordinary classical dynamics. In the case of small connection ( $|c| \ll 1$ ) we recover the constraint

$$\tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}_{\text{FRW}} \propto -p^2 c(c - 1). \quad (12.116)$$

The appearance of the ordinary dynamics for small values of the connection  $c$  is in agreement with the claim that a coarse triangulation well approximates the classical theory for a low-curvature space-time. The constraint (12.115) can be seen as the effective constraint of the standard LQC. We have thus obtained a “natural” effective Hamiltonian constraint by using the discretized scalar constrained of the theory without need of a polymerization of the classical theory. In the triangulated loop quantum cosmology the polymerization is a direct consequence of the existence of the triangulation.

The quantization of the isotropic sector of the triangulated model is straightforward. The variable  $c$  multiplies the generator of a  $U(1)$  subgroup of the compact group  $SU(2)^4$ :  $c \in [0, 4\pi]$ . The kinematic Hilbert space is that of the square integrable functions on a circle as

$$\mathcal{F}_{\text{iso}} = L^2(S^1, dc/4\pi), \quad (12.117)$$

and the eigenstates of  $p$  are labeled by an integer  $\mu$  and read as  $\langle c|\mu\rangle = e^{i\mu c/2}$ . The wave functions  $\psi(c)$  can be decomposed in a Fourier series of eigenstates of  $p$ , labeled by an integer  $\mu$ , as

$$\psi(c) = \sum_n \psi_\mu e^{i\mu c/2}. \quad (12.118)$$

The fundamental operators on this representation are  $p$  and  $\exp(ic/2)$ , whose action on generic states reads as

$$p|\mu\rangle = (\mu/2)|\mu\rangle, \quad \exp(ic/2)|\mu\rangle = |\mu+1\rangle. \quad (12.119)$$

This way, the quantum constraint operator corresponding to that in Eq. (12.115) rewrites as a difference equation for the coefficients  $\psi_\mu = \langle c|\mu\rangle$ , explicitly reading as

$$D^+(\mu)\langle c|\mu+2\rangle + D^0(\mu)\langle c|\mu\rangle + D^-(\mu)\langle c|\mu-2\rangle = 0, \quad (12.120)$$

where  $D^{\pm 0}(\mu)$  are some coefficients. Equation (12.120) has the structure of the LQC difference equation (12.57). Notice that the discrete dynamics is recovered in this manner without recurring to the minimal area gap argument.

### 12.4.3 Anisotropic sector: Bianchi IX

The dynamics of a homogeneous anisotropic cosmological model is described by three scale factors which identify three independent directions (in the time evolution) of the Cauchy surfaces. In the connection formalism

(see Section 12.3), relaxing the isotropy condition corresponds to consider three different connections  $c_a = c_a(t)$  and momenta  $p_a = p_a(t)$ . The triangulated model can then be extended to an anisotropic setting by demanding that the variables of the theory are given by

$$U_f = \exp((c^a + \delta)\omega_f^a\tau_a), \quad E_f = p^a\omega_f^a\tau_a. \quad (12.121)$$

Let us notice a difference with respect to the usual LQC formulation of an anisotropic model (see Sec. 12.3). In that case, the holonomies  $h_a$  are directional objects computed along the edges parallel to the three axes individuated by the anisotropies, see Eq. (12.85). On the other hand, the variables  $U_f$  in Eq. (12.121) are non-directional objects, because the four faces of the triangulation do not have any special orientation with respect to the three isotropy axes. The connection components are summed over and they are thus independent on the  $a$ -direction. The  $U_f$  are in fact group element of  $SU(2)$  that depends on the face  $f$ .

As in the isotropic case, the dynamics of this model is summarized in the scalar constraint. The Gauss one does not carry out any informations since it identically vanishes thanks to Eq. (12.113) thus leaving three degrees of freedom. The Hamiltonian constraint is quite complex but, also in this case, can be regarded as the effective one of the standard LQC. In particular, the proper classical limit is recovered as soon as small connections  $|c_a| \ll 1$  are taken into account. For the complete analysis of this sector we refer to the original papers.

#### 12.4.4 *Full dipole model*

The dipole model has nine degrees of freedom and we have already identified three of them which correspond to be Bianchi IX ones. The remaining six are necessarily inhomogeneous, due to the generality of Bianchi IX, and can be regarded as inhomogeneous perturbations to the Bianchi IX Universe. This way, the inhomogeneous degrees of freedom are naturally captured by a truncated LQG model.

As we have seen in Sec. 9.1, inhomogeneous perturbations to the Mixmaster Universe are expressed in terms of Wigner  $D$ -functions  $D_{mm'}^j(g)$ . Let us translate this language to a first order formalism. The inhomogeneity implies that the basic 1-forms  $\omega_\alpha^a(x^\gamma)$  are replaced by time dependent 1-forms  $\tilde{\omega}_\alpha^a(x, t)$  which, in turn, are decomposed on the homogeneous basis, that is

$$\omega_\alpha^a(x) \rightarrow \tilde{\omega}_\alpha^a(x, t) = \omega_\alpha^a(x) + \varphi_\alpha^a(x, t). \quad (12.122)$$

The functions  $\varphi_\alpha^a(x, t)$  define the inhomogeneous perturbations to the model. Similarly to Sec. 9.1, the projection of the perturbations on the homogeneous basis  $\omega_\alpha^a(x^\gamma)$  and the subsequent expansion on the Wigner  $D$ -functions is accomplished by

$$\varphi_\alpha^a(x, t) = \varphi^{ab}(x, t) \omega_\alpha^b(x), \quad (12.123)$$

and

$$\varphi^{ab}(x, t) = \sum_{jm} \left( \sum_{m'=-j}^j \varphi_{jmm'}^{ab}(t) D_{mm'}^j(g(x)) \right), \quad (12.124)$$

where  $g(x)$  are  $SU(2)$  group elements which coordinatize  $S^3$ . Let us consider the lowest nontrivial integer modes, i.e. the ones with spin and magnetic numbers are given by  $j = 1$  and  $m' = 0$ , respectively. Assuming that the dynamical inhomogeneous degrees of freedom  $\varphi_{jmm'}^{ab}(t)$  are diagonal in the internal indices  $(a, b)$ , the lowest ones are given by

$$\varphi_{1,0,m}^{ab}(t) = \delta^{ab} \varphi_m^a(t) \quad m = -1, 0, 1, \quad (12.125)$$

providing nine degrees of freedom.

This framework can be straightforwardly casted into the triangulation formalism of the dipole model. The building blocks of the model are the  $su(2)$  vectors  $\omega_f^a$  which, by means of the decomposition (12.122), rewrite as

$$\tilde{\omega}_f^a = \frac{1}{2} \int_f \epsilon^a{}_{bc} \tilde{\omega}^b \wedge \tilde{\omega}^c \simeq \omega_f^a + \int_f \epsilon^a{}_{bc} \omega^b \wedge \varphi^c = \omega_f^a + \varphi_f^a, \quad (12.126)$$

where we have neglected the second order terms, giving the inhomogeneous discrete fiducial element. The dipole degrees of freedom are completely specified in terms of the three connections  $c_a(t)$  and the nine inhomogeneities  $\varphi_f^a(t)$  as

$$U_f(c^a, \varphi_m^a) = \exp(c^a \omega_f^a \tau_I) \exp(\varphi_f^a \tau_a) \exp(\delta \omega_f^a \tau_a). \quad (12.127)$$

Thus, all the 12 degrees of freedom of the dipole cosmology have been univocally identified with geometrical quantities.

It is worth noting that the the Gauss constraint (12.105) does not identically vanish any longer but can be split into the homogeneous and the inhomogeneous terms as

$$G^a = \sum_f \omega_f^a + \sum_f \varphi_f^a = 0. \quad (12.128)$$

The first part is the constraint which appears within the Bianchi IX framework and identically vanishes because of the Stokes theorem. The second

one gives three conditions on the inhomogeneous perturbations leading to nine degrees of freedom.

Summarizing, we have analyzed a finite dimensional truncation of LQG. In particular, a coarse triangulation of the physical space has been fixed and we have considered a discretization and quantization of GR on this triangulation. The model of a dipole  $SU(2)$  lattice theory, triangulating a topological three-sphere by means of two tetrahedra, has been related to a Bianchi IX cosmological model perturbed by six inhomogeneous degrees of freedom. By means of this scheme, the LQC dynamics arises directly from LQG *without the need of heuristic arguments*, like the minimal area gap or polymerization and in this sense this framework represents a way for merging LQC in LQG.

#### 12.4.5 Quantization of the model

Let us finally perform the quantization of the triangulated model. We deal with the generic triangulated model for which the dipole cosmology is a particular case. The quantization procedure follows the methods developed in lattice gauge theories.

A quantum representation of the observable algebra (12.102) is provided in the auxiliary Hilbert space

$$\mathcal{F}_{\text{aux}} = L_2[SU(2)^{2n}, dU_f], \quad (12.129)$$

where  $dU_f$  is the Haar measure, i.e. the states have the form  $\psi(U_f)$ . The operators  $U_f$  are diagonal and the operators  $E_f$  are given by the left invariant vector fields on each  $SU(2)$  element.

Gauss invariance is achieved by  $SU(2)$  spin network states on the dual graph  $\Delta_n^*$ . In the dual triangulation only four-valent vertices appear, which intertwine spin- $j_f$  representations  $D_{j_f}(U)$  associated to the holonomies around the links  $l_f$ . The basis of these spin-network states is labeled by

$$|j_f, \iota_t\rangle, \quad (12.130)$$

where  $\iota_t$  denotes the intertwiner quantum number at the given node. The spin network basis states are explicitly given by

$$\Psi_{j_f \iota_t}(U_f) = \langle U_f | j_f, \iota_t \rangle = \otimes_f D^{(j_f)}(U_f) \cdot \otimes_t \iota_t, \quad (12.131)$$

in which the dot operator “ $\cdot$ ” indicates the contraction of the indices of the matrices  $D^{(j_f)}(U)$  with the indices of the intertwiners  $\iota_t$ .



The quantum dynamics of the model is obtained following the Dirac prescription (see Sec. 12.1). The Hamiltonian constraint (12.106) can be directly implemented as a quantum operator rescaled by the volume  $V_t$ . The physical states are those satisfying the equation

$$\tilde{\mathcal{H}}_t \Psi = \sum_{ff' \in t} \text{Tr}[U_{ff'} E_{f'} E_f] \Psi = 0, \quad (12.132)$$

which corresponds to the early proposal by Rovelli and Smolin to perform the quantization of the Hamiltonian constraint in LQG. On the other hand, the Hamiltonian constraint can be defined *à la* Thiemann, as in Eq. (12.28), rewriting the Hamiltonian constraint (12.106) in the form

$$\mathcal{H}_t = \sum_{ff'f'' \in t} \epsilon^{ff'f''} \text{Tr}[U_{ff'} U_{f''}^{-1} \{U_{f''}, V_t\}] = 0 \quad (12.133)$$

and then defining the corresponding quantum operator by replacing the Poisson brackets with the commutators. Here the operator associated to the volume  $V_t$  turns out to be the standard LQG volume operator.

## 12.5 Snyder-Deformed Quantum Cosmology

This Section is devoted to present some results obtained in a recent approach to quantum cosmology. We discuss the implementation of a deformed Heisenberg algebra on the FRW cosmological models, having a clear contact with non-commutative geometries. In particular, we quantize a deformed symplectic structure of the minisuperspace and investigate the dynamics of this quantum Universe, finally comparing it with LQC. Since the algebra realizes the Snyder non-commutative space, we denote such approach as the *Snyder-deformed quantum cosmology*.

Let us consider the modified symplectic geometry arising from the classical limit of the Snyder-deformed Heisenberg algebra (see Sec. 11.5). It is then possible, considering  $s$  as an independent constant with respect to  $\hbar$ , to replace the quantum-mechanical commutator (11.74) via the Poisson brackets

$$-i[\tilde{q}, p] \implies \{\tilde{q}, p\} = \sqrt{1 - sp^2}, \quad (12.134)$$

where,  $\tilde{q}$  refers to the non-commutative coordinate and  $s \in \mathbb{R}$  is the deformation parameter with the dimension of a squared length. The ordinary algebra is recovered as  $s = 0$ . The relation (12.134) corresponds exactly to the *unique* (up to a sign) possible realization of the Snyder non-commutative space. In order to obtain the deformed Poisson brackets, one

requires that they possess the same properties as the quantum mechanical commutators, i.e. to be anti-symmetric, bilinear and to satisfy the Leibniz rules as well as the Jacobi identity. This way, the Poisson brackets (for any two-dimensional phase space function) are

$$\{F, G\} = \left( \frac{\partial F}{\partial \tilde{q}} \frac{\partial G}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial \tilde{q}} \right) \sqrt{1 - sp^2}. \quad (12.135)$$

The time evolution of the coordinate and momentum variables with respect to a given deformed Hamiltonian  $\mathcal{H}(\tilde{q}, p)$ , are specified as

$$\dot{\tilde{q}} = \{\tilde{q}, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p} \sqrt{1 - sp^2}, \quad \dot{p} = \{p, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial \tilde{q}} \sqrt{1 - sp^2}. \quad (12.136)$$

It is straightforward to implement such framework in a cosmological setting. Let us consider the FRW model in the presence of a generic matter, described by an energy density  $\rho = \rho(a)$  as in Sec. 10.8. The dynamics of this model is encoded in the Hamiltonian constraint (10.127). The FRW minisuperspace is then assumed to be Snyder-deformed and the commutator between the isotropic scale factor  $a$  and its conjugate momentum  $p_a$  is uniquely fixed by the relation

$$\{a, p_a\} = \sqrt{1 - sp_a^2}. \quad (12.137)$$

Considering such approach, the classical equations of motion of the model are modified as

$$\dot{a} = N\{a, \mathcal{H}_{\text{RW}}\} = N \left[ -\frac{\kappa}{12\pi^2} \frac{p_a}{a} \sqrt{1 - sp_a^2} \right], \quad (12.138a)$$

$$\begin{aligned} \dot{p}_a &= N\{p_a, \mathcal{H}_{\text{RW}}\} \\ &= -N \left( \frac{\kappa}{24\pi^2} \frac{p_a^2}{a^2} - \frac{6\pi^2}{\kappa} K + 6\pi^2 a^2 \rho + 2\pi^2 a^3 \frac{d\rho}{da} \right) \sqrt{1 - sp_a^2}. \end{aligned} \quad (12.138b)$$

As in the standard case, the equation of motion for the Hubble function ( $H = \dot{a}/a$ ) can be obtained solving the constraint (3.81) with respect to  $p_a$  and then considering Eq. (12.138a). Taking  $N = 1$ , it explicitly becomes

$$\left( \frac{\dot{a}}{a} \right)^2 = \left( \frac{\kappa}{3} \rho - \frac{K}{a^2} \right) \left[ 1 - \frac{48\pi^4 s}{\kappa} a^2 \left( a^2 \rho - \frac{3}{\kappa} K \right) \right], \quad (12.139)$$

providing the deformed Friedmann equation which entails the modifications arising from the Snyder-deformed Heisenberg algebra.

In order to make a comparison with respect to the LQC model described in Sec. 12.2, it is interesting to consider the flat FRW Universe, i.e. with  $K = 0$ . In this case, Eq. (12.139) reduces to

$$\left( \frac{\dot{a}}{a} \right)_{K=0}^2 = \frac{\kappa}{3} \rho \left( 1 - \text{sign } s \frac{\rho}{\rho_c} \right), \quad \rho_c = \frac{\kappa}{48\pi^4 |s|} \rho_{\text{P}}, \quad (12.140)$$

where  $\rho_P$  denotes the Planck energy density  $\rho_P = 1/l_P^4$ . In the last step the existence of a fundamental minimal length has been assumed, i.e. that the scale factor (the energy density) has a minimum (maximum) at the Planck scale.

The modifications arising from the deformed Heisenberg algebra in Eq. (12.140) are manifested in the  $\rho^2$  term. As soon as  $\rho$  reaches the critical value  $\rho_c$  (and  $s > 0$ ), the Hubble function vanishes and the Universe experiences a Big Bounce in the scale factor. For energy densities much smaller than the critical one, the standard Friedmann dynamics is recovered. In the same way, as the deformation parameter  $s$  vanishes, the correction term disappears and the ordinary behavior of the Hubble function is obtained.

Two interesting features have to be stressed.

- (i) The deformed Friedmann equation (12.140) in the  $s > 0$  case is equivalent, at a phenomenological level, to the one obtained for the effective LQC dynamics (12.69).
- (ii) The string inspired Randall-Sundrum braneworld scenario leads to a modified Friedmann equation as in Eq. (12.140) with  $s < 0$ . The opposite sign of the  $\rho^2$  term, is the well-known key difference between LQC and the Randall-Sundrum framework. In fact, the former approach leads to a non-singular bouncing cosmology while in the latter, because of the positive sign,  $\dot{a}$  cannot vanish and a cosmological bounce cannot take place. Of course, to obtain a bounce, the correction term should be negative, i.e. make a repulsive contribution.

Let us analyze the flat FRW model filled with a massless scalar field  $\phi$ . As usual (see Sec. 10.8), the energy density  $\rho_\phi$  is given by  $\rho_\phi = p_\phi^2/a^6$  while the phase space is four-dimensional with coordinates  $(a, p_a, \phi, p_\phi)$ . Since  $p_\phi$  is a constant of motion, each classical trajectory can be described in the  $(a, \phi)$ -plane. The scalar field  $\phi$  is considered as an internal clock ( $\dot{\phi} = 1$ ) as soon as the condition (10.129) for the lapse function holds. In this case, the deformed Friedmann Eq. (12.139) rewrites as

$$\left(\frac{\dot{a}}{a}\right)^2 = B^2 \left(1 - \frac{s}{B^2} \frac{p_\phi^2}{a^2}\right), \quad (12.141)$$

where  $B$  is defined in Eq. (10.130). The solution to Eq. (12.141) is given by

$$a(\phi) = \text{const} \times e^{-B\phi} \left[ \frac{s}{2} \left( \frac{1}{\pi B^2} \right)^2 p_\phi^2 + e^{2B\phi} \right]. \quad (12.142)$$

This equation clearly predicts a Big Bounce if  $s > 0$  and from now on we consider this case only.

As discussed in Sec. 10.8, the (effective) Hamiltonian in the internal time  $\phi$  description is expressed as

$$\mathcal{H}_e = B p_a a. \quad (12.143)$$

Given any observable  $\mathcal{O}$ , its evolution is governed by (10.133) taking into account the deformed commutators as in (12.137). The equations of motion

$$\begin{aligned} \frac{d}{d\phi} \langle a \rangle &= B \left\langle a \sqrt{1 - \frac{s}{B^2} \frac{p_\phi^2}{a^2}} \right\rangle, \\ \frac{d}{d\phi} \langle p_a \rangle &= -B \left\langle p_a \sqrt{1 - \frac{s}{B^2} \frac{p_\phi^2}{a^2}} \right\rangle \end{aligned} \quad (12.144)$$

hold and the trajectories are in agreement with the deformed classical ones. As before, to discuss the fate of the cosmological singularity at quantum level, we have to analyze the evolution of a semiclassical initial state. Let us remember that a semiclassical observable  $\mathcal{O}$  requires an expectation value close to the classical one with negligible fluctuations  $(\Delta\mathcal{O})^2$ , i.e.  $(\Delta\mathcal{O})^2 \ll \langle \mathcal{O} \rangle^2$ . The dynamics of the relative scale factor fluctuations is described by the equation

$$\frac{d}{d\phi} \left( \frac{(\Delta a)^2}{\langle a \rangle^2} \right) = 2B \frac{1}{\langle a \rangle^2} \left( \left\langle a^2 \sqrt{1 - \frac{s}{B^2} \frac{p_\phi^2}{a^2}} \right\rangle - \frac{\langle a^2 \rangle}{\langle a \rangle} \left\langle a \sqrt{1 - \frac{s}{B^2} \frac{p_\phi^2}{a^2}} \right\rangle \right). \quad (12.145)$$

As we have seen in Sec. 10.8, such quantity is conserved during the whole evolution in the ordinary framework ( $s = 0$ ) and thus the semiclassicality of an initial state is there preserved. Such property is also valid in the deformed scheme at late times  $|\phi| \rightarrow \infty$ , i.e. for large scale factor values  $a \gg \sqrt{s} p_\phi / l_P$ . At the bouncing time, i.e. when the scale factor reaches its minimum value

$$a_{\min} = \frac{\sqrt{s}}{B} p_\phi, \quad (12.146)$$

Eq. (12.145) vanishes. Although the relative scale factor fluctuations are in general not constant during the evolution, it is possible to show that the difference in the asymptotic values

$$D = \lim_{\phi \rightarrow \infty} \left| \left( \frac{(\Delta a)^2}{\langle a \rangle^2} \right)_{-\phi} - \left( \frac{(\Delta a)^2}{\langle a \rangle^2} \right)_\phi \right| \quad (12.147)$$

vanishes, since either the fluctuations  $(\Delta a)^2(\phi)$  either the mean value  $\langle a \rangle(\phi)$  are symmetric in time. This way, starting with a semiclassical state, for example Gaussian, such that  $(\Delta a)^2/\langle a \rangle^2 \ll 1$  at late times, this property is satisfied on the other side of the bounce when the Universe approaches large scales ( $a \gg \sqrt{s}p_\phi/l_P$ ).

Summarizing, a bouncing cosmology is predicted by a Snyder-deformed Friedmann dynamics and this model can be regarded as an attempt to mimic the original LQC system by a simpler one.

Two remarks are however in order.

- (i) LQC is based on a Weyl representation of the canonical commutation relations which is inequivalent to the Schrödinger representation. On the other hand, the Snyder-deformed algebra cannot be obtained by a canonical transformation of the ordinary Poisson brackets of the system.
- (ii) The  $\rho^2$  term in the effective Friedmann Eq. (12.69) is not the only correction from LQC unless the only matter source is a massless scalar field. If it has mass or is self-interacting, there are infinitely many other correction terms involving the pressure also. In the deformed quantum cosmology, the structure of Eq. (12.140) is independent of the detailed matter content.

## 12.6 GUP and Polymer Quantum Cosmology: The Taub Universe

In this Section we will compare the dynamics of the Taub Universe resulting from two different quantization techniques. The purpose is to quantize a cosmological model by implementing in the formalism a minimal length and to discuss the fate of the classical singularity. The model will be analyzed at classical and quantum level in both schemes. The two (quantum) frameworks are the so-called generalized uncertainty principle (GUP) and the polymer ones. For the first case, the model is quantized according to the commutation relations associated to an extended formulation of the Heisenberg algebra which reproduces the GUP (see Sec. 11.6). The polymer quantum dynamics (see Sec. 11.2) of the Taub Universe will then be considered. We stress that the polymer quantization is closely related to the LQC techniques discussed in Sec. 12.2. We first analyze the classical modified dynamics and then the quantum one.

### 12.6.1 Deformed classical dynamics

As seen in Sec. 10.10.1, the Taub model can be interpreted as a massless scalar relativistic particle (namely a photon) moving in the Lorentzian minisuperspace  $(\tau, x)$ -plane. The classical evolution corresponds to its light-cone in the configuration space. More precisely, the incoming particle ( $\tau < 0$ ) bounces on the wall ( $x = x_0 = \ln(1/2)$ ) and falls into the classical cosmological singularity ( $\tau \rightarrow \infty$ ). Investigations on the modifications of the dynamics within the GUP and polymer frameworks will show that those two behaviors can be regarded as *complementary*.

Let us firstly discuss the GUP case. The GUP (classical) dynamics is contained in the deformed phase space geometry arising from the classical limit of (11.77). The fundamental minisuperspace Poisson brackets are thus given by

$$\{x, p\} = 1 + sp^2. \quad (12.148)$$

Applying this scheme to the Hamiltonian (10.153), we obtain the equations of motion for the model, i.e.

$$x(\tau) = (1 + sA^2)\tau + \text{const}, \quad p(\tau) = \text{const} = A, \quad (12.149)$$

where  $x \in [x_0, \infty)$ .

Let us consider at the classical level, the effects of the deformed Heisenberg algebra (12.148) on the Taub Universe. The angular coefficient in the  $x(\tau)$  trajectory is given by  $(1 + sA^2) > 1$  for  $s \neq 0$ . Thus the angle between the two straight lines  $x(\tau)$ , for  $\tau < 0$  and  $\tau > 0$ , decreases as  $s$  grows. The trajectories of the particle (Universe), before and after the bounce on the potential wall at  $x = x_0 \equiv \ln(1/2)$ , are closer to each other than in the canonical case ( $s = 0$ ).

Let us discuss the polymer case. The polymer (classical) dynamics of the model can be summarized as the substitution of Eq. (11.48) in the Hamiltonian (10.153). This way, the equations of motion read as

$$\frac{dx}{d\tau} = \{x, \mathcal{H}_{\text{ADM}}^T\} = \cos(\mu_0 p), \quad \frac{dp}{d\tau} = \{p, \mathcal{H}_{\text{ADM}}^T\} = 0, \quad (12.150)$$

and are solved by

$$x(\tau) = \cos(\mu_0 p)\tau, \quad p(\tau) = \text{const} = A. \quad (12.151)$$

In the discretized (polymer) case, i.e. for  $\mu_0 \neq 0$ , the one-parameter family of trajectories flattens, as the angle between the incoming and the outgoing trajectories is greater than  $\pi/2$ , since  $p \in (-\pi/\mu_0, \pi/\mu_0)$ . Since these trajectories diverge rather than converge, we expect the polymer quantum effects to be reduced in comparison to the classical case, as we will verify below.

### 12.6.2 Deformed quantum dynamics

The quantum dynamics of the Taub Universe is here investigated according to the GUP and polymer approaches above considered. Particular attention is paid to the wave-packet evolution and to the fate of the classical cosmological singularity. In both frameworks, the variable  $\tau$  is regarded as a time coordinate and therefore  $(\tau, p_\tau)$  are treated in the canonical way. The deformed quantization (GUP or polymer) is then implemented *only* on the submanifold describing the degrees of freedom of the Universe, i.e. the phase space spanned by  $(x, p)$ . Thus, we deal with a Schrödinger-like equation as in the WDW case (see Sec. 10.10) given by

$$i\partial_\tau \Psi(\tau, p) = \hat{\mathcal{H}}_{\text{ADM}}^T \Psi(\tau, p), \quad (12.152)$$

where the operator  $\hat{\mathcal{H}}_{\text{ADM}}^T$  accounts for the modifications specific of the two frameworks. As above, we have to square the eigenvalue problem in order to correctly impose the boundary conditions: we will use the well-grounded hypothesis that the eigenfunctions form be independent of the presence of the square root, since its removal implies the square of the eigenvalues only. Wave packets of the form (10.157) are then constructed for both models. Some differences with respect to the ordinary scheme, as well as between the two generalized approaches, however occur. In particular, the differences are due to the distinct eigenfunctions and to the domain of definition of the variables. Analyzing such evolutions, the GUP Taub Universe appears to be probabilistically singularity-free. In the polymer case the cosmological singularity is not tamed by the cut-off-scale effects.

Let us consider the model in the GUP approach. All the information on the position is lost (see Sec. 11.6), so that the boundary conditions have to be imposed on the quasiposition wave function (11.87), that is

$$\psi(\zeta = \zeta_0) = 0. \quad (12.153)$$

Here  $\zeta_0 = \langle \psi_\zeta^{ml} | x_0 | \psi_\zeta^{ml} \rangle$ , in agreement with the discussion in Sec. 11.6. The form of the solution of Eq. (12.152) is the same as in the ordinary framework (see Sec. 10.10), i.e.

$$\psi_k(p, t) = \psi_k(p) e^{-ik\tau}, \quad \psi_k(p) = \delta(p^2 - k^2), \quad (12.154)$$

where  $k$  is the momentum conjugate to  $\tau$ . The functions  $\psi_\omega(p)$  are however modified with respect to the WDW case which in terms of the quasiposition wave function (11.87) read as

$$\begin{aligned} \psi_k(\zeta) = \frac{A}{k(1 + sk^2)^{3/2}} & \left[ \exp \left( i \frac{\zeta}{\sqrt{s}} \arctan(\sqrt{s}k) \right) \right. \\ & \left. - \exp \left( i \frac{(2\zeta_0 - \zeta)}{\sqrt{s}} \arctan(\sqrt{s}k) \right) \right], \end{aligned} \quad (12.155)$$

where  $A$  is a constant. In Eq. (12.155) the boundary conditions (12.153) have already been imposed.

The deformation parameter  $s$ , i.e. a non-zero minimal uncertainty in the anisotropy of the Universe, is responsible for the GUP effects on the dynamics. The modifications induced by the deformed Heisenberg algebra on the Universe dynamics are summarized in different  $s$ -regions. As soon as  $s$  becomes more and more important, i.e. when we are at a scale such as to appreciate the GUP effects, the evolution of the wave packets is different from the canonical case. These effects are present when the product  $k_0\sqrt{s}$  becomes remarkable (namely as  $k_0\sqrt{s} \sim \mathcal{O}(1)$ ),  $k_0$  being the energy at which the weighting function  $A(k)$  is peaked:

$$A(k) = k(1 + sk^2)^{3/2} e^{-\frac{(k-k_0)^2}{2\sigma^2}}. \quad (12.156)$$

For fixed  $k_0$  and for growing  $s$  values, the wave packets begin to spread and a constructive and destructive interference between the incoming and outgoing waves arises. They *escape* from the classical trajectories and approach a stationary (independent of  $\tau$ ) state close to the potential wall. A probability peak “near” the potential wall thus appears (see Fig. 12.2).

Such behavior reflects what happens at the classical level. In fact, the incoming and the outgoing trajectories shrink each other, so that a quantum probability interference is *a fortiori* predicted. On the other hand, the stationarity of the dynamics is a purely quantum GUP effect, and this behavior cannot be inferred from a deformed classical analysis. From this point of view, the classical singularity ( $\tau \rightarrow \infty$ ) is strongly probabilistically suppressed, because the probability to find the Universe is peaked just around the potential wall. This way, *the GUP-Taub Universe is singularity-free*.

Let us consider the polymer case. The quantum analysis of the model is obtained by choosing a discretized  $x$  space and solving the corresponding eigenvalue problem in the  $p$  polarization. Considering the time evolution of the wave function  $\Psi$ , one obtains the following eigenvalue problem

$$(p^2 - k^2)\psi_k(p) = \left[ \frac{2}{\mu_0^2} (1 - \cos(\mu_0 p)) - k^2 \right] \psi_k(p), \quad (12.157)$$

whose solution stands as

$$k^2 = k^2(\mu_0) = \frac{2}{\mu_0^2} (1 - \cos(\mu_0 p)) \leq k_{\max}^2 = \frac{4}{\mu_0^2} \quad (12.158a)$$

$$\psi_{k,\mu_0}(p) = A\delta(p - p_{k,\mu_0}) + B\delta(p + p_{k,\mu_0}) \quad (12.158b)$$

$$\psi_{k,\mu_0}(x) = A[\exp(ip_{k,\mu_0}x) - \exp(ip_{k,\mu_0}(2x_0 - x))]. \quad (12.158c)$$



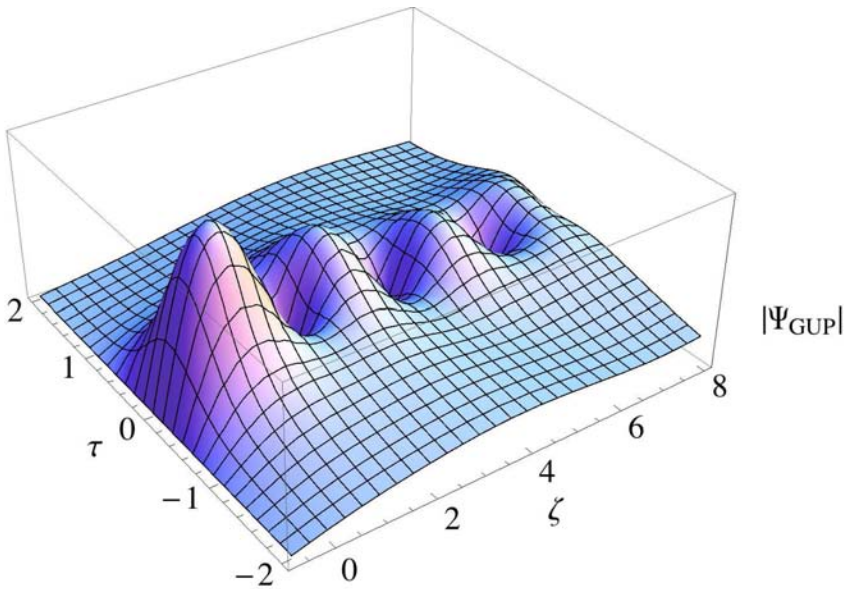


Figure 12.2 Wave packets  $|\Psi(\tau, \zeta)|$  of the Taub Universe in the GUP framework as  $sk_0^2 = 1$  ( $k_0 = 1$  and  $\sigma = 4$ ).

Here Eq. (12.158b) provides the momentum wave function, with  $A$  and  $B$  being two arbitrary integration constants, and Eq. (12.158c) is the coordinate wave function, where an integration constant has been dropped by imposing suitable boundary conditions. Moreover, the modified dispersion relation

$$p_{k, \mu_0} = \frac{1}{\mu_0} \arccos \left( 1 - \frac{k^2 \mu_0^2}{2} \right) \quad (12.159)$$

has been obtained from (12.158a). Let us stress that  $k^2$  is bounded from above, as provided by Eq. (12.158a). However, its square root, considered for its positive determination, accounts for the time evolution of the wave function.

The next step is to construct suitable wave packets  $\Psi(x, \tau)$ , weighted with a Gaussian centered at  $k_0$ , accounting for the previous discussion (note that a maximum energy  $k_{\max}$  is now predicted). At a fixed  $k_0$ , an interference phenomenon between the wave and the wall appears and it becomes more relevant as  $\mu_0$  increases. Nevertheless, this interference cannot tame the singularity ( $\tau \rightarrow \infty$ ), as it takes place in the “outer” region, in a way complementary to the GUP approach (see Fig. 12.3). Then, *the polymer-*

*Taub Universe is then still a singular cosmological model.*

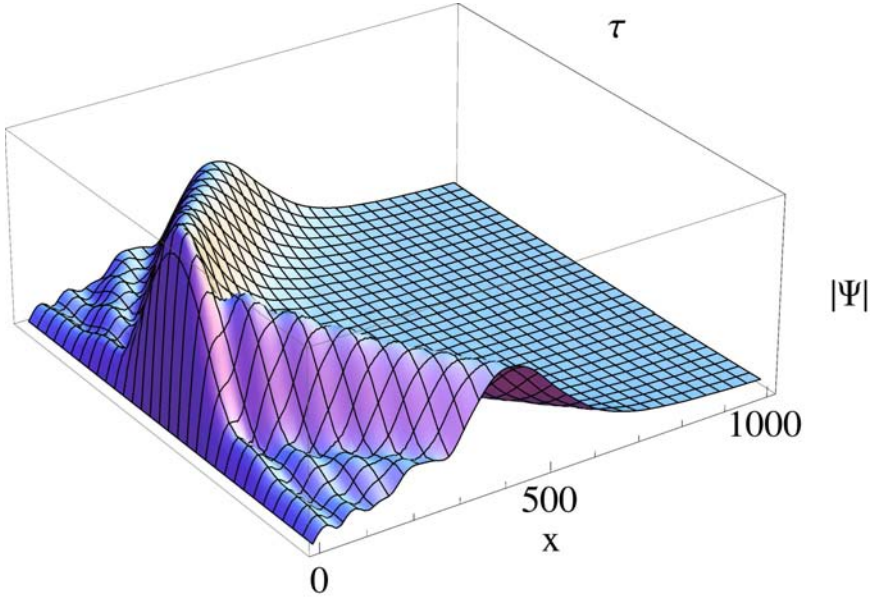


Figure 12.3 The spread polymer wave packet  $|\Psi(x, \tau)|$  for the Taub Universe as  $k_0\mu_0 = 1/2$  ( $\mu_0 = 50$ ,  $k_0 = 0.01$ ,  $\sigma = 0.125$ ).

Summarizing, the Taub cosmological model offers a suitable scenario to apply and compare different quantization techniques. It is possible to single out a time variable, so that the anisotropy describes the real degree of freedom of the Universe and therefore to investigate the fate of the cosmological singularity without modifying the time variable.

The non-removability of the cosmological singularity within the polymer framework could seem apparently in contradiction with other models (see Sec. 12.2), but there are at least two fundamental differences:

- (i) The variable  $\tau$ , which describes the isotropic expansion of the Universe, is not discretized but treated in the ordinary way. The only anisotropy variable is discretized without modifying the volume (time). In the FRW case, the scale factor of the Universe is directly quantized by the use of the polymer (loop) techniques, thus the evolution itself of the wave packet of the Universe is deeply modified.
- (ii) In the Taub case the variable  $p$ , conjugated to the anisotropy, is a

constant of motion and, from the Schrödinger equation, it describes also  $k$ , namely the energy of the system. According to the polymer scheme (see Sec. 11.2), it is always possible to choose a scale  $\mu_0$  for which the polymer effects are negligible during the whole evolution, at classical level. On the other hand, the Hamiltonian constraint in the FRW case does not allow for a constant solution of the variable conjugate to the scale factor, and it is not possible to choose a scale such that the polymer modifications are negligible throughout the whole evolution.

Comparing the GUP to the polymer approach allows us to infer that it is not always sufficient to “deform” the anisotropy variable to obtain significant modifications on the Universe evolution. However, the polymer paradigm is a Weyl representation of the commutation relations, while the generalized commutation relations cannot be obtained by a canonical transformation of the Poisson brackets.

## 12.7 Mixmaster Universe in the GUP Approach

As we have seen in the previous Section, the classical singularity of the Taub cosmological model is tamed by GUP effects. Recalling that the Taub is a special case of the Bianchi IX model, it is natural to investigate the GUP effects on the Mixmaster dynamics. This Section addresses the Mixmaster in the GUP framework, paying particular attention to the fate of its chaotic behavior.

The ADM Hamiltonian of a homogeneous cosmological model (see Section 8.2.4) is given by

$$-p_\alpha = \mathcal{H}_{\text{ADM}} = (p_+^2 + p_-^2 + \mathcal{V})^{1/2}, \quad (12.160)$$

where the lapse function  $N = N(t)$  has been fixed by the time gauge  $\dot{\alpha} = 1$  as in Eq. (8.41). The function  $\mathcal{H}_{\text{ADM}}$  is a time-dependent Hamiltonian from which it is possible to extract, for a given symplectic structure, all the dynamical information.

Let us investigate the modifications to the classical dynamics induced by the GUP algebra (see Sec. 11.6). This way, only the phase space spanned by the anisotropy variables (and their conjugate momenta) is deformed. The time variable  $\alpha$ , i.e. the isotropic volume of the Universe, is treated in the standard way. Considering the symplectic algebra (11.89), the time

evolution of the anisotropies and momenta, with respect to the ADM Hamiltonian (12.160), is given by ( $i, j = \pm$ )

$$\dot{\beta}_i = \{\beta_i, \mathcal{H}_{\text{ADM}}\} = \frac{1}{\mathcal{H}_{\text{ADM}}} [(1 + sp^2)\delta_{ij} + 2sp_i p_j] p_j, \quad (12.161a)$$

$$\dot{p}_i = \{p_i, \mathcal{H}_{\text{ADM}}\} = \frac{-1}{2\mathcal{H}_{\text{ADM}}} [(1 + sp^2)\delta_{ij} + 2sp_i p_j] \frac{\partial \mathcal{V}}{\partial \beta_j}, \quad (12.161b)$$

where the dot denotes differentiation with respect to  $\alpha$  and  $p^2 = p_+^2 + p_-^2$ . These are the deformed equations of motion for the homogeneous Universes, while the ordinary ones are recovered for the  $s = 0$  case.

Let us discuss the Bianchi I model which corresponds to the case  $\mathcal{V} = 0$  and thus, from Eq. (12.160), it is described by a two-dimensional massless scalar relativistic particle. The velocity of the particle (Universe) is modified by the deformed symplectic geometry and, from Eq. (12.161a), it reads as

$$\dot{\beta}^2 = \dot{\beta}_+^2 + \dot{\beta}_-^2 = \frac{p^2}{\mathcal{H}_{\text{ADM}}^2} (1 + 6\mu + 9\mu^2) = 1 + 6\mu + 9\mu^2, \quad (12.162)$$

where  $\mu = sp^2$ . For  $s \rightarrow 0$  ( $\mu \ll 1$ ), the standard Kasner velocity  $\dot{\beta}^2 = 1$  is recovered (see Sec. 8.2.2). The effect of a cut-off on the anisotropies then implies that the point-Universe moves faster than the ordinary case. In such deformed scheme the solution is still Kasner-like, that is

$$\dot{\beta}_{\pm} = C_{\pm}(s), \quad \dot{p}_{\pm} = 0, \quad (12.163)$$

but this behavior is modified by Eq. (12.162). In particular, the second relation between the Kasner indices (see Eq. (7.51b))  $p_1, p_2, p_3$  is deformed as

$$p_1^2 + p_2^2 + p_3^2 = 1 + 4\mu + 6\mu^2, \quad (12.164)$$

while the first one  $p_1 + p_2 + p_3 = 1$  remains unchanged (see Eq. (7.51a)).

Two considerations are in order.

- (i) The GUP acts in an opposite way with respect to a massless scalar field (or stiff-fluid, with pressure equal to density) in the standard model (see Sec. 8.7.1). In that case the chaotic behavior of the Mixmaster Universe is tamed. On the other hand, in the GUP framework, all the terms on the right-hand side of Eq. (12.164) are positive, thus the Universe cannot isotropize, i.e. it cannot reach the stage such that the Kasner indices are all equal.

- (ii) For every non-zero  $\mu$ , two indices can be negative at the same time. Thus, as the volume of the Universe contracts toward the classical singularity, the distances can shrink along one direction and grow along the other two. In the ordinary case the contraction is along two directions.

The natural bridge between Bianchi I and the Mixmaster Universe is represented by the Bianchi II model. Bianchi II is described by a potential term  $\mathcal{V}(\alpha, \beta_{\pm}) \propto e^{4\alpha} e^{-8\beta_{+}}$  which can be directly recovered from the one of Bianchi IX in the asymptotic region  $\beta_{+} \rightarrow -\infty$ . The BKL map of Bianchi IX is obtained considering such a simplified model since it is, in the ordinary framework, an integrable system. As we have seen in Sec. 7.4.2, the BKL map is at the basis of the chaotic analysis of the Mixmaster Universe and it is given by the reflection law of the particle (the point Universe) against the potential walls.

A fundamental difference between the deformed and the ordinary frameworks is that the ADM Hamiltonian  $\mathcal{H}_{\text{ADM}}$  is no longer a constant of motion near the classical singularity, since the wall velocity  $\dot{\beta}_{\text{wall}}$  is modified as

$$\dot{\beta}_{\text{wall}} = \frac{1}{36\mu} \left( -4 + 22^{1/3} \Xi^{-1/3} + 2^{2/3} \Xi^{1/3} \right), \quad (12.165)$$

where

$$\Xi = 2 + 81\mu\dot{\beta}^2 + 9\sqrt{\mu\dot{\beta}^2(4 + 81\mu\dot{\beta}^2)}. \quad (12.166)$$

From the two velocity equations (12.162) and (12.165), it is possible to understand the details of the bounce. In the standard case, the particle (Universe) moves twice as fast as the receding potential wall, independently of its momentum (namely, of its energy). In the GUP framework, the particle velocity, as well as the velocity of the potential wall, depends on the anisotropy momentum and on the deformation parameter  $s$ . Also in this case the particle moves faster than the wall since the relation

$$\dot{\beta}_{\text{wall}} < \dot{\beta} \quad (12.167)$$

is always verified (see Fig. 12.4). A bounce also takes place in the deformed picture. Furthermore, in the asymptotic limit  $\mu \gg 1$  the maximum angle such that the bounce against the wall occurs is given by

$$|\theta_{\text{max}}| = \arccos \left( \frac{\dot{\beta}_{\text{wall}}}{\dot{\beta}} \right) = \frac{\pi}{2}. \quad (12.168)$$

This is in contrast to the ordinary case ( $\dot{\beta}_{\text{wall}}/\dot{\beta} = 1/2$ ) where the maximum incidence angle is given by  $|\theta_{\text{max}}| = \pi/3$ . The particle bounce against the

wall is thus *improved* in the sense that a maximum limit angle doesn't appear anymore. The main difference with respect to the ordinary picture is however that the deformed Bianchi II model is not analytically solvable. No reflection map can be in general inferred since in the GUP picture it is no longer possible to identify two constants of motion.

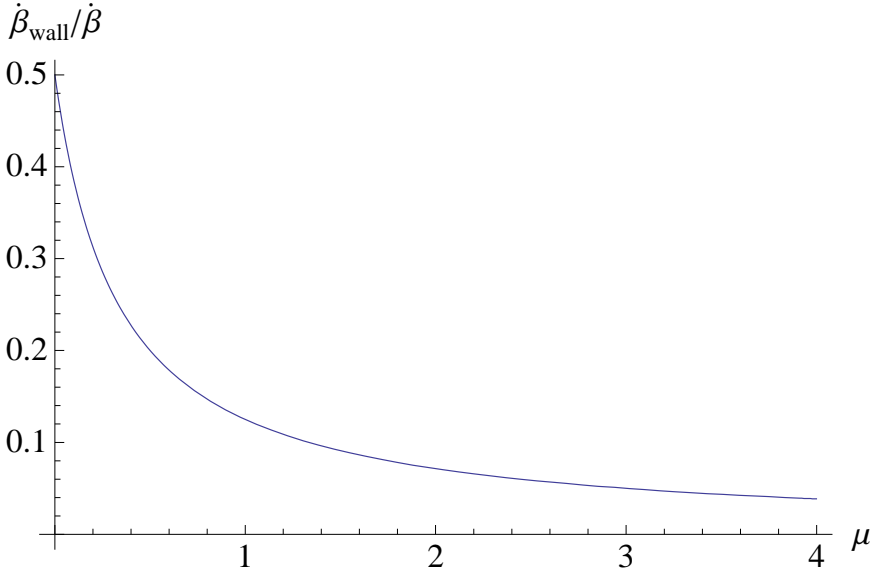


Figure 12.4 The potential wall velocity  $\dot{\beta}_{\text{wall}}$  of the Bianchi IX model with respect to the particle one  $\dot{\beta}$  in function of  $\mu = sp^2$ . In the  $\mu \rightarrow 0$  limit, the ordinary behavior  $\dot{\beta}_{\text{wall}}/\dot{\beta} = 1/2$  is recovered.

On the basis of the previous analysis, we get several features of the GUP Mixmaster Universe. The potential term of Bianchi IX is given by (8.37b) and its evolution is that of a two-dimensional particle bouncing an infinite number of times against three walls which rise steeply toward the singularity, with every single bounce described by the Bianchi II model. Between two subsequent bounces the system is described by a Kasner evolution and the permutations of the expanding-contracting directions is given by the BKL map showing a chaotic behavior (see Sec. 7.4.1).

Two conclusions on the GUP-deformed Mixmaster Universe can be inferred.

- (i) When the ultra-deformed regime is reached ( $\mu \gg 1$ ), i.e. when the

- point Universe has a momentum larger than the cut-off one, the triangular closed domain appears to be stationary with respect to the particle itself. The bounces of the particle are then increased by the presence of a non-zero minimal uncertainty in the anisotropies.
- (ii) In general, a BKL map (reflection law) cannot be obtained. This arises analyzing the single bounce against any wall of the equilateral-triangular domain, but the Bianchi II model is no longer a solvable system in the deformed picture. The chaotic behavior of the Bianchi IX model is not tamed by GUP effects, i.e. *the GUP Mixmaster Universe is still a chaotic system.*

In conclusion, it is interesting to point out the differences between this model and the Mixmaster dynamics in LQC described in Sec. 12.3. In the LQC scheme, the classical reflections of the point particle stop after a finite amount of time and the Mixmaster chaos is suppressed. In that framework, although the analysis is performed through the ADM reduction of the dynamics similarly to the GUP case, all the three scale factors are quantized using the LQG techniques. On the other hand, the time variable (related to the volume of the Universe) is here treated in the standard way and only the two physical degrees of freedom of the Universe (the anisotropies) are deformed.

## 12.8 Guidelines to the Literature

The LQC theory presented in Sec. 12.1 is described in the books of Rovelli [398] and of Thiemann [438] and in the reviews [28, 384, 422, 437]. Pedagogical expositions of the subject can be found in [120, 341], while for a critical point of view, see [364]. The first paper on the loop representation in GR is [400], while the spectrum of the area operator was firstly obtained in [401]. The original construction of the Hamiltonian constraint is in [436]. The original paper on the Wilson loop technique is [466], while a good textbook is that of Makeenko [334].

LQC, discussed in Sec. 12.2, is reviewed in [23–25, 94]. The absence of singularity in LQC was shown in [91, 92] and developed in [29, 30]. The effective equations of motion are described in [432]. For the improved dynamics in LQC see [31, 127]. For a comparison between LQC and WDW approach, see [98].

The Bianchi IX model (Sec. 12.3) has been analyzed in LQC in [95–97]. For recent developments in homogeneous LQC, see for example, [33, 34].

The triangulated version of LQC analyzed in Sec. 12.4 has been proposed in [49] developing the basic idea presented in [402].

The implementation of a Snyder-deformed Heisenberg algebra in quantum cosmology as discussed in Sec. 12.5 has been developed in [46].

The GUP and polymer quantizations of the Taub Universe (Sec. 12.6) are discussed in [54] and [48] respectively. The GUP quantum dynamics of the FRW is studied in [53]. Different implementations of the GUP paradigm in quantum cosmology can be found, for example, in [444–446].

The dynamics of the Mixmaster Universe in the GUP approach as presented in Sec. 12.7 is in [55]. For related studies, see [359].



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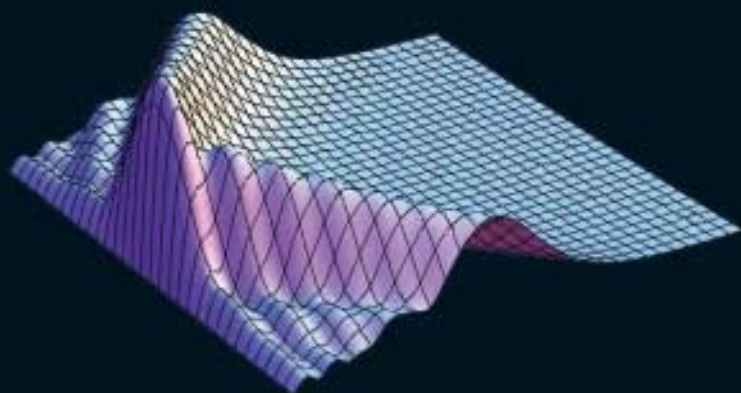
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# PRIMORDIAL COSMOLOGY

*Primordial Cosmology* deals with one of the most puzzling and fascinating topics debated in modern physics — the nature of the Big Bang singularity. The authors provide a self-consistent and complete treatment of the very early Universe dynamics, passing through a concise discussion of the Standard Cosmological Model, a precise characterization of the role played by the theory of inflation, up to a detailed analysis of the anisotropic and inhomogeneous cosmological models. The most peculiar feature of this book is its uniqueness in treating advanced topics of quantum cosmology with a well-traced link to more canonical and pedagogical notions of fundamental cosmology.

This book traces clearly the backward temporal evolution of the Universe, starting with the Robertson–Walker geometry and ending with the recent results of loop quantum cosmology in view of the Big Bounce. The reader is accompanied in this journey by an initial technical presentation which, thanks to the fundamental tools given earlier in the book, never seems heavy or obscure.

